Archivum Mathematicum

Juhani Nieminen

Operations on graphs determining congruences on graphs

Archivum Mathematicum, Vol. 14 (1978), No. 4, 215--218

Persistent URL: `http://dml.cz/dmlcz/107013`

Terms of use:

© Masaryk University, 1978

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://project.dml.cz`

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIV: 215—218, 1978

OPERATIONS ON GRAPHS DETERMINING CONGRUENCES ON GRAPHS

JUHANI NIEMINEN, Oulu

(Received August 15, 1977)

The purpose of this paper is to characterize by means of concepts and operations of graph theory partitions of the elements of a finite modular lattice H that determine congruence relations on H. By the aid of the characterization we construct thereafter a class of congruence relations on graphs. We recall first some concepts of graph theory and apply thereafter them to the Hasse diagram of H in order to obtain the characterization.

We shall consider finite undirected and connected graphs $G = (P(G), L(G))$ only without loops and multiple lines, where $P(G)$ is the set of points of G and $L(G)$ its set of lines. SP is a mapping $P(G) \times P(G) \to 2^{P(G)}$ defined as follows:

$$SP(x, y) = \{z \mid z \in P(G) \textit{ and } z \textit{ is on a shortest path joining } x \textit{ and } y \textit{ in } G\}.$$

We shall call SP a binary operation on $P(G)$, although the mapping induced by the operation is a one-to-many mapping, as the name operation helps us to find some useful analogies we shall apply. In particular, $\{x, y\} \subseteq SP(x, y)$ and $SP(x, x) = \{x\}$, $x, y \in P(G)$. In general, let U and W be two subsets of $P(G)$, then $SP(U, W)$ denotes the union of the sets $SP(u, w)$, where $u \in U$ and $w \in W$; formally $SP(U, W) = \{z \mid z \in SP(u, w) \textit{ for some } u \textit{ and } w, u \in U \textit{ and } w \in W\}$. A set $U \subset P(G)$ is called an *ideal of* G, if $U \neq \emptyset$ and $SP(U, U) = U$. By the notation $SP^n(x, y)$ we denote the operation $SP(SP^{n-1}(x, y), SP^{n-1}(x, y))$. Thus $SP^2(x, y) = SP(SP(x, y), SP(x, y))$. As we consider finite graphs only, there is for any pair $x, y \in P(G)$ a value of n such that $SP^n(x, y)$ is an ideal of G. The graph of Figure 1 illuminates the case where $SP^2(x, y)$ is not an ideal of G but $SP^3(x, y)$ is. It is important to construct from a pair $x, y \in P(G)$ an ideal of G by means of sequential applying of the SP-operation and in order to use a brief notation, $SU(x, y)$ denotes the ideal obtained from x, y by applying the SP-operation enough many times.

Ideals of graphs and the SP-operation were introduced in [4] and briefly considered in [5]. These concepts are natural generalizations of corresponding concepts defined for trees by Nebeský in [3].

In this paper we consider the Hasse diagram of a lattice H as an undirected graph and denote it by G_H. Lemma 1 and Theorem 1 are proved in a more general form than we need later. A lattice H is locally finite, if its every interval is finite.

Lemma 1. *Let H be a locally finite lattice. Then $SU(x, y) = [x \wedge y, x \vee y]$ for any two elements $x, y \in H$ if and only if H is modular.*

Proof. If H is modular, then according to the metric properties of finite modular lattices, $x \wedge y$, $x \vee y \in SP(x, y)$ (see e.g. Draškovičová [1]). As the lengths of any two chains between a and b in a finite modular lattice are equivalent when $a < b$, each $z \in [x \wedge y, x \vee y]$ belongs to a shortest path from $x \wedge y$ to $x \vee y$ and so $z \in SU(x, y)$. Obviously $SU(x, y) \subseteq [x \wedge y, x \vee y]$, and thus $SU(x, y) = [x \wedge y, x \vee y]$.

Let H satisfy the condition of the lemma for any pair $x, y \in H$. If H were non-modular, then it contains the well known non-modular sublattice (in Figure 2 the sublattice of elements a, b, c, d, e), where the set $\{a, b, c\} = SU(b, c) \neq [a, e] = [b \wedge c, b \vee c]$. This completes the proof.

