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Categories of models of infinitary Horn theories

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Our aim is to characterize underlying functors of categories of models of infinitary Horn theories. The characterization is, in fact, an infinitary version of one result of O. Kean ([2], Prop. 1.4.1).

Infinitary Horn theories are theories of a language $L_{\infty, \infty}$. The language $L_{\infty, \infty}$ has a set (possibly empty) of $n$-ary function symbols for each cardinal number $n \geq 1$, a set (possibly empty) of $n$-ary relation symbols for each cardinal number $n \geq 1$ and a set of constant symbols. Further, we have a proper class $V$ of variables. If $n$ is a cardinal number, then the string $(x_i)_{i \in n}$ of variables will be denoted by $x$ and sometimes $x$ will be identified with a map $x : n \to V$. Terms and atomic formulas are defined as usual. Formulas are built up from atomic formulas by means of a negation, conjunctions $\bigwedge_{i \in I} \varphi_i$, where $I$ can be an arbitrary set and quantifiers $\forall x$, where $x : n \to V$ and $n$ is an arbitrary cardinal number. Remark that no genuine occurrence of a quantifier will appear in our considerations because all formulas will be universal. Concerning infinitary logic consult [1].

An infinitary Horn theory $H$ is a theory of $L_{\infty, \infty}$ whose axioms are all of the form (where we will assume that the following formulas all have their free variables universally quantified in front):

1. $\varphi$ where $\varphi$ is an atomic formula
2. $\bigwedge_{i \in I} \varphi_i \to \Theta$ where $\varphi_i$, $i \in I$ and $\Theta$ are atomic formulas.

Let $\mathcal{A}_H$ be the category of all models of a given infinitary Horn theory $H$ (morphisms are homomorphisms, i.e. maps which preserve atomic formulas). Let $U_H : \mathcal{A}_H \to \text{Set}$ be the forgetful functor. Our permission of a class of function and relation symbols can cause two inconveniences. The functor $U_H$ need not have a left adjoint and $U_H$ need not be fibre-small (i.e. there can be a proper class of models on the same underlying set). The first inconvenience can be easily excluded syntactically by the assumption that there is only a set of $n$-ary terms in $H$ for each cardinal number $n$. Namely, then the algebraic reduct of $\mathcal{A}_H$ (if we consider operations only) is varietal
in the sense of [3] and if we endow the free algebra over a set $X$ by the weakest relational structure we get the free $\mathcal{A}_H$-object over $X$ (see [2], 1.6.). The syntactical counterpart of the second inconvenience is not clear and so we adopt the following convention.

**Definition:** A fibre-small functor $U : \mathcal{A} \to \text{Set}$ will be called a Horn functor if there is an infinitary Horn theory $H$ such that for each cardinal number $n$ there is only a set of $n$-ary terms and an equivalence $M : \mathcal{A} \to \mathcal{A}_H$ such that $U_H \cdot M = U$.

We are going to give a characterization of Horn functors analogous to the characterization of varietal functors from [3]. We say that pushouts preserve onto morphisms if in a pushout

![Diagram of pushout](image1)

$Uf$ onto infers that $U\bar{f}$ is onto. This condition implies that $U$ carries coequalizers on epics for

![Diagram of coequalizer](image2)

is a coequalizer iff the following diagram is a pushout

![Diagram of pushout for coequalizer](image3)
**Theorem:** \( U : \mathcal{A} \to \text{Set} \) is a Horn functor iff \( \mathcal{A} \) is cocomplete and co-well-powered, \( U \) is faithful, has a left adjoint and the following conditions hold:

(i) Pushouts preserve onto morphisms

(ii) If \( U f_i : UA_i \to UB_i \) are onto, then \( U \sum_i f_i : U \sum_i A_i \to U \sum_i B_i \) is onto.

