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ON ASYMPTOTIC PROPERTIES AND DISTRIBUTION OF ZEROS OF SOLUTIONS OF $y'' + f(t, y, y') = 0$

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1. Consider a differential equation

$$
\begin{align*}
  y'' + f(t, y, y') &= 0, \\
  t &\in [t_0, \infty), y \in \mathbb{R}, v \in \mathbb{R}, f(t, y, v) > 0 \text{ for } y \neq 0.
\end{align*}
$$

(1)

where the function $f$ is continuous in $D = \{(t, y, v) : t \in [t_0, \infty), y \in \mathbb{R}, v \in \mathbb{R}\}$.

It is evident that Cauchy initial problem for (1) has a solution but we do not suppose its uniqueness. In the present paper we shall omit the trivial solution $y = 0$ from our considerations.

A solution $y$ of (1) is called oscillatory if there exists a sequence of numbers $\{t_k\}_{k=1}^\infty$ such that $t_0 \leq t_k < t_{k+1}$, $y(t_k) = 0$, $y(t) \neq 0$ on $(t_k, t_{k+1})$, $k = 1, 2, 3, \ldots$

In this paper we shall deal only with oscillatory solutions of (1) that exist on the whole interval $[t_0, \infty)$, i.e. $\lim_{k \to \infty} t_k = \infty$. We shall study some asymptotic properties of them and the distribution of their zeros.

If $y$ is an oscillatory solution of (1), then the distribution of its zeros is characterized by the sequence $\{A_k\}_{k=1}^\infty$. $A_k = t_{k+1} - t_k$ where $\{t_k\}_{k=1}^\infty$ is the sequence of all zeros of $y$. There exists exactly one sequence $\{\tau_k\}_{k=1}^\infty$ called the sequence of extremants of $y$, with the property $t_k < \tau_k < t_{k+1}$, $y'(\tau_k) = 0$ (see [2]). The symbols $t_k, \tau_k, A_k$ have the above mentioned meaning in the present paper.

We shall need the simple lemma the proof of which can be found in [2], [3].

**Lemma 1.** Let $y$ be an arbitrary non-trivial solution of (1) and $t_1 < t_2$ its consecutive zeros $(y(t) \neq 0$ on $(t_1, t_2))$. Then there exists exactly one number $\tau$, $t_1 < \tau < t_2$ such that $y'(\tau) = 0$ holds, the function $y' \text{ sgn } y$ is decreasing on $(t_1, t_2)$ and the inequalities

$$
|y'(t_1)| (\tau - t_1) > |y(\tau)|, \quad |y'(t_2)| (t_2 - \tau) > |y(\tau)|
$$

are valid.

2. **Theorem 1.** Let $y$ be an oscillatory solution of (1) and let there exist a constant $M > 0$ such that for an arbitrary number $M_1 > 0$ the following relation holds:
\[
\lim_{t \to \infty} f(t, y, v) = 0, \quad \text{uniformly for } \quad |y| \leq M_1, |v| \leq M.
\]

Then at least one of the following assertions is valid

(i) \( \lim_{t \to \infty} y'(t) = 0, \)

(ii) \( y \) is unbounded on \([t_0, \infty).\)

If, in addition, constants \( M_2, M_3 \) exist such that

\[ 0 < M_2 \leq |y(t_k)| \leq M_3, \quad k = 1, 2, 3, \ldots, \]

then \( \lim_{k \to \infty} \Delta_k = \infty. \)

Proof. Assume that \( y \) is bounded on \([t_0, \infty).\) We shall prove at first the relation

\( \lim_{t \to \infty} y'(t) = 0. \) Let

\[ |y(t)| \leq N = \text{const.} < \infty, \quad t \in [t_0, \infty). \]

Put

\[ H_k(v) = \max |f(t, y, v)| > 0 \quad \text{for } v \in R, \]

\[ |y| \leq N, \quad t_k \leq t \leq t_{k+1}. \]

It follows from the assumptions of the theorem that

(2) \[ \lim_{k \to \infty} H_k(v) = 0 \quad \text{uniformly for } |v| \leq M. \]

By multiplying the equation (1) by \( -\frac{y'}{H_k(y')} \) and by integration we obtain

