A reflexive and symmetric binary relation $T$ on a non-empty set $A$ is called a tolerance relation (or shortly tolerance) on $A$ and the ordered pair $(A, T)$ is called a tolerance space. By the symbol $I$ we denote the identity relation on $A$, i.e. such a relation that $xTy$ if and only if $x = y$ for any $x$ and $y$ from $A$. Denote $T^0 = I, T^1 = T, T^{n+1} = T \cdot T^n$ for each positive integer $n$.

Definition 1. Let $(A, T)$ be a tolerance space. A non-empty subset $B$ of $A$ is called $T$-connected in $A$, if for any $x \in B$, $y \in B$ there exists a positive integer $p$ such that $xT^py$. If $A$ is $T$-connected in $(A, T)$, then $(A, T)$ is called a connected tolerance space.

Proposition 1. Let $(A, T)$ be a tolerance space, let $B$ be a $T$-connected set in $A$ and let $\delta_T(x, y)$ be an integer-valued function on $B \times B$ given by the rule

\begin{equation}
\delta_T(x, y) = 0 \iff xT^0y,
\end{equation}

\begin{equation}
\delta_T(x, y) = p \iff xT^py \quad \text{and} \quad \neg xT^qy \quad \text{for} \quad q < p.
\end{equation}

Then $\delta_T(x, y)$ is an integer-valued metric on $B$.

Proposition 2. Let $(A, \mu)$ be a quasimetric space and $\varepsilon$ a positive real number. The relation $T_{\mu(\varepsilon)}$ defined on $A$ by the rule

\begin{equation}
xT_{\mu(\varepsilon)}y \iff \mu(x, y) \leq \varepsilon
\end{equation}

is a tolerance on $A$ and the tolerance space $(A, T_{\mu(\varepsilon)})$ is $T_{\mu(\varepsilon)}$-connected.

Definition 2. Let $(A, T)$ be a tolerance space, let $B$ be a $T$-connected set in $A$. The metric $\delta_T$ on $B$ is called induced by the tolerance $T$. Let $\varepsilon > 0$ and let $(A, \mu)$ be a quasimetric space. Then the tolerance $T_{\mu(\varepsilon)}$ is called induced by the quasimetric $\mu$ with the unit $\varepsilon$. 

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Proposition 3. Let \((A, T)\) be a connected tolerance space, \(\delta_T\) a metric induced by the tolerance \(T\) and \(T_\delta(1)\) the tolerance induced by the metric \(\delta_T\) with the unit \(\varepsilon = 1\). Then \(T = T_\delta(1)\).

Proposition 4. Let \((A, \mu)\) be a quasimetric space, let \(0 < \varepsilon \leq 1\), let \(T_\mu(\varepsilon)\) be the tolerance induced by the quasimetric \(\mu\) with the unit \(\varepsilon\) and \(\delta_T\) the metric induced by the tolerance \(T_\mu(\varepsilon)\). Then \(\delta_T(x, y) \geq \mu(x, y)\) for any \(x \in A, y \in A\).

Proposition 5. Let \((A, \pi)\) be a metric space with an integer-valued metric \(\pi\), let \(T_\pi(1)\) be a tolerance on \(A\) induced by the metric \(\pi\) with the unit \(\varepsilon = 1\) and \(\delta_T\) the metric induced by the tolerance \(T_\pi(1)\). Then \(\pi = \delta_T\).

Proposition 6. Let \((A, \mu)\) be a quasimetric space and \(\varepsilon_1, \varepsilon_2\) positive real numbers. If \(\varepsilon_1 < \varepsilon_2\), then \(T_\mu(\varepsilon_1) \subseteq T_\mu(\varepsilon_2)\). If \(x \in A, y \in A\) and \(\varepsilon_1 < \mu(x, y) < \varepsilon_2\), then \(T_\mu(\varepsilon_1) \neq T_\mu(\varepsilon_2)\), i.e.

\[ \varepsilon_1 < \varepsilon_2 \Rightarrow T_\mu(\varepsilon_1) \subseteq T_\mu(\varepsilon_2). \]

