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DISTRIBUTION OF ZEROS OF SOLUTIONS OF CERTAIN PERIODIC DIFFERENTIAL EQUATIONS

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Consider a differential equation

$$y'' = q(t)\,y) \tag{q}$$

where q is a continuous function on an interval (a, b) and periodic with period π, $-\infty \leqq a < b \leqq \infty$. If Floquet Theory gives for the equation (q) a real characteristic exponent $\varrho > 0$, we can find two independent solutions of the differential equation (q) in the form

$$\begin{aligned} u(t) &= e^{\varrho t}\, p_1(t), \\ v(t) &= e^{-\varrho t}\, p_2(t), \end{aligned} \tag{1}$$

where p_1, p_2 are real periodic functions with period π having continuous derivatives up to and including the order 2, e.g. see [1]. In this paper we shall deal with the mentioned case. We shall investigate the distribution of all zeros of solutions of an oscillatory differential equation of the type (q), supposing we know the zeros of the solutions u, v on the interval $\langle t_0, t_0 + \pi)$.

First of all we determine the asymptotic behaviour of the first phase α corresponding to the pair of the independent solutions u, v.

Then from the form of the asymptotic behaviour of the phase we derive the final results.

The first phase α corresponding to the pair u, v is defined as a continuous function on the interval (a, b) that satisfies the relation

$$\tan \alpha(t) = \frac{u(t)}{v(t)} \tag{2}$$

everywhere, where $v(t) \neq 0$, see [1]

The above defined function always exists and it has the two following properties:

$$\alpha'(t) = \frac{-W(u, v)}{u^2(t) + v^2(t)} \neq 0, \tag{3}$$

where $W(u, v) = \text{const.} \neq 0$ denotes the Wronskian of u, v, and

(4) $$\alpha \in C^3(a, b).$$

Remark: Let u, v be two independent solutions of the equation (q) in the form (1). Let $a_{00}, a_{10}, \ldots, a_{k0}$ be the zeros of the solution u and $b_{00}, b_{10}, \ldots, b_{k0}$ be the zeros of the solution v on the interval $\langle t_0, t_0 + \pi)$. Let $k \geqq 1$.

From the periodicity of the functions p_1, p_2 all zeros of the solution u are $a_{in} = a_{i0} + n\pi$ and all zeros of the solution v are $b_{in} = b_{i0} + n\pi$, $i = 0, 1, \ldots, k$, $n = \ldots, -1, 0, 1, \ldots$

On each of the intervals $\langle t_0 + n\pi, t_0 + (n + 1)\pi)$, $n = \ldots, -1, 0, 1, \ldots$, zeros of the solutions u, v must fulfil one of the following inequalities:
either

(5) $$a_{0n} < b_{0n} < a_{10} < b_{10} < \ldots a_{k-1,n} < b_{k-1,n}$$

or

(6) $$b_{0n} < a_{0n} < b_{10} < a_{10} < \ldots b_{k-1,n} < a_{k-1,n}$$

Without loss of generality (by a suitable choise of t_0) we suppose that the inequality (5) holds.

Definition: Let

$$\alpha^+(t) := -\text{sign } W(u, v)\,\pi\left[\frac{1}{2} + i + nk\right], \quad t \in (a_{in}, a_{i+1,n}),\ i = 0, 1, \ldots, k - 1,$$
$$t \in (a_{kn}, a_{0,n+1}),$$
$$:= -\text{sign } W(u, v)\,\pi[i + nk], \quad t = a_{in},\ i = 0, 1, \ldots, k$$

for $n = \ldots -1, 0, 1, \ldots$ and

$$\alpha^-(t) := -\text{sign } W(u, v)\,\pi[i + nk], \quad t \in (b_{i-1,n}, b_{in}) \qquad i = 1, 2, \ldots, k$$
$$t \in (b_{kn}, b_{0n})$$
$$:= -\text{sign } W(u, v)\,\pi\left[\frac{1}{2} + i + nk\right], \quad t = b_{in},\ i = 0, 1, \ldots, k,$$
$$\text{for } n = \ldots -1, 0, 1 \ldots$$