(3) \[ \int_{t_k}^{t_k} \frac{t \, dt}{H_k(t \, \text{sgn} \, y'(t_k))} = - \int_{t_k}^{t_k} \frac{y''(t) \, y'(t)}{H_k(y'(t))} \]

\[ = \int_{t_k}^{t_k} \frac{|f(t, y(t), y'(t))| \, |y'(t)|}{H_k(y'(t))} \, dt \leq \int_{t_k}^{t_k} \frac{|y'(t)| \, dt}{H_k(y'(t))} \leq N, \quad k = 1, 2, \ldots \]

Suppose that \( y' \) does not converge to zero for \( t \to \infty. \) Then there exists a sequence of integers \( \{k_i\}_i \) such that \( |y'(t_{k_i})| \geq \varepsilon > 0, \quad i = 1, 2, 3, \ldots \) holds in case \( \varepsilon, \varepsilon \leq M \) is a suitable number. According to (2)

\[ \lim_{i \to \infty} \int_{0}^{t_{k_i}} \frac{t \, dt}{H_k(t \, \text{sgn} \, y'(t_{k_i}))} \leq \lim_{i \to \infty} \int_{0}^{t_{k_i}} \frac{t \, dt}{H_k(t \, \text{sgn} \, y'(t_{k_i}))} = \infty \]

and we get a contradiction to the inequality (3). Thus the first part of the statement is proved.

Let \( M_2, M_3 \) be constants such that \( 0 < M_2 \leq |y(t_k)| \leq M_3, \quad k = 1, 2, 3, \ldots \)

Thus according to the proved part of the theorem \( \lim_{i \to \infty} y'(t) = 0 \) holds and the rest of the statement follows from Lemma 1:
Theorem 2. Let $y$ be an oscillatory solution of (1) and let a constant $M, 0 < M$ exist such that for arbitrary numbers $M_1, M_2, 0 < M_2 \leq M, 0 < M_1$ the following relation holds

$$\lim_{t \to \infty} \left| f(t, y, v) \right| = \infty \quad \text{uniformly for } M_2 \leq |y| \leq M, |v| \leq M_1.$$ 

Then at least one of the following assertions is valid

(i) $\lim_{t \to \infty} y(t) = 0$,

(ii) $y'$ is unbounded on $[t_0, \infty)$.

Proof. Suppose that (i) is not valid. Then there exists a sequence of integers $\{k_i\}_{i=1}^\infty$ such that $|y(t_{k_i})| \geq \varepsilon > 0, i = 1, 2, 3, \ldots$ where $\varepsilon, \varepsilon \leq M$ is a suitable number. Put

$$H_i(v) = \min \left| f(t, y, v) \right| > 0, \quad \text{for} \quad \varepsilon/2 \leq |y| \leq \varepsilon, \quad t_{k_i} \leq t \leq t_{k_{i+1}}, \quad v \in R; \quad i = 1, 2, \ldots$$

With respect to the assumptions of the theorem we have

$$\lim_{i \to \infty} H_i(v) = \infty \quad \text{uniformly for } |v| \leq \text{const.}$$

By multiplying (1) by $-\frac{y'}{H_i(y')}$ and by integrating we obtain

$$\int_0^{t_{k_i}} \frac{t \, dt}{H_i(t \, \text{sgn} \, y(t_{k_i}))} = \int_{t_{k_i}}^{t_{k_{i+1}}} \left| f(t, y(t), y'(t)) \right| \, dt \geq$$

$$\geq \int_{t_i}^{t_i} \frac{t \, dt}{H_i(y'(t))} \geq \int_{t_i}^{t_{i+1}} \left| y'(t) \right| \, dt = |y(t_{i+1})| - |y(t_i)| = \frac{\varepsilon}{2},$$

where $t_i, t_i$ are such numbers that $t_{k_i} < t_i^1 < t_i^2 \leq t_{k_i}, \quad |y(t_i^1)| = \varepsilon/2, \quad |y(t_i^2)| = \varepsilon, \quad i = 1, 2, 3, \ldots$