Remark. Evidently each equivalence on \(A\) is a tolerance on \(A\). By Definition 1 it is evident that for an equivalence \(E\) on \(A\) a set \(B\) such that \(\emptyset \neq B \subseteq A\) is \(E\)-connected in \(A\) if and only if there exists a partition class \([a] \in A/E\) such that \(B \subseteq [a]\). Therefore if \(x, y, z\) are elements of \([a]\), then for \(\delta_E\) the triangle inequality holds. Further, if \(x = y\), evidently \(\delta_E(x, y) = 0\) and for \(x \in [a], y \in [a], x \neq y\) we have \(\delta_E(x, y) = 1\), because the transitivity of \(E\) implies \(T_k \subseteq T\) for \(k = 0, 1, 2, \ldots\). This implies that if \(T\) is an equivalence on \(A\), \(xTy, yTz, x \neq y, y \neq z\), then the sharp triangle inequality (T)

\[ \delta_T(x, z) < \delta_T(x, y) + \delta_T(y, z) \]

holds, because \(\delta_T(x, z) \leq 1\) and \(\delta_T(x, y) + \delta_T(y, z) = 2\). We shall show that also the converse assertion holds.

Proposition 7. Let \((A, T)\) be a connected tolerance space with at least three elements and let \(\delta_T\) be the metric induced by the tolerance \(T\). If for any three elements (pairwise distinct) \(x, y, z\) of \(A\) the sharp triangle inequality (T) holds, then \(T\) is an equivalence on \(A\).

Proof. Let \(x, y, z\) be pairwise distinct elements of \(A\) and let \(xTy, yTz\). Then by (P) we have \(\delta_T(x, y) = 1, \delta_T(y, z) = 1\) and (T) implies \(\delta_T(x, z) < 2\), i.e. \(\delta_T(x, z) \leq 1\). As \(x \neq z\), we have \(\delta_T(x, z) \neq 0\), because \(\delta_T\) is a metric (by Proposition 1), therefore \(\delta_T(x, z) = 1\) and (P) implies \(xTz\). As \(x, y, z\) were chosen arbitrarily, \(T\) is transitive, i.e. it is an equivalence.

§2

Definition 3. Let \((A, \mu)\) be a quasimetric space and \(\{\varepsilon_i\}_{i=1}^\infty\) a decreasing sequence of positive real numbers. Denote
\[ T_{\text{lim}} = \bigcap_{i=1}^{\infty} T_{\mu(\varepsilon_i)}, \]

where \( T_{\mu(\varepsilon_i)} \) is a tolerance on \( A \) induced by the quasimetric \( \mu \) with the unit \( \varepsilon_i \).

**Proposition 8.** Let \((A, \mu)\) be a quasimetric space, let \( \{\varepsilon_i\}_{i=1}^{\infty} \) be a decreasing sequence of positive real numbers and \( \varepsilon = \lim_{i \to \infty} \varepsilon_i \). Then \( T_{\mu(\varepsilon)} = T_{\text{lim}} \) and \( T_{\text{lim}} \) is a tolerance on \( A \).

**Proof.** Evidently \( T_{\text{lim}} \) is a reflexive and symmetric relation on \( A \), i.e. it is a tolerance. If \( xT_{\mu(\varepsilon)}y \), then \( \mu(x, y) \leq \varepsilon \), therefore \( \mu(x, y) \leq \varepsilon_i \) for each \( i \). Hence \( xT_{\mu(\varepsilon_i)}y \) for each \( i \) and \( xT_{\text{lim}}y \). We have proved \( T_{\mu(\varepsilon)} \subseteq T_{\text{lim}} \). Conversely, if \( xT_{\text{lim}}y \), then \( xT_{\mu(\varepsilon_i)}y \) for each \( i \) and thus \( \mu(x, y) \leq \varepsilon_i \) for each \( i \). Hence \( T_{\mu(\varepsilon)} \subseteq T_{\text{lim}} \). We have proved \( T_{\mu(\varepsilon)} \subseteq T_{\text{lim}} \). Hence \( T_{\text{lim}} \leq T_{\mu(\varepsilon)} \).

**Proposition 9.** Let \((A, \mu)\) be a quasimetric space and \( \{\varepsilon_i\}_{i=1}^{\infty} \) a decreasing sequence of positive real numbers such that \( \lim_{i \to \infty} \varepsilon_i = 0 \). Then the following two assertions are equivalent:

1. \( \mu \) is a metric.
2. \( T_{\text{lim}} = I \).