Now we are ready to prove the characterization.

Theorem 1. *Let H be a locally finite modular lattice and $\mathfrak{C} = \{C_1, \ldots, C_m\}$ a partition of its elements. $\mathfrak{C}$ is a congruence partition of H with respect to the operations $\vee$ and $\wedge$ on H if and only if the condition (A) holds.*

(A) If $x, y \in C_i$ and $a, b \in C_j$ in $\mathfrak{C}$, then $SU(x, a) \cap C_k \neq \emptyset$ holds for some k in G_H if and only if $SU(y, b) \cap C_k \neq \emptyset$ holds, $1 \leqq k \leqq m$.

Proof. Assume that $\mathfrak{C}$ is a partition of the points $P(G_H)$ such that $SU(x, a) \cap C_k \neq \emptyset$ if and only if $SU(y, b) \cap C_k \neq \emptyset$. We show that R is a latticecongruence on H, with the classes $C_1, \ldots, C_m$. Clearly R is reflexive, symmetric and transitive. Thus it remains to show the compatibility of R, i.e. to show that xRy implies $x \wedge zRy \wedge z$ and $x \vee zRy \vee z$ for any $z \in H$. Moreover, if $qRp \Leftrightarrow q \wedge pRq \vee p$, we may assume that $x \leqq y$.

Let $x < y$ (the case $x = y$ is trivial), xRy and $z \in H$. Thus $x \vee z \leqq y \vee z$. We assume that in the partition $\mathfrak{C}$ of $Hx \vee z$ and $y \vee z$ belong to different sets of $\mathfrak{C}$. As $x \leqq y \wedge (x \vee z) \leqq y$ and yRx, $y \wedge (x \vee z)\, Ry$ holds, too. The relations $y \vee zRy \vee z$ and yRx imply that (A) holds for $SU(y \vee z, y)$ and $SU(y \vee z, x)$. $x \vee z \in SU(y \vee z, x)$ and we assume that $x \vee z \in C_h$. As $x \vee z < y \vee z$, $x \vee z \notin SU(y \vee z, y)$. Then according to (A), $SU(y \vee z, y) \cap C_h \neq \emptyset$, and let t be the greatest element of the set $SU(y \vee z, y) \cap C_h$; such an element exists as $SU(y \vee z, y)$ is finite and for any two elements of $SU(y \vee z, y)$ (of C_h), $SU(y \vee z, y)$ (C_h) contains the join of these elements. But $x \vee z \vee t \in C_h$ and $x \vee z \vee t \leqq y \vee z$, whence $x \vee z \vee t \in SU(y \vee z, y)$. Thus we can assume that $x \vee z \leqq t$, and as $t \in SU(y \vee z, y)$, $t \geqq y$. But then $y \vee x \vee z = y \vee z \leqq t$, whence $y \vee z$, $x \vee z \in C_h$, which is a contradiction. Hence $y \vee z$, $x \vee z \in C_k$ for some value k of i. The proof is similar for $y \wedge zRx \wedge z$.

Conversely, we assume that $\mathfrak{C}$ generates a latticecongruence on H. Let $x, y \in C_i$,

$a, b \in C_j$ and $i \neq j$. Accordingly, we may assume that $x \leqq y$ and $a \leqq b$. As R is a congruence relation on H, $y \vee bRx \vee a$. If there is an element $q \in C_k$, $y \leqq q \leqq$ $\leqq y \vee b$, then $x \leqq q \wedge (x \vee a) \leqq x \vee a$, $q \wedge (x \vee a) Rq$ and thus $q \wedge (x \vee a) \in C_j$. By applying this technique to the intervals $[y \wedge b, y \vee b]$ and $[x \vee a, x \wedge a]$ we see that the condition (A) holds for $SU(y, b)$ and $SU(x, a)$. The proof is similar for $SU(x, b)$ and $SU(y, a)$. This completes the proof.

As the example of Figure 2 shows, a partition of a non-modular lattice H satisfying the condition (A) need not be either a $\wedge$-congruence or a $\vee$-congruence on H.

In the next theorem we show how the condition (A) generalizes by a natural way the construction of compatible tolerances on graphs introduced by Zelinka in [7].