Let $y'$ be bounded on $[t_0, \infty)$. Thus there exists a constant $N > 0$ such that $|y'(t)| \leq N, t \in [t_0, \infty)$. Then according to (4) we can conclude

$$\int_0^{t_{k_i}} \frac{t \, dt}{H_i(t \, \text{sgn} \, y(t_{k_i}))} \leq N \int_0^{t_{k_i}} \frac{dt}{H_i(t \, \text{sgn} \, y(t_{k_i}))} \to 0.$$

But this fact is in contradiction with the inequality (5) and the function $y'$ is unbounded on $[t_0, \infty)$. The theorem is proved.
Remark 1. It is evident from (3) and (5) that under the assumptions of Theorem 1
\[
\liminf_{k \to \infty} |y(\tau_k)| < \infty \Rightarrow \liminf_{k \to \infty} |y'(t_k)| = 0
\]
holds, too. Similarly, if the assumptions of Theorem 2 are valid, then
\[
\liminf_{k \to \infty} |y(\tau_k)| > 0 \Rightarrow \liminf_{k \to \infty} |y'(t_k)| = \infty
\]
holds.

Remark 2. Let the assumptions of Theorem 1 be valid and let \(y\) be bounded on \([t_0, \infty)\). Then
\[
\lim_{t \to \infty} y'(t) = 0, \quad \lim_{t \to \infty} y''(t) = 0.
\]
The last result follows from the previous one and from the relation
\[
\lim_{t \to \infty} y''(t) = -\lim_{t \to \infty} f(t, y(t), y'(t)) = 0.
\]
Similarly, let \(y\) be an oscillatory solution of (1) and let for an arbitrary constant \(M_2 > 0\) the following relation hold
\[
\lim_{t \to \infty} |f(t, y, v)| = \infty \quad \text{uniformly for} \quad |v| < \infty, \quad M_2 \leq |y| < \infty.
\]
Let \(\limsup_{k \to \infty} |y(\tau_k)| > 0\). Then the functions \(y', y''\) are unbounded on \([t_0, \infty)\).

If, in addition, \(\liminf_{k \to \infty} |y(\tau_k)| > 0\), then
\[
\lim_{k \to \infty} |y'(\tau_k)| = \lim_{k \to \infty} \sigma_k = \infty \quad \text{where} \quad \sigma_k = \max_{t \in [t_k, t_{k+1}]} |y''(t)|.
\]

Remark 3. The results of Theorems 1 and 2 were studied in [2], [3] for the differential equation
\[
(p(t)y')' + f(t, y, y') = 0.
\]
The statements of Theorems 1 and 2 generalize some conclusions of the above mentioned papers for (1).

Corollary 1. Let \(y\) be an oscillatory solution of a differential equation
\[
\begin{align*}
y'' + a(t)f(y, y') &= 0, \\
\text{where} \quad a(t), f(y, v) \text{ are continuous functions} \\
\text{for} \quad t \in [t_0, \infty), \quad y \in R, \quad v \in R, \quad a > 0, \\
f(y, v)y > 0 \quad \text{for} \quad y \neq 0.
\end{align*}
\]

(i) Let \(\lim_{t \to \infty} a(t) = 0\). If \(y\) is bounded on \([t_0, \infty)\), then
\[
\lim_{t \to \infty} y'(t) = 0, \quad \lim_{t \to \infty} y''(t) = 0.
\]
If, in addition, lim inf \( \{ |y(r_k)| \} > 0 \), then \( \lim_{k \to \infty} A_k = \infty \) holds.

(ii) Let \( \lim_{t \to \infty} a(t) = \infty \). If \( y' \) is bounded on \([t_0, \infty)\), then
\[
\lim_{t \to \infty} y(t) = 0.
\]

(iii) Let \( \lim_{t \to \infty} a(t) = \infty \). If there exist constants \( M, M_1 \) such that \( 0 < M \leq \leq |y(\tau_k)| \leq M_1 < \infty \) holds, then \( \lim_{k \to \infty} |y'(t_k)| = \infty \).

If, in addition, for an arbitrary constant \( N > 0 \) there exists a number \( M_2 > 0 \) such that
\[
|f(y, v)| \geq M_2 > 0, \quad |y| \geq N, \quad |v| < \infty
\]
holds, then \( \lim_{k \to \infty} \sigma_k = \infty, \quad \sigma_k = \max_{t \in [t_k, t_{k+1}]} |y''(t)|. \)

3. In this paragraph some well-known results for the linear differential equation (see [15], [1], [9])
\[
y'' + q(t)y = 0, \quad t \in [t_0, \infty)
\]
\( q \) continuous, \( 0 < M = \text{const.} \leq q(t) \leq M_1 = \text{const.} < \infty, \ t \in [t_0, \infty) \) are extended to the equation (1).