**Proof.** Let \( \mu \) be a metric and \( xT_{\text{lim}}y \) for some \( x \in A, y \in A \). Then \( xT_{\mu(\varepsilon_i)}y \) for each \( i \), thus \( \mu(x, y) \leq \lim_{i \to \infty} \varepsilon_i = 0 \). As \( \mu \) is a metric, \( \mu(x, y) = 0 \) implies \( x = y \) and hence \( T_{\text{lim}} \subseteq I \). Evidently \( I \subseteq T_{\text{lim}} \), therefore \( T_{\text{lim}} = I \). Now let \( T_{\text{lim}} = I \) and \( \mu(x, y) = 0 \) for some \( x \in A, y \in A \). Then \( \mu(x, y) = \lim_{i \to \infty} \varepsilon_i \), i.e. \( xT_{\text{lim}}y \), hence by (2) we have \( x = y \) and \( \mu \) is a metric.

**Proposition 10.** Let \((A, \mu)\) be a quasimetric space and \( T_{\mu(\varepsilon)} \) a tolerance induced by the quasimetric \( \mu \) with the unit \( \varepsilon \). Then for \( \varepsilon = 0 \) the relation \( T_{\mu(0)} \) is an equivalence on \( A \) and \( \{A, T_{\mu(0)}\} \) is a metric space.

**Proof.** If \( xT_{\mu(0)}y, yT_{\mu(0)}z \), then \( \mu(x, y) = 0, \mu(y, z) = 0 \) and this implies \( 0 \leq \mu(x, z) \leq \mu(x, y) + \mu(y, z) = 0 \), hence \( \mu(x, z) = 0 \) and \( xT_{\mu(0)}z \). The second assertion is evident.

§ 3

**Lemma 1.** Let \( L \) be a lattice with the least element \( 0 \), let \( T \) be a compatible tolerance on \( L \) (see for example [3]). If \( a \in L, b \in L \) and \( aT^r 0, bT^s 0 \) for some non-negative integers \( r, s \), then \( (a \lor b) T^{\max(r, s)} 0, (a \land b) T^{\min(r, s)} 0 \).

**Proof.** If \( T \) is a compatible relation on \( L \), then (by Theorem 3 in [1]) \( T^k \) is also a compatible relation on \( L \) for each non-negative integer \( k \). If \( aT^r 0, bT^s 0 \), then by Corollary 5 in [1] we have \( aT^q 0, bT^q 0 \) for \( q = \max(r, s) \) and the compatibility of \( T^q \) implies \( (a \lor b) T^q 0 \). Further let \( p = \min (r, s) \); without less of generality let \( p = r \). Then \( aT^r 0 \Rightarrow (a \land b) T^0 (0 \land b) = 0 \) and thus \( (a \land b) T^p 0 \).
Definition 3. Let $L$ be a lattice with the least element $0$. A tolerance $T$ on $L$ is called disjunctive, if $(a \wedge b) T^k 0$ implies $a T^k 0$ or $b T^k 0$.

In [2] the concept of a valuation on a lattice is introduced. A real-valued function $v$ on $L$ is called a valuation, if for any two elements $a, b$ of $L$ 

$$v(a) + v(b) = v(a \wedge b) + v(a \vee b).$$

A valuation is called order-preserving, if $a \leq b$ implies $v(a) \leq v(b)$ and positive, if $a < b$ implies $v(a) < v(b)$ for any $a$ and $b$. If there exists an order-preserving (or positive) valuation on $L$, then $L$ is called a quasimetric (or metric respectively) lattice. (see [2], p. 108).

Theorem 1. Let $L$ be a lattice with the least element $0$ and let $T$ be a compatible disjunctive tolerance on $L$ such that $(L, T)$ is a connected tolerance space. Then $L$ is a quasimetric lattice.