**Proof:** Necessity is a matter of a direct verification. Let \( U \) fulfil the mentioned properties. Denote by \( F \) a left adjoint of \( U \), by \( \varphi = \varphi_{n,A} : \mathcal{A}(Fn, A) \to \text{Set}(n, UA) \) the adjunction isomorphism, by \( \eta : 1 \to UF \) the unit and by \( \varepsilon : FU \to 1 \) the counit of the adjunction. Consider the language \( L_{\infty, \infty} \) which has morphisms \( f : Fn \to Fm \) as \( n \)-ary function symbols (constants will be treated as 0-ary function symbols) and morphisms \( p : Fn \to X \) such that \( Up \) is onto as \( n \)-ary relation symbols. If \( g : Fn \to Fm \) and \( i : 1 \to n \) maps the unique element of 1 on \( i \in n \), then the composition \( g . Fi \) will be denoted by \( g_i \). Consider the Horn theory \( H \) with the following axioms:

(A1) (a) If \( F1 \xrightarrow{f} Fm \xrightarrow{g} Fn \), then

\[
(gf)(x) = f(g_1(x), g_2(x), \ldots)
\]

(b) If \( i : 1 \to n \), then \( (Fi)(x) = x_i \).

(A2) If

\[
\begin{array}{c}
Fm \\
\downarrow p \\
X \\
\downarrow f \\
Fn \\
\uparrow q \\
Y
\end{array}
\]

commutes and \( Up, Uq \) are onto, then

(a) \( p(x) \to q(xg) \)

Moreover, if the square is a pushout, then

(b) \( p(x) \leftrightarrow q(xg) \)

(A3) If \( Fn \xrightarrow{f} Fm \xrightarrow{p} X \) is a coequalizer, then

\[
p(x) \leftrightarrow \bigwedge_{i \in n} f_i(x) = g_i(x)
\]

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(A4) If Fig. 5 is a pushout, $U_p, U_q$ are onto and $r = w \cdot p$, then

$$r(x) \leftrightarrow p(x) \land q(x).$$

(A5) If $I$ is a set and $p_i : F_{n_i} \to X_i$, $U_{p_i}$ onto for any $i \in I$ and $u_i : n_i \to \sum_{i \in I} n_i$ are injections, then

$$(\sum_{i \in I} p_i)(x) \leftrightarrow \bigwedge_{i \in I} p_i(x \cdot u_i).$$

Here, (i) was used in (A3), (A4) and (ii) in (A5).

Define the functor $M : \mathcal{A} \to \mathcal{A}_H$ as follows. Consider $A \in \mathcal{A}$. Let $M(A)$ have $UA$ as the underlying set, interpret $f : F_n \to X$ as $^h(A)$, any $u \in X$ such that $p \cdot u = a$. It is easy to verify that $M(A)$ is a model of $H$. Clearly $U_h : M(A) \to M(B)$ carries a homomorphism of models for any $h : A \to B$. Thus $M$ is a functor. $M$ is faithful and we will show that it is full. Consider a homomorphism $h : M(A) \to M(B)$ of models. Since $e^{MA}_A(1_{UA})$ holds, we get $e^{MB}_B(U_h h)$. Thus there is $g : A \to B$ such that $U_h h = \varphi(g \cdot e_A) = U_g$. Hence $h = M(g)$ and $M$ is full. It remains to show that $M$ is an equivalence, i.e. that any $C \in \mathcal{A}_H$ is isomorphic to $M(A)$ for some $A \in \mathcal{A}$.

Let $C \in \mathcal{A}_H$ and denote by $p^C$ the interpretation of in $C$ for each relation symbol $p$. The map $Ue_A$ is onto for each $A \in \mathcal{A}$ because $Ue \cdot \eta U = 1$ (see [4] p. 80) and thus we may put $C(A) = (e_A)^C$. By (A2) (a) applied to the square $e_B$. $FUF = f \cdot e_A$ we get that $Set(Uf, U_H C)$ induces a map $C(f) : C(B) \to C(A)$ for any $f : A \to B$ in $\mathcal{A}$. Hence we get a functor $C : \mathcal{A}^{op} \to Set$. Let $A = \sum_{i \in I} A_i$ in $\mathcal{A}$, $t_i : A_i \to A$ be injections and denote by $k : \sum_{i \in I} UA_i \to UA$ the canonical map. Let $e : FUA \to E$ be the coequalizer of $FUF \sum_{i \in I} A_i \xrightarrow{\varepsilon_{FUA \cdot FUF}} FUA$. Since $e_A$ equalizes $FUE_A, \varepsilon_{FUA}$, there is a unique morphism $v : E \to A$ such that
\[ v \cdot e = e_A. \] Hence \( v \cdot e \cdot Fk = e_A \cdot Fk = \Sigma e_{A_i}. \) Let the left square in the following diagram be a pushout and \( \overline{u} \) be the unique morphism such that \( \overline{u} \cdot \overline{v} = v \cdot e \) and \( \overline{u} \cdot u = 1. \)