**Theorem 3.** Let \( y \) be an oscillatory solution of (1). Let constants \( M, M_1, M_3, 0 < M, 0 < M_1, 0 < M_3 \) exist such that for an arbitrary number \( M_2, 0 < M_2 \leq M \) there hold
\[
|f(t, y, v)| \leq M_3, \quad t \in [t_0, \infty), \quad |y| \leq M, \quad |v| \leq M_1,
\]
\[
0 < M_4 \leq |f(t, y, v)|, \quad t \in [t_0, \infty), \quad M_2 \leq |y| \leq M, \quad |v| \leq M_1,
\]
where \( M_4 \) is a constant (depending on \( M_2 \)).

Then
\[
\lim_{t \to \infty} y(t) = 0 \quad \text{if, and only if} \quad \lim_{t \to \infty} y'(t) = 0.
\]

**Proof.** Let \( \lim_{t \to \infty} y(t) = 0 \). Then there exists a number \( t \in [t_0, \infty) \) and an integer \( k \)
such that \( |y(t)| \leq M, \ t \in [t, \infty), \ \tau_k \geq t \). We can define the function \( H_k(v) \) in the same way as in Theorem 1 \( (N = M) \), the estimation (3) holds, thus especially
\[
|y'(t_k)| \leq \frac{t dt}{H_k(t \text{ sgn } y'(t_k))} \leq |y(t_k)|, \quad k = k, k + 1, k + 2, \ldots
\]

If \( \lim_{k \to \infty} y(t_k) = 0 \), then according to the estimation
\[
H_k(v) \leq M_3 = \text{const.} < \infty, \quad |v| \leq M_1, \quad k = k, k + 1, k + 2, \ldots
\]
(the validity of which follows from the assumptions of the theorem) we can conclude \( \lim_{k \to \infty} y'(t_k) = 0 \).
On the contrary, let $\lim_{t \to \infty} y'(t) = 0$. We shall use the indirect proof for proving the relation $\lim_{t \to \infty} y(t) = 0$. If this relation is not valid, then there exists a sequence $\{k_i\}_{i=1}^{\infty}$ and a number $\varepsilon$, $0 < \varepsilon$ such that $|y(\tau_k)| \geq \varepsilon$, $i = 1, 2, \ldots$ and (5) hold. Thus

$$\int_0^t \frac{t \, dt}{H_i(t \, \text{sgn} \, y'(t_k))} \geq \frac{\varepsilon}{2} > 0.$$  

From this and according to $\lim_{t \to \infty} y'(t) = 0$ we obtain a contradiction because from the assumptions of the theorem there follows

$$0 < N_2 = \text{const.} \leq H_i(v), \quad i = 1, 2, \ldots$$  

uniformly in some neighbourhood of $v = 0$. The theorem is proved.

**Remark 4.** It is evident from the proof of Theorem 3 that the following assertion is valid, too

$$\liminf_{k \to \infty} |y(t_k)| = 0 \iff \liminf_{k \to \infty} |y'(t_k)| = 0.$$

**Corollary 2.** Let $y$ be an oscillatory solution of (6). Let constants $M, M_1$ exist such that

$$0 < M \leq a(t) \leq M_1, \quad t \in [t_0, \infty)$$

holds. Then

$$\lim_{t \to \infty} y(t) = 0 \quad \text{if and only if} \quad \lim_{t \to \infty} y'(t) = 0.$$

**Theorem 4.** Let $y$ be an oscillatory solution of (1). Let a continuous function $g(v)$, $v \in \mathbb{R}$, $g > 0$, $\int_0^\infty \frac{t \, dt}{g(\pm t)} = \infty$ exist such that for arbitrary constants $M, M_1, M_2$,

$$0 < M_1 \leq M < \infty \text{ there hold } |f(t, y, v)| \leq M_3 g(v), \quad t \in [t_0, \infty), \quad |y| \leq M, \quad v \in \mathbb{R}$$

and

$$(8) \quad 0 < M_4 \leq |f(t, y, v)|, \quad t \in [t_0, \infty), \quad M_1 \leq |y|, \quad |v| \leq M_2.$$  

