Jiří Močkoř

A note on approximation theorems

Archivum Mathematicum, Vol. 15 (1979), No. 2, 107--117

Persistent URL: http://dml.cz/dmlcz/107029

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XV: 107—118, 1979

A NOTE ON APPROXIMATION THEOREMS

JIŘÍ MOČKOŘ, Ostrava
(Received October 31, 1977)

1. INTRODUCTION

One of the most useful tool in valuation theory is the approximation theorem for independent valuations on the field. This theorem was subsequently generalized by Jaffard [5] and Fukawa [2], some generalization and the independent theorem were given by Ribenboim [13]. A lot of interesting results concerning approximation theorems were given by Griffin [4]. On the other hand, Müller [10], Jaffard [6] and Nakano [12] extended approximation theorems for lattice ordered groups and their results are in close relation with approximation theorems for valuations.

T. Nakano [11] introduced ring-like system called "d-*group*" which includes lattice ordered groups and rings and he showed that many theorems generalizing theorems in both systems there should be proved. Hence it is natural to find some approximation theorem in this sýstem. In this note we show the existence of several types of approximation theorems for d-groups and it has been shown that d-groups have similar properties with respect to these approximation theorems to those of ordinary integral domains and abelian lattice ordered groups. It should be noted that some of the proofs are adaptions of well-known proofs of approximation theorems for valuations.

2. DEFINITIONS AND BASIC FACTS

In this note all rings and groups are commutative integral domains and abelian groups,

At first, we repeat some basic facts about d-groups (see [11]).

A d-*group* is a partially ordered group $(G, .)$ with an element $\infty \notin G$ which admits a multivalued addition $\oplus$, i.e. to every ordered pair of elements $(a, b) \in (G \cup \{\infty\}) \times \times (G \cup \{\infty\})$ is assigned a no-void subset $a \oplus b$ of $G \cup \{\infty\}$ such that

(1) $a \oplus b = b \oplus a$,
(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,
(3) $a \in b \oplus c$ implies $b \in a \oplus c$,
(4) $a . (b \oplus c) = a . b \oplus a . c$,
(5) $\infty \in a \oplus b$ if and only if $a = b$,
(6) $a, b \geqq c$ and $x \in a \oplus b$ imply $x \geqq c$,

for any $a, b, c \in G$.

An *m-ring* is a commutative semigroup $(M, .)$ with identity and an element $\infty \notin M$ that admits a multivalued addition and satisfies (1)–(5). In this note all m-rings are required to obey the cancellation law. Let A be an m-ring, $U(A)$ its group of units. Then all the quotients ab^{-1} with $a, b \in A$, form a group $Q(A)$. It is easy to see that the factor group $G(A) = Q(A)/U(A)$ is partially ordered and becomes a d-group. $G(A)$ is called a *d-group relative to A*.

A subset J of an m-ring A is called a m-*ideal* of A provided that $a \oplus b \subseteq J$, $ar \in J$, for every $a, b \in J$, $r \in A$, and it is called a *prime* m-*ideal* if $ab \in J$ implies $a \in J$ or $b \in J$ for $a, b \in A$.

Let $(A, \oplus)$ be a m-ring and let J be a m-ideal of A. For two elements $a, b \in A$ we define

$$a \equiv b (\text{mod } J) \qquad \text{iff} \qquad (a \oplus b) \cap J \neq \emptyset.$$

By [11]; § 4, this relation is an equivalence relation on A. All the cosets $a \oplus J$, $a \in A$, form a *factor* m-*ring* A/J with respect to the multivalued addition

$$(a \oplus J) \oplus' (b \oplus J) = \{c \oplus J : c \in a \oplus b\}$$

and multiplication

$$(a \oplus J) . (b \oplus J) = a . b \oplus J$$

and infinity element J as a coset.

Let G be a d-group, $G_+ = \{g \in G : g \geqq 1_G = 1\}$. A subgroup H of G is called d-*convex* if it is convex and $H . G_+ \oplus H . G_+ = H . G_+$. Any directed convex subgroup of G is d-convex ([11]; Lemma 5) and for any d-convex subgroup H of G it is easy to see that the factor group G/H plus the infinity element $\infty = \infty H$ becomes a d-group (called *factor* d-*group*) with respect to the multivalued addition

$$aH \oplus' bH = (aH \oplus bH)/H$$

and the factor ordering.