\[
\begin{array}{ccc}
\Sigma e_{A_i} & \xrightarrow{u} & \Sigma e_{A_i} \\
\downarrow \overline{v} & & \downarrow \overline{v} \\
A & \xrightarrow{\overline{u}} & A
\end{array}
\]

Then the outer rectangle is a pushout. Namely, we have to prove that \( r \cdot e \cdot Fk = s \cdot \Sigma e_{A_i} \) implies \( s \cdot v = r. \) But it follows from \( r \cdot e \cdot FU \Sigma e_{A_i} = r \cdot e \cdot FUFk = s \cdot e_{F\Sigma e_{A_i}} = s \cdot e_{F\Sigma e_{A_i}} = s \cdot v \cdot e \cdot Fk \). Hence \( FUFk = s \cdot v \cdot e \cdot Fk \) because \( FUFk \) is epi by (ii). Hence the right square is a pushout. Following (A2) (b), (A5), (A4) and (A3) we have that

\[ (1) \quad \varepsilon_A(x) \leftarrow (\bigwedge_{i \in I} e_{A_i}(x \cdot U_i)) \wedge (\bigwedge_{j \in U F \Sigma e_{A_i}} x_{\Sigma e_{A_i}}(j)) = \varphi^{-1}(j)(x \cdot k) \]

Consider the canonical map \( t : C(A) \rightarrow \prod_{i \in I} C(A_i) \) which is given by \( t(c) = \langle c \cdot U_{i_1} \rangle_{i_1} \) for any \( c : UA \rightarrow U_H \) from \( C(A) \). Since \( U \Sigma e_{A_i} \) is onto, \( r = U \Sigma e_{A_i}(j) \) for any \( r \in U A \). Following (1) \( c_r = \varphi^{-1}(j)(c \cdot k) \) for any \( c \in C(A) \). Hence \( t \) is injective. Let \( \langle c^i \rangle_i \in \prod_{i \in I} C(A_i) \) and let \( \overline{c} : \Sigma U A_i \rightarrow U_H C \) be determined by \( c^i \). By (A5) \( (\Sigma e_{A_i}) \varepsilon (\overline{c}) \) holds. Let \( j_1, j_2 \in U F \Sigma U A_i \) and \( U \Sigma e_{A_i}(j_1) = U \Sigma e_{A_i}(j_2) \). Then \( (\Sigma e_{A_i}) \varphi^{-1}(j_1) = (\Sigma e_{A_i}) \varphi^{-1}(j_2) \) \( Fj_1 = e_A \cdot FUFk = s \cdot e_{F\Sigma e_{A_i}} = s \cdot e_{F\Sigma e_{A_i}} = s \cdot v \cdot e \cdot Fk \). Hence \( \overline{c} \) is a coequalizer of \( \varphi^{-1}(j_1), \varphi^{-1}(j_2) \). Since \( \Sigma e_{A_i} \) can be factorized through \( e, p^C(\overline{c}) \) holds by (A2) (a) and \( \varphi^{-1}(j_1)(\overline{c}) = \varphi^{-1}(j_2)(\overline{c}) \) by (A3). Hence \( c_r = \varphi^{-1}(j_1)(\overline{c}) \), where \( r = U \Sigma e_{A_i}(j) \) defines \( c : UA \rightarrow U_H C \) and \( c \in C(A) \) by (1). Thus \( t \) is bijective and \( C \) preserves products.