Here $M_3$ ($M_4$) is a suitable number that depends on $M$ (on $M_1, M_2$). Then

(i) $y$ is bounded on $[t_0, \infty)$ if and only if $y'$ is bounded on $[t_0, \infty)$

(ii) $\lim_{k \to \infty} |y(\tau_k)| = \infty$ if and only if $\lim_{k \to \infty} |y'(\tau_k)| = \infty$

(iii) If $0 < M_5 = \text{const.} \leq |y(\tau_k)| \leq M_6 = \text{const.}, \quad k = 1, 2, 3, \ldots$, then there exist numbers $C, C_1$ such that

$$(9) \quad 0 < C \leq A_k \leq C_1, \quad k = 1, 2, 3, \ldots$$

holds.

**Proof.** (i) Let $y$ be bounded $|y(t)| \leq N, \quad t \in [t_0, \infty)$. Then we can define the function $H_k(v)$ in the same way as in Theorem 1 and (3) holds. Thus, especially
According to the assumptions of the theorem a constant $N_1 < \infty$ exists such that $H_k(v) \leq N_1 g(v), \ k = 1, 2, \ldots$ From this

$$\frac{1}{N_1} \int_0^{y'(t_k)} t \, dt \leq \int_0^{y'(t_k)} g(t \text{ sgn} \ y'(t_k)) \, dt \leq N.$$ 

Thus with respect to the assumptions of the function $g$ we obtain that $y'$ must be bounded on $[t_0, \infty)$. 

On the contrary let $y'$ be bounded $|y'(t)| \leq N_2, \ t \in [t_0, \infty)$. The statement will be proved by the indirect proof. Thus suppose that there exists a sequence of integers $\{k_i\}_{i=1}^\infty$ such that $\lim_{i \to \infty} |y(t_{k_i})| = \infty$. Let $\varepsilon > 0, \sigma_i, \ i = 1, 2, \ldots$ be such numbers that $|y(t_{k_i})| > \varepsilon, \ |y(\sigma_i)| = \varepsilon, \ t_{k_i} < \sigma_i < \tau_{k_i}, \ i = 1, 2, \ldots$ hold. Put

$$A_i(v) = \min \{ f(t, y, v) : t \leq y(t_{k_i}), \ t_{k_i} \leq t \leq t_{k+1} \}. $$

There exists a constant $N_3$ such that

$$A_i(v) \geq N_3, \ |v| \leq N_2, \ i = 1, 2, \ldots$$ 

and thus (according to (1))

$$\infty > \int_0^{t_{k_i}} \frac{t \, dt}{N_3} \leq \int_0^{\tau_{k_i}} \frac{t \, dt}{N_3} \leq \int_{\sigma_i}^{t_{k_i}} \frac{t \, dt}{N_3} \leq \int_{\sigma_i}^{t_{k_i}} \frac{t \, dt}{N_3} = \int_{\sigma_i}^{t_{k_i}} \frac{t \, dt}{N_3} = \frac{1}{N_3} \int_{\sigma_i}^{t_{k_i}} t \, dt \leq \int_{\sigma_i}^{t_{k_i}} f(t, y(t), y'(t)) \, dt \geq A_i(y'(t)) \geq \int_{\sigma_i}^{t_{k_i}} |y'(t)| \, dt = |y(t_{k_i})| - \varepsilon \to \infty.$$ 

But this is a contradiction.

(ii) The result follows from the proved part of the theorem and from the following conclusion which can be proved in the same way as (i):

Let $\{k_i\}_{i=1}^\infty$ be a sequence of integers. The sequence $\{|y(t_{k_i})|\}_{i=1}^\infty$ is bounded on $[t_0, \infty)$ iff $\{|y'(t_{k_i})|\}_{i=1}^\infty$ is bounded on this interval.