Theorem 1: *Let an oscillatory periodic differential equation* (q) *with real non-zero characteristic exponents be given. Let* u, v *denote two independent solutions of* (q) *in the form* (1) *with zeros* a_{in}, b_{in}, $i = 0, 1, \ldots, k$, $n = \ldots -1, 0, 1, \ldots$ *The phase* α *corresponding to the pair* u, v *has the following asymptotic behaviour*:

(7) $$\lim_{n \to \infty} [\alpha(t + n\pi) - \alpha^+(t + n\pi)] = 0,$$
$$\lim_{n \to -\infty} [\alpha(t + n\pi) - \alpha^-(t + n\pi)] = 0,$$

Proof. Let ϱ denote the positive characteristic exponent of (q). For the phase α we have

(8) $$\alpha(t) = \operatorname{arc\,tan}\frac{u(t)}{v(t)} - \operatorname{sign} W(u, v)\,\pi[i + nk],$$

for $t \in (b_{i-1,n}, b_{in})$, $i = 1, 2, \ldots, k$, $t \in (b_{k,n-1}, b_{0n})$

where $$-\frac{\pi}{2} \leqq \operatorname{arc\,tan}\frac{u(t)}{v(t)} \leqq \frac{\pi}{2}$$

Hence:

(9) $$\alpha(a_{in}) = -\operatorname{sign} W(u, v)\,\pi[i + nk]$$

(10) $$\alpha(b_{in}) = -\operatorname{sign} W(u, v)\,\pi\left[\frac{1}{2} + i + nk\right],$$

for $i = 0, 1, \ldots, k$, $n = \ldots -1, 0, 1, \ldots$

Let $W(u, v) < 0$. Then

$$\frac{p_1(t)}{p_2(t)} > 0 \quad \text{for } t \in (a_{in}, b_{in}), \quad i = 0, 1, \ldots, k, \quad n = \ldots, -1, 0, 1, \ldots$$

$$\frac{p_1(t)}{p_2(t)} < 0 \quad \text{for } t \in (b_{in}, a_{i+1,n}), \quad i = 0, 1, \ldots, k - 1$$

$$t \in (b_{kn}, a_{0,n+1})$$

$$n = \ldots -1, 0, 1, \ldots$$

From (2) we have: $$\tan\alpha(t) = \frac{u(t)}{v(t)} = e^{2\varrho t}\frac{p_1(t)}{p_2(t)}$$

and

$$\lim_{n\to\infty}\tan\alpha(t + n\pi) = \lim_{n\to\infty} e^{2\varrho(t+n\pi)}\frac{p_1(t)}{p_2(t)} = \infty$$

for $t \in (a_{i0}, b_{i0})$, $i = 0, 1, \ldots, k$

$$\lim_{n\to\infty}\tan\alpha(t + n\pi) = \lim_{n\to\infty} e^{2\varrho(t+n\pi)}\frac{p_1(t)}{p_2(t)} = -\infty$$

for $t \in (b_{i0}, a_{i+1,0})$, $i = 0, 1, \ldots, k - 1$,

and $t \in (b_{k0}, a_{01})$

For $W(u, v) > 0$ we have

$$\lim_{n\to\infty}\tan\alpha(t + n\pi) = -\infty \quad \text{for } t \in (a_{i0}, b_{i0}), \ i = 0, 1, \ldots, k$$

$$\lim_{n\to\infty}\tan\alpha(t + n\pi) = \infty \quad \text{for } t \in (b_{i0}, a_{i+1,0}), \ i = 0, 1, \ldots, k - 1,$$

and $t \in (b_{k0}, a_{01})$

From the last four relations we obtain:

$$\lim_{n\to\infty}\left[\alpha(t+n\pi) - \operatorname{sign} W(u,v)\,\pi\left(\frac{1}{2}+i+nk\right)\right] = 0 \tag{11}$$

$$\text{for } t\in(a_{i0}, b_{i0}), \quad i = 0, 1, \ldots, k$$

$$\text{for } t\in(b_{i0}, a_{i+1,0}), \quad i = 0, 1, \ldots, k-1$$

$$\text{for } t\in(b_{k0}, a_{0,n+1})$$

The relations (9), (10), (11) can be simply expressed in the form:

$$\lim_{n\to\infty}\left[\alpha(t+n\pi) - \alpha^+(t+n\pi)\right] = 0$$

The relation (7) for $n \to -\infty$ is thus proved.