Let \( A \xrightarrow{f} B \xrightarrow{g} D \) be a coequalizer diagram in \( A \). Since \( U e \) is epi, the canonical map \( t \) from \( C(D) \) into an equalizer of \( C(B) \xrightarrow{f} C(A) \) is injective. We will prove that it is onto. Let \( y : UB \rightarrow U_H C \in C(B) \) and \( y \cdot Uf = y \cdot Ug \). Let \( h : UB \rightarrow E \) be an
equalizer of \( Uf, Ug \) and \( k : E \to UD \) be the unique map such that \( k \cdot h = Ue \). We are going to show that the following square is a pushout

![Fig. 7](image)

Consider \( u : B \to X \) and \( v : FE \to X \) with \( u \cdot \varepsilon_B = v \cdot Fh \). It holds \( u \cdot f \cdot \varepsilon_A = u \cdot \varepsilon_B \cdot FUf = v \cdot Fh \cdot FUf = v \cdot Fh \cdot FUg = u \cdot g \cdot \varepsilon_A \) and thus \( u \cdot f = u \cdot g \). There is a unique \( r : D \to X \) such that \( r \cdot e = u \). Further \( r \cdot \varphi^{-1}(k) \cdot Fh = r \cdot \varepsilon_D \cdot F(k \cdot h) = = r \cdot \varepsilon_D \cdot FUe = r \cdot \varepsilon_B \cdot \varepsilon_B = u \cdot \varepsilon_B = v \cdot Fh \) and thus \( r \cdot \varphi^{-1}(k) = v \) because \( Fh \) is epi. By (A4)

\[
(\varepsilon \cdot \varepsilon_B)(y) \leftrightarrow (\varepsilon_B(y) \land (Fh)(y))
\]

Since \( Fh \) is a coequalizer of \( FUf, FUg \), (A3) implies that

\[
(F(h)(y)) \leftrightarrow \bigwedge_{i \in UA} y_{Uf(i)} = y_{Ug(i)}
\]

Since we have supposed that \((\varepsilon_B)^C(y)\) and \(y \cdot Uf = y \cdot Ug\), we get by (2) and (3) that \((\varepsilon \cdot \varepsilon_B)^C(y)\) and hence \((FUe)^C(y)\) holds following (A2)(a). Further, \( Ue \) is a coequalizer of its kernel pair \( r, s : Z \to UB \). Thus \( FUe \) is a coequalizer of \( Fr, Fs \) and by (A3)

\[
(FUe)(y) \leftrightarrow \bigwedge_{i \in Z} y_{r(i)} = y_{s(i)}
\]

Hence \( y \cdot r = y \cdot s \) and there is a unique \( x : UD \to U_HC \) such that \( x \cdot Ue = y \). The following rectangle is a pushout because we have proved that the left square is a pushout and the right square is a pushout for \( Fk \)

![Fig. 8](image)

epi. By (A2)(b) \((\varepsilon_B)^C(x \cdot Ue) \leftrightarrow (\varepsilon_B)^C(x)\). Hence \( y = t(x) \).

We have proved that \( \mathcal{C} \) preserves limits. Since \( \mathcal{A}^{op} \) is complete, well-powered and \( F1 \) is its cogenerator, \( \mathcal{C} \) is representable by the Freyd's theorem (see [4], p. 126).
Denote by $N(C)$ a representing object and by $\zeta : \mathcal{A}(-, N(C)) \to \mathcal{C}$ a representing isomorphism. We will prove that $\mathcal{C} \cong MN(C)$. Namely, we will show that the following mapping carries the isomorphism of models $MN(C) \to \mathcal{C}$.

$$\alpha : UN(C) \xrightarrow{\varphi^{-1}} \mathcal{A}(F1, N(C)) \xrightarrow{\zeta} C(F1) \xrightarrow{(U_HC)^{n_1}} U_HC$$

Clearly $\epsilon_{F_n}$ is a coequalizer of $1_{FUf_n}, F\eta_{n} \cdot \epsilon_{F_n}$ for any set $n$. By (A3)