A d-group $(G, \oplus)$ is called *local* provided that the multivalued addition $\oplus$ is *exact*, i.e. for every $a, b \in G$, $a > b$, $a \oplus b = \{b\}$ holds. A d-convex subgroup H of G is called *prime* if the factor d-group G/H is local. For a d-group G we denote by $\mathfrak{M}(G)$ the set of all directed prime d-convex subgroups of G. By [8]; Lemma 2.1., there is a bijection ψ between $\mathfrak{M}(G)$ and the set of all prime m-ideals of G_+ such that $H_1 \subseteq H_2 \Leftrightarrow \psi(H_1) \supseteq \psi(H_2)$.

Let $(G, \oplus_1)$, $(H, \oplus_2)$ be d-groups. A map f from G into H is called a *d-homomorphism* if it is an o-homomorphism of partially ordered groups (i.e. $f(G_+) \subseteq H_+$ and f is a group homomorphism) and $f(a \oplus_1 b) \subseteq f(a) \oplus_2 f(b)$ for every $a, b \in G$; f is called a *d-epimorphism* if it is an o-epimorphism (i.e. o-homomorphism and $f(G_+) = H_+$) and d-homomorphism, and it is called a *d-isomorphism* if it is an o-isomorphism (i.e. group isomorphism and $f(G_+) = H_+$) and $f(a \oplus_1 b) = f(a) \oplus_2 f(b)$ for every $a, b \in G$.

Further, a d-group G is called a *Prüfer d-group* if for any minimal (with respect to inclusion) $H \in \mathfrak{M}(G)$, G/H is totally ordered.

An important example of a d-group was given in [11]: Let $G = (G, ., \leqq)$ be a lattice ordered group (notation: l-group). Then it is possible to define a multivalued addition $\oplus_m$ on G in the following way:

$x \oplus_m y = \{z \in G : x \wedge y = z \wedge y = z \wedge x\}$, where $x \wedge y = \inf(x, y)$ in G. Then

$(G \cup \{\infty\}, ., \leqq, \oplus_m)$ is a d-group and every prime l-ideal of G is a prime d-convex subgroup.

3. APPROXIMATION THEOREMS FOR D-GROUPS

We begin this section with several simple lemmas.

3.1. *Let G be a totally ordered local d-group, G_1 totally ordered group such that there exists an o-homomorphism $f: G \to G_1$. Then f is a d-homomorphism with respect to the multivalued addition $\oplus_m$ on G_1.*

The proof will be omited.

3.2. *Let $(G_1, \oplus_1)$, $(G_2, \oplus_2)$ be totally ordered local d-groups and let $f: G_1 \to G_2$ be a d-epimorphism such that $\ker f \neq \{1\}$. Then $\ker f$ is a prime d-convex subgroup of G_1 and $G_1/\ker f$ is d-isomorphic with G_2.*

Proof. Since $\ker f$ is a convex directed subgroup of G_1, it is d-convex and $G_1/\ker f$ is o-isomorphic with G_2. Let $\bar{f}$ be the canonical o-isomorphism of $G_1/\ker f$ onto G_2. It is easy to see that $\bar{f}$ is a d-homomorphism. Let $\bar{f}(gH) \in \bar{f}(aH) \oplus_2 \bar{f}(bH)$, where $H = \ker f$. If $aH > bH$, then $b \in a \oplus_1 b$, $\bar{f}(aH) \oplus_2 \bar{f}(bH) = \{\bar{f}(bH)\}$, hence $gH = bH \in aH \oplus' bH$, where $\oplus'$ is the factor multivalued addition on G_1/H. If $aH = bH$, then $a = bh$ for some $h \in H$. Since $H \neq \{1\}$, there exists $h' \in H$, $h' < 1$, and we may find an element $h'' \in H$ with $g > ah' . h