The proof for $n \to -\infty$ is analogous.

Theorem 2: *Let w be a non-trivial solution of the oscillatory periodic differential equation (q), which is independent on the solution u and v. Let the assumptions of the Theorem I be satisfied. Denote the zeros of w by c_{in}, $n = \ldots -1, 0, 1, \ldots$ such that $c_{in} \in (a_{in}, a_{i+1,n})$ for $i = 0, 1, 2, \ldots, k-1$, and $c_{kn} \in (a_{kn}, a_{0,n+1})$ for $n = \ldots -1, 0, 1, \ldots$*

If $c_{rs} \in (a_{rs}, b_{rs})$ for some $r = 0, 1, \ldots, k-1$ and $s = \ldots -1, 0, 1, \ldots,$ then $(c_{in} - a_{in}) \to 0_+$ for $n \to \infty$ and $(c_{in} - b_{in}) \to 0_-$ for $n \to -\infty$

If $c_{rs} \in (b_{rs}, a_{r+1,s})$ or $c_{k-1,s} \in (b_{k-1,s}, a_{0,s+1})$, then $(c_{in} - a_{i+1,n}) \to 0_-$ for $n \to \infty$ and $(c_{in} - b_{in}) \to 0_+$ for $n \to -\infty$.

Proof. According to Sturm Comparison Theorem the zeros of the solution w must be mutual separated with the zeros of the solution u and also with the zeros of the solution w. The points c_{in} can therefore be either only in the intervals (a_{in}, b_{in}) for $i = 0, 1, \ldots, k$, $n = \ldots -1, 0, 1, \ldots$ or only in the intervals $(b_{in}, a_{i+1,n})$, $i = 0, 1, \ldots, k-1$, $(b_{kn}, a_{0,n+1})$, $n = \ldots -1, 0, 1, \ldots$

On the interval (a, b) Abel's equation is fulfilled

$$\alpha(\varphi(t)) = \alpha(t) + \pi \operatorname{sign} \alpha'(t),$$

where φ is the dispersion of the given differential equation, see again [2] or [3].

Hence

$$\alpha(\varphi_k(t)) = \alpha(t) + k\pi \operatorname{sign} \alpha'(t),$$

where φ_k is the k-th iteration of the function φ.

The last equation can be transformed into the form:

$$\alpha(\varphi_k(t)) = \alpha(t) - k\pi \operatorname{sign} W(u, v).$$

We shall consider the case $c_{ir} \in (a_{ir}, b_{ir})$. Suppose $W(u, v) < 0$. Then the first phase α is increasing with respect to (3). We have

$$\alpha(c_{ir}) < \alpha(b_{ir}) = \left(\frac{1}{2} + i + rk\right)\pi. \tag{12}$$

Let $c_{i,r+n} - a_{i,r+n} \nrightarrow 0_+$ as $n \to \infty$. Then there exists an ε_0 such that $c_{i,r+n} - a_{i,r+n} > \varepsilon_0$ for infinity many indices $n \in N_0$. Hence we have

$$\alpha(a_{i,r+n} + \varepsilon_0) < \alpha(c_{i,r+n}) \qquad \text{for } n \in N_0.$$

Since $a_{i,r+n} = a_{i,r} + n\pi$, and $c_{i,r+n} = \varphi_{nk}(c_{i,r})$, we get

$$\alpha(a_{i,r} + n\pi + \varepsilon_0) < \alpha(\varphi_{nk}(c_{i,r})) = \alpha(c_{i,r}) + nk\pi.$$

Applying Theorem 1 for $t := a_{i,r} + \varepsilon_0$ and $n \in N_0$, $n \to \infty$, the last relation gives

$$\alpha(a_{i,r} + \varepsilon_0 + n\pi) - \alpha^+(a_{i,r} + \varepsilon_0 + n\pi) <$$

$$< \alpha(c_{i,r}) + nk\pi - \left(\frac{1}{2} + i + (r + nk)\right)\pi, \qquad \text{or}$$

$$0 \leqq \alpha(c_{i,r}) - \left(\frac{1}{2} + i + rk\right)\pi \qquad \text{or} \qquad \left(\frac{1}{2} + i + rk\right)\pi \leqq \alpha(c_{i,r}),$$

that is a contradiction to (12). Hence for the case Theorem 2 is proved. Other cases can be proved analogously.