(4)

$$\epsilon_{F_n}(x) \mapsto \bigwedge_{i \in Uf_n} x_i = \varphi^{-1}(i) \cdot \eta_n$$

for $(F\eta_{n} \cdot \epsilon_{F_n})_i(x) = (F\eta_{n} \cdot \varphi^{-1}(i))_i(x) = \varphi^{-1}(i) \cdot \eta_n)$. Hence $(U_HC)^{n_1} : C(F1) \to U_HC$ is bijective and therefore $\alpha$ is bijective. Let $f : F1 \to F_n$ be an $n$-ary function symbol. We denote by $f^D$ the interpretation of $f$ in a model $D$ of $H$. The diagram

commutes by the definition of $f^{N(C)}$, the naturality of $\zeta$ and by (4) because $f^C(c, \eta_n) = \epsilon_{Uf \cdot \eta}(c)$ for any $c : UF_n \to U_HC \in C(Fn)$. Hence $\alpha$ preserves $f$ because for any $x : n \to UN(C)$ and $i \in n$ it holds $\alpha^n(x)(i) = \alpha(x \cdot i) = \zeta(\varphi^{-1}(x \cdot i)) \cdot \eta_1 = \zeta(\varphi^{-1}(x)) \cdot \eta_1 = \zeta(\varphi^{-1}(x)) \cdot UFi \cdot \eta_1 = (\zeta(\varphi^{-1}(x)) \cdot \eta_n)(i)$.

Let $p : F_n \to X$ be an $n$-ary relation symbol and consider $\alpha : n \to UN(C)$. Let $p^{N(C)}(a)$ hold. Then there is $g : X \to N(C)$ such that $\varphi(g \cdot p) = a$. Further, $\alpha^n(a) = \zeta(\varphi^{-1}(a)) \cdot \eta_n = \zeta(g \cdot p) \cdot \eta_n = \zeta(g) \cdot Up \cdot \eta_n$. Since $\epsilon_X \cdot F(Up \cdot \eta_n) = p$, following (A2) $\zeta_X(x) = p(x \cdot Up \cdot \eta_n)$. Since $(\epsilon_X)^C(\zeta(g))$, we have $p^C(\alpha^n(a))$.

Let the both squares in the following diagram be pushouts
Then the outer rectangle is a pushout and since the top row is equal to $1_{F_n}$, one gets that $v = p$. Hence $(p(x, \eta_n) \land \varepsilon_{F_n}(x)) \iff v(x) \land \varepsilon_{F_n}(x) \iff (p \cdot \varepsilon_{F_n}) (x) \rightarrow (FU_p)(x)$.

Let $p^{\xi}(x^{\eta_n}(a))$. Then $p^{\xi}(\zeta(\varphi^{-1}(a)) \cdot \eta_n)$ and $\varepsilon_{F_n}^{\xi}(\zeta(\varphi^{-1}(a)))$. Therefore $(FU_p)^{\xi}(\zeta(\varphi^{-1}(a)))$. In the same way as in the proof that $C$ preserves equalizers it can be shown that there is $b : UX \rightarrow UH_C$ such that $b \cdot Up = \zeta(\varphi^{-1}(a))$. Now,

![Fig. 11](image)

is a pushout because $u \cdot p \cdot \varepsilon_{F_n} = v \cdot FU_p$ implies $u \cdot \varepsilon_X \cdot FU_p = u \cdot p \cdot \varepsilon_{F_n} = v \cdot FU_p$. Hence $\varepsilon_X(x) \iff (p \cdot \varepsilon_{F_n}) (x \cdot Up)$. All these facts together yield $\varepsilon_X^{\xi}(b)$. Finally, $\varphi^{-1}(a) = \zeta^{-1}(b \cdot Up) = \zeta^{-1}(b)$. $p$ and $p^{N(C)}(a)$ is true.

We have proved that $\alpha$ carries an isomorphism and thus $M$ is an equivalence.

To compare the just proved Theorem with Prop. 1.4.1 of [2] we remark that $\mathcal{A}^{\text{op}}$ plays a role of Kean’s abstract Horn theory with $F_1$ as its $M$ and onto morphism as its monies. We gave a complete proof of the Theorem for the proof is only sketched in [2]. The associated Horn theory $H$ in our paper differs slightly from that one of the paper [2]. The reason for this change is the fact that the author was unable to succeed with the Kean’s original $H_t$.

Our Theorem shows that topological spaces are given by an infinitary Horn theory. It would be useful to find a convenient presentation of it.

REFERENCES


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