We call a binary, reflexive, symmetric and transitive relation R on a graph a SU-compatible congruence relation on G when aRb and xRy imply $SU(a, x) RSU(b, y)$. The notation $SU(a, x)\, RSU(b, y)$ means that for any $z \in SU(a, x)$ there is a point $u \in SU(b, y)$ such that zRu, and for any $w \in SU(b, y)$ there is a point $v \in SU(a, x)$ such that vRw.

Theorem 2. *Let $\mathfrak{C}$ be a partition of the pointset $P(G)$ of a graph $G = (P(G), L(G))$. The relation R given by $\mathfrak{C}$ determines a SU-compatible congruence relation on G if and only if $\mathfrak{C}$ satisfies the condition (A).*

Proof. If $\mathfrak{C}$ is a partition of $P(G)$ such that the relation R given by $\mathfrak{C}$ satisfies the condition (A), the SU-compatibility of R follows directly from (A). The converse proof follows similarly directly from the definition of the SU-compatibility.

By using the terminology of Theorem 2, we can say, according to Theorem 1, that R is a latticecongruence on a finite modular lattice H if and only if R is a SU-compatible congruence relation on G_H.

We obtain also a characterization of finite modular lattices as given in the next theorem.

Theorem 3. *Let H be a finite lattice and $\mathfrak{C}$ a partition of H determining a SU-compatible relation R on G_H. H is modular if and only if each R defined above is a lattice-congruence on H.*

Proof. If H is modular, then the assertion follows from Theorems 1 and 2. Thus let each R of the theorem be a congruence relation on H. If H is non-modular, it contains as a sublattice the lattice of the elements a, b, c, d, e in Figure 2, where the subset $\{a, b, c\}$ of the partition $\mathfrak{C} = \{\{a, b, c\}, \{d, e\}\}$ shows that $\mathfrak{C}$ does not determine a congruence relation on H although R is SU-compatible on G_H.

As a model for constructing a SU-compatible congruence on G were latticecongruences on a finite modular lattice H. This model is used in the following theorem where an analogy is presented between SU-compatible congruences on G and congruences on algebras. Its proof is a direct copy of the corresponding proof for algebras given e.g. in [6, Thm. 96 and its supplement], and hence we omit it.

Theorem 4. *Let G be a given graph. G is a Cartesian product of two non-trivial graphs G_1 and G_2, i.e. $G = G_1 \times G_2$, if and only if there are two non-trivial SU-compatible congruences $R_1, R_2 \in H(G)$ which are permutable and complements of each other in $H(G)$. $H(G)$ is the lattice of SU-compatible congruences on G.*

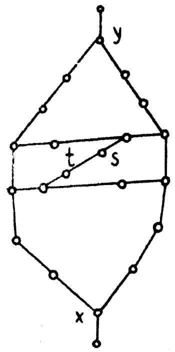

Fig. 1

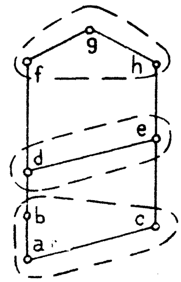

Fig. 2

REFERENCES

[1] H. Draškovičová: *Über die Relation „zwischen" in Verbänden,* Mat.-fyz. čas. SAV 16 (1966), 13—20.

[2] F. Harary and G. W. Wilcox: *Boolean operations on graphs,* Math. Scand. 20 (1967), 41—51.

[3] L. Nebeský: *Algebraic properties of trees,* Acta Univ. Carol. Philologica — Monographia XXV, 1969.

[4] J. Nieminen: *On ideals in a graph; a communication problem,* Arch. Elektr. Übertragungstech. 27, (1973), 213—217.

[5] J. Nieminen: *The lattice of connected subgraphs of a connected graph,* Comment. Math. (Poznań), to appear.

[6] G. Szász: *Introduction to lattice theory,* Academic Press, New York—London, 1963.

[7] B. Zelinka: *Tolerance graphs,* Comment. Math. Univ. Carol. 9 (1968), 121—131.

Juhani Nieminen
University of Oulu
Dept. Math. in Faculty of Technology
90101 OULU 10
Finland