Proof. Let $v$ be an integer-valued function on $L$ defined so that $v(a) = 0$ for each $a \in L$ such that $a T 0$ and $v(a) = p$ for each $a \in L$ such that $a T^{p+1} 0$ and $\neg a T^q 0$ for all $q \leq p$. As $(L, T)$ is connected, $v$ is defined for all elements of $L$. If $a \leq b$, then $a \vee b = b$, $a \wedge b = a$ and thus $v(a \wedge b) + v(a \vee b) = v(a) + v(b)$. Now let $a, b$ be two incomparable elements of $L$, let $v(a) = p$, $v(b) = q$; without loss of generality let $q \leq p$. Then $a T^{p+1} 0$, $b T^{q+1} 0$ and by Lemma 1 we have $a \vee b T^p 0$. From this and from $a T^p a$ we obtain $a = a \wedge (a \vee b) T^p 0 \wedge a = 0$ and $v(a) < p$, which is a contradiction. Therefore $v(a \vee b) = p$. Further from Lemma 1 we have $(a \wedge b) T^{s+1} 0$. Let $j \leq q$; then $\neg a T^j 0$, $\neg b T^j 0$ and the disjunctivity of $T$ implies $\neg (a \wedge b) T^j 0$, therefore $v(a \wedge b) = q$. We have $v(a \wedge b) + v(a \vee b) = p + q = v(a) + v(b)$. We have proved that $v$ is a valuation on $L$. Now let $x \leq y$, $v(y) = q$. Then $y T^{q+1} 0$ and $\neg y T^r 0$ for $r \leq q$. We have $x \wedge y = x$ and from the compatibility of $T^{q+1}$ we obtain

$$x T^{q+1} x, \quad y T^{q+1} 0 \Rightarrow x = (x \wedge y) T^{q+1} (x \wedge 0) = 0$$

and thus $v(x) \leq q = v(y)$ and $v$ is order-preserving. This means that $L$ is quasimetric.

Remark. We shall show that in the case when $T$ is not disjunctive the function $v$ defined in this proof is not a valuation. If $T$ is not disjunctive, then there exist elements $a, b$ of $L$ and a non-negative integer $s$ such that $\neg a T^{s+1} 0$, $\neg b T^{s+1} 0$, $a \wedge b T^{s+1} 0$. Then $v(a \wedge b) \leq s$. Let $v(a) = p$, $v(b) = q$; then $p \geq s + 1$, $q \geq s + 1$. Without loss of generality let $p \geq q$. We have $a T^{p+1} 0$, $b T^{q+1} 0$, thus by Lemma 1 $(a \vee b) T^{p+1} 0$ and $v(a \vee b) \leq p$. Then

$$v(a) + v(b) = p + q > p + s + 1,$$

$$v(a \wedge b) + v(a \vee b) \leq p + s$$

and thus $v(a) + v(b) \neq v(a \wedge b) + v(a \vee b)$. 

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We have proved that the valuation $v$ defined in the proof of Theorem 1 is order-preserving. The following proposition shows, when it is positive.

**Proposition 11.** Let $L$ and $T$ be given as in Theorem 1, let $v$ be the valuation defined in the proof of Theorem 1. The valuation $v$ is positive, only if $L$ is a chain embeddable into the chain of all non-negative integers (naturally ordered).

**Proof.** Suppose that $L$ is not a chain. Then there exist two elements $x, y$ of $L$ which are incomparable. Let $v(x) = p$, $v(y) = q$ and without loss of generality $p \geq q$. In the proof of Theorem 1 it is proved that then $v(x \land y) = q$, $v(x \lor y) = p$. But then $x \land y < y$, $v(x \land y) = v(y)$ and $v$ is not positive. Therefore $L$ is a chain.

If $v$ is a positive valuation on a chain, it is evidently an embedding of this chain into the chain of all non-negative integers.

Now it seems to be reasonable to consider the valuation in which $v(a) = 0$ only for $a = 0$.

**Theorem 2.** Let $L$ be a lattice with the least element 0 and with the property that $a \land b = 0$ in $L$ if and only if $a = 0$ or $b = 0$. Let $T$ be a compatible disjunctive tolerance on $L$ such that $(L, T)$ is a connected tolerance space. Then there exists an order-preserving valuation $v$ on $L$ such that $v(a) = 0$ only for $a = 0$.