(iii) It follows from the proved part of the theorem that

$$|y'(t_k)| \leq N_5 = \text{const.}, \ k = 1, 2, \ldots$$ 

holds. Denote by $\sigma_k, \bar{\sigma}_k$ such numbers that $t_k < \sigma_k < \bar{\sigma}_k \leq \tau_k, \ |y(\sigma_k)| = \frac{M_5}{2}, \ |y(\bar{\sigma}_k)| = M_5.$

As the function $y'(t) \text{ sgn} \ y(t)$ is decreasing in the interval $(t_k, \tau_k)$ (see Lemma 1), the following inequalities are valid

$$0 < \sigma_k - t_k < \bar{\sigma}_k - \sigma_k \leq \tau_k - \sigma_k.$$
Put

\[ A_k(v) = \min |f(t, y, v)| > 0 \]

for

\[ M_5/2 \leq |y| \leq M_6, \quad t_k \leq t \leq \tau_k, \quad v \in \mathbb{R}; \quad k = 1, 2, \ldots \]

Then \( A_k(v) \geq N_6 = \text{const.} > 0 \) for \( |v| \leq N_5 \). By multiplying (1) by \(-A_k^{-1}(y'(t))\) and by integration we can conclude

\[
\infty > \frac{N_5}{N_6} \geq \int_0^\tau_k \frac{|f(t, y(t), y'(t))|}{A_k(t \operatorname{sgn} y'(t_k))} \, dt \leq \\
\geq \int_{\sigma_k}^{\tau_k} \frac{|f(t, y(t), y'(t))|}{A_k(y'(t))} \, dt \geq \int_{\sigma_k}^{\tau_k} \, dt = \tau_k - \sigma_k.
\]

Thus \( \tau_k - t_k = (\tau_k - \sigma_k) + (\sigma_k - t_k) \) is bounded above for \( k = 1, 2, 3, \ldots \). It can be proved similarly that \( \{t_k + 1 - \tau_k\}_1^\infty \) is bounded.

The boundedness of \( \{A_k\}_1^\infty \) from below by a positive constant is a consequence of Lemma 1 and (11). The theorem is proved.

**Remark 5.** It is evident from the proof that under the assumptions of Theorem 4 the following assertions are valid

(i) \( \liminf_{k \to \infty} |y(t_k)| < \infty \iff \liminf_{k \to \infty} |y'(t_k)| < \infty \)

(ii) \( \limsup_{t \to \infty} |y(t)| = \infty \iff \limsup_{t \to \infty} |y'(t)| = \infty \)

**Remark 6.** It can be easily seen from the proof of Theorem 4 that the conclusion (iii) holds even if we suppose

\[ 0 < M_1 \leq |f(t, y, v)|, \quad t \in [t_0, \infty), \quad M_1 \leq |y| \leq M, \quad |v| \leq M_2 \]

instead of (8).

**Theorem 5.** Let \( y \) be an oscillatory solution of (1) and let a continuous function \( g \)

exist, \( g(t) > 0, \quad t \in [t_0, \infty), \quad \int_0^\infty \frac{t \, dt}{\int_0^\infty g(\pm t)} < \infty \) such that for an arbitrary constant \( M_1 \), \( 0 < M_1 \) it holds

\[ 0 < M_2 g(v) \leq |f(t, y, v)|, \quad t \in [t_0, \infty), \quad M_1 \leq |y| \leq M, \quad |v| \leq M_2 \]

where \( M_2 > 0 \) is a suitable number (depending on \( M_1 \)). Then \( y \) is bounded on \([t_0, \infty)\). If, in addition, \( 0 < M_3 = \text{const.} \leq |y(t_k)|, \quad k = 1, 2, 3, \ldots \), then \( \{A_k\}_1^\infty \) is bounded.

**Proof.** The first part of the theorem can be proved similarly to the second part of the statement (i) in Theorem 4 (i.e. \( y' \) is bounded \( \Rightarrow y \) is bounded). We must use the estimation

\[ A_i(v) \geq N_3 g(v), \quad v \in \mathbb{R}, \quad i = 1, 2, 3, \ldots \]
instead of (10). The boundedness of $A_k$ can be proved similar to the same result in Theorem 4.

4. Consider a differential equation

$$y'' + f(t, y) g(y') = 0,$$

where $f(t, y), g(v)$ are continuous functions in $D = \{(t, y) : t \in [t_0, \infty), y \in \mathbb{R}\}, v \in \mathbb{R}, g > 0, f(t, y) y > 0$ for $y \neq 0$.