Note. On the basis of Theorem 1 and 2 can observe that for the studied differential equations there always exist two special solutions, u and v, zeros of which are distributed with the same density, whereas zeros of other (linearly independent on u and v) solutions cumulate near zeros of u and grow distant from zeros of v for $t \to \infty$. Zeros of u are of some kind of attractors and zeros of v are accessors of zeros of other solutions for $t \to \infty$. For $t \to -\infty$ the role of u and v is interchanged.

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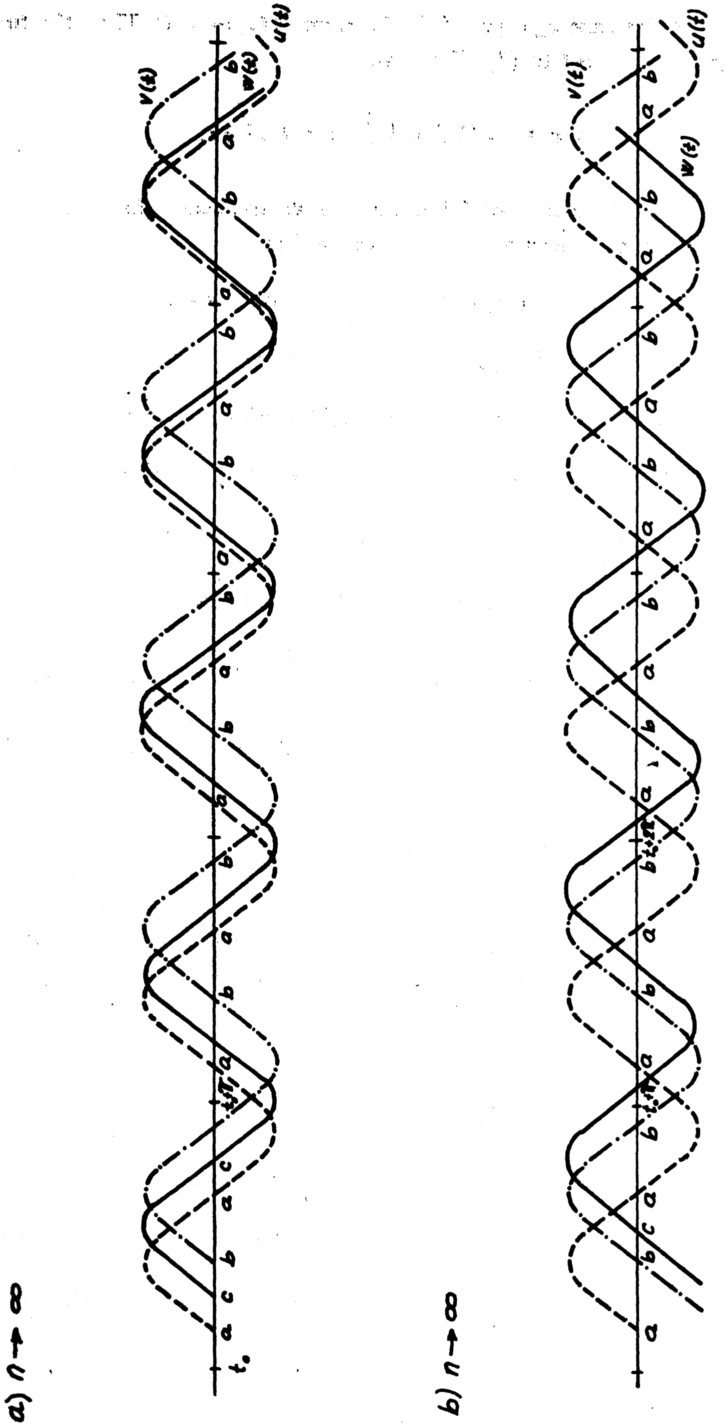
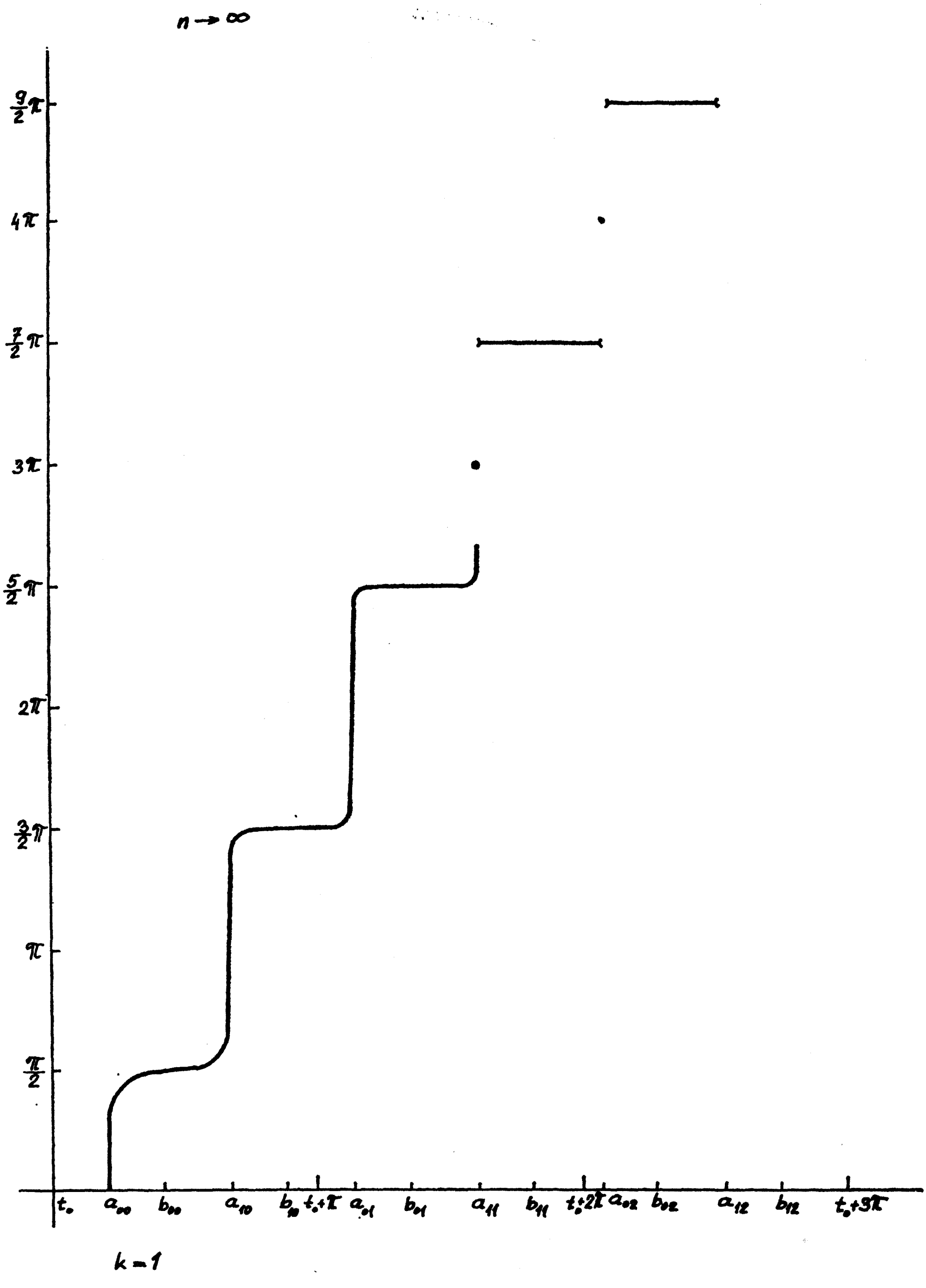

a) n→∞
b) n→∞
v(t)
w(t)
u(t)



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