**Proof.** Let $v$ be the valuation from the proof of Theorem 1. Put $v'(0) = 0$, $v'(a) = v(a) + 1$ for each $a \neq 0$. Let $x, y$ be two elements of $L$. If $x \neq 0$, $y \neq 0$, then also $x \land y \neq 0$, $x \lor y \neq 0$ and we have

$$v'(x \land y) + v'(x \lor y) = v(x \land y) + v(x \lor y) + 2 = v(x) + v(y) + 2 = v'(x) + v'(y).$$

If $x = 0$, $y \neq 0$, then $x \land y = 0$, $x \lor y \neq 0$ and

$$v'(x \land y) + v'(x \lor y) = v(x \land y) + v(x \lor y) + 1 = v(x) + v(y) + 1 = v'(x) + v'(y).$$

Analogously for $x \neq 0$, $y = 0$. For $x = y = 0$ the equality is evident. Therefore $v'$ is the required valuation.

Before proving the last theorem, we shall prove a lemma.

**Lemma 2.** Let $m_1, m_2, n_1, n_2$ be four non-negative integers, let $|m_1 - n_1| \leq 1$, $|m_2 - n_2| \leq 1$. Then

$$|\max (m_1, m_2) - \max (n_1, n_2)| \leq 1,$$

$$|\min (m_1, m_2) - \min (n_1, n_2)| \leq 1.$$
then \[ |m_1 - n_2| = n_2 - m_1 \leq n_2 - m_2 = |m_2 - n_2| \leq 1. \] Analogously we do the proof for \( m_1 \leq m_2, n_1 \geq n_2 \) and \( m_1 \leq m_2, n_1 \leq n_2 \). The proof for the minimum is dual.

**Theorem 3.** Let \( L \) be a quasimetric lattice with the valuation \( v \) satisfying \( v(x \lor y) = \max(v(x), v(y)), v(x \land y) = \min(v(x), v(y)) \), for any two elements \( x, y \) of \( L \). Let \( T \) be the tolerance on \( L \) defined so that \( xTy \) if and only if \( v(x \lor y) - v(x \land y) \leq 1 \). Then \( T \) is a compatible tolerance on \( L \).

**Proof.** Let \( a, b \) be two elements of \( L \). Let \( aTb \). This means \( v(a \lor b) - v(a \land b) \leq 1 \) and according the assumption \( \max(v(a), v(b)) - \min(v(a), v(b)) \leq 1 \). But one of the numbers \( v(a), v(b) \) is the maximum and the other is the minimum of these two numbers, therefore \( |v(a) - v(b)| \leq 1 \). On the other hand, if \( |v(a) - v(b)| \leq 1 \), then \( \max(v(a), v(b)) - \min(v(a), v(b)) \leq 1 \) and \( aTb \). We have proved that \( aTb \) if and only if \( |v(a) - v(b)| \leq 1 \). Now let \( x_1, x_2, y_1, y_2 \) be four elements of \( L \) such

![Fig. 1](image-url)
Fig. 2

Fig. 3
that $x_1Ty_1$, $x_2Ty_2$; this means $|v(x_1) - v(y_1)| \leq 1$, $|v(x_2) - v(y_2)| \leq 1$. Then $v(x_1 \lor x_2) = \max (v(x_1), v(x_2))$, $v(x_1 \land x_2) = \min (v(x_1), v(x_2))$, $v(y_1 \lor y_2) = \max (v(y_1), v(y_2))$, $v(y_1 \land y_2) = \min (v(y_1), v(y_2))$. By Lemma 2 we have $|v(x_1 \lor x_2) - v(y_1 \lor y_2)| \leq 1$, $|v(x_1 \land x_2) - v(y_1 \land y_2)| \leq 1$ and thus $(x_1 \lor x_2) T(y_1 \lor y_2)$, $(x_1 \land x_2) T(y_1 \land y_2)$ and $T$ is compatible.

Remark. Fig. 1 shows a lattice with the valuation satisfying the conditions of Theorem 3. On Fig. 2 we see a lattice with a valuation which does not satisfy them; for the elements $x_1, x_2, y_1, y_2$ of this lattice we have $x_1Ty_1$, $y_2Ty_2$, but not $(x_1 \land x_2) T(y_1 \land y_2)$. Fig. 3 presents a lattice which satisfies the conditions, but not the assertion; therefore the converse assertion to Theorem 3 is not true.

The tolerance $T$ in Theorem 3 is derived from a metric mentioned in [2].

REFERENCES