In our further considerations we must suppose very often the validity of the conditions

$$f(t, y) = -f(t, -y), \quad g(v) = g(-v).$$

This paragraph contains especially some consequences of the previous paragraphs and of some results of [5]. At first we mention here necessary results of [5].

**Theorem.** Let $y$ be an oscillatory solution of (12). Let the function $|f(t, y)|$ be non-increasing (non-decreasing) with respect to $t$ in $D$.

(i) If $g(v) = g(-v)$ for $v \in (-\infty, \infty)$, then the sequence $\{y'(t_k)\}_{k=1}^\infty$ is non-increasing (non-decreasing) and $t_k - t_k \leq t_{k+1} - t_k$ ($t_k - t_k \geq t_{k+1} - t_k$) holds.

(ii) If $f(t, y) = -f(t, -y)$ in $D$, then the sequence $\{y'(t_k)\}_{k=1}^\infty$ is non-decreasing (non-increasing).

**Theorem 6.** Let $y$ be an oscillatory solution of (12) and let (13) hold. Let

(i) $|f(t, y)|$ be non-decreasing with respect to $t$ in $D$

(ii) a constant $M > 0$ exist such that $f(t, y)$ is non-decreasing with respect to $y$ in $D_1 = \{(t, y) : t \in [t_0, \infty), |y| \leq M\}$.

If $\lim_{t \to \infty} y(t) = 0$, then $\lim_{k \to \infty} A_k = 0$.

**Proof.** It follows from Theorem that $\{y(t_k)\}_{k=1}^\infty$ is non-increasing and $\{y'(t_k)\}_{k=1}^\infty$ is non-decreasing. Thus

$$|y'(t_k)| \geq |y'(t_1)| = N > 0, \quad k = 1, 2, 3, \ldots$$

Let $\varepsilon > 0$ be an arbitrary number, $\varepsilon \leq \frac{\min_{|t| \leq N} g(t)}{\max_{|t| \leq N} g(t)} \frac{N}{2}$. Denote by $\sigma_k, k = 1, 2, 3, \ldots$ numbers such that $t_k < \sigma_k < t_k, |y'(\sigma_k)| = \varepsilon$. First we prove that $\lim_{k \to \infty} \sigma_k - t_k = 0$ holds. According to the Rolle theorem there exists a number $\xi_k \in (t_k, \sigma_k)$ such that

$$y'(\xi_k) = \frac{y(\sigma_k) - y(t_k)}{\sigma_k - t_k}.$$ 

From this

$$\sigma_k - t_k = \frac{|y(\sigma_k)|}{|y'(\xi_k)|} \leq \frac{|y(\sigma_k)|}{\varepsilon} \to 0$$

and thus $\lim_{k \to \infty} \sigma_k - t_k = 0$. 


According to \( \lim_{t \to \infty} y(t) = 0 \) there exists an integer \( n \) such that \( |y(t_k)| \leq M, k \geq n \).

Consider the function

\[
F(t) = \int_{t_k}^{\tau_k} \frac{y''(t)}{g(y(t))} dt = -\int_{0}^{\tau_k} \frac{dt}{g(t)} = -\int_{0}^{\tau_k} f(t, y(t)) dt, \quad t \in [t_k, \tau_k].
\]

If \( \sigma_k \leq t_1 \leq t_2 \leq \tau_k \), then \( |y(t_1)| \leq |y(t_2)| \) and

\[
F'(t_2) - F'(t_1) = f(t_2, y(t_2)) - f(t_1, y(t_1)) = [f(t_2, y(t_2)) - f(t_2, y(t_1))] + [f(t_2, y(t_1)) - f(t_1, y(t_1))] \geq 0 \quad \text{for } y(t_1) > 0
\]

\[
\leq 0 \quad \text{for } y(t_1) < 0.
\]

As \( F'(t) > 0 \) \((< 0)\) if \( y > 0 \) \((y < 0)\), the function \( |F'| \) is non-decreasing on \((t_k, \tau_k)\).

Denote by \( \sigma_k \) such number that \( t_k \leq \sigma_k < \tau_k \), \( |F(\sigma_k)| = 2 |F(\sigma_k)| \). \( \sigma_k \) really exists because \( |F| \) is non-increasing and

\[
|F(\sigma_k)| = \int_{0}^{\sigma_k} \frac{dt}{g(t)} \leq \frac{\varepsilon}{\min_{|t| \leq N} g(t)} \leq \frac{\varepsilon}{2 \max_{|t| \leq N} g(t)},
\]

\[
|F(t_k)| = \int_{0}^{y(t_k)} \frac{dt}{g(t)} \leq \frac{N}{\max_{|t| \leq N} g(t)}.
\]

According to the mean value theorem we have

\[
|F(\sigma_k)| = |F(t_k) - F(\sigma_k)| = |F'(\xi_1)| (\sigma_k - t_k), \quad \xi_1 \in (\sigma_k, \tau_k),
\]

\[
|F(\sigma_k)| = |F(\sigma_k) - F(\sigma_k)| = |F'(\xi_2)| (\sigma_k - \sigma_k), \quad \xi_2 \in (\sigma_k, \sigma_k).
\]

From this and with respect to \( |F'| \) being non-decreasing, we can conclude

\[
\tau_k - \sigma_k \leq \sigma_k - \sigma_k \leq \sigma_k - t_k \overset{k \to \infty}{\longrightarrow} 0.
\]

Thus finally \( \lim_{k \to \infty} \tau_k - t_k = 0 \).

By Theorem there holds \( t_{k+1} - \tau_k \leq \tau_k - t_k \) and the theorem is proved.

Remark 7. Katranov [11], [12] deals with the problem of Theorem 6. He proved the statement of this theorem but under the more restrictive assumptions. He must, in addition, suppose that

1° there exists \( \frac{\partial}{\partial t} f(t, y) \) and it is continuous

2° for an arbitrary \( M \neq 0 \) there holds \( \lim_{t \to \infty} |f(t, M)| = \infty \)

3° there exists a constant \( g_1 > 0 \) such that \( g(v) > g_1, v \in R \)

4° the uniqueness of the Cauchy initial problem holds.
Theorem 7. Let \( y \) be an oscillatory solution of (12), (13) and let \( |f(t, y)| \) be non-increasing with respect to \( t \) in \( D \). Further, let for an arbitrary constant \( 0 < M \) there holds \( \lim_{t \to \infty} f(t, y) = 0 \) uniformly for \( |y| \leq M \). Then

\[
\lim_{k \to \infty} \Delta_k = \infty.
\]

Proof. It is evident that the assumptions of Theorem 1 are fulfilled. Thus either \( \lim y'(t) = 0 \) or \( y \) is unbounded. From this and according to Lemma 1 and Theorem we have

\[
\Delta_k > \frac{|y(t_k)|}{|y'(t_k)|} \quad \text{as } k \to \infty.
\]

The theorem is proved.

The following Corollary is a consequence of Theorem 4 and Theorem.

Corollary 3. Let \( y \) be an oscillatory solution of (12), (13). Let \( |f(t, y)| \) be non-increasing (non-decreasing) with respect to \( t \) in \( D \) and let \( \int_0^\infty \frac{t}{g(t)} \, dt = \infty \). Further suppose that for an arbitrary constant \( M, 0 < M \) there exists a number \( M_1 \) such that

\[
0 < M_1 \leq \lim_{t \to \infty} |f(t, y)|, \quad M \leq |y|,
\]

\[
(\lim_{t \to \infty} |f(t, y)| \leq M_1 \quad \text{for } |y| \leq M)
\]

holds. Then the sequences \( \{ |y(t_k)| \}_1^\infty \), \( \{ |y'(t_k)| \}_1^\infty \), \( \{ \Delta_k \}_1^\infty \) are bounded above and are bounded away from zero.

Remark 8. The problems concerning the boundedness of \( y \) and \( y' \) are studied in [6], [7], [14], [18], [19] (these papers deal with the differential equation (1), \( f(t, y, v) = a(t) r(y) h(y') \)) and in [8], [10], [13], [16], [17] (for \( f(t, y, v) = a(t) r(y) \)), but mostly without the assumptions (13) and \( \int_0^\infty \frac{t}{g(t)} = \infty \). The other assumptions of Corollary 3 are supposed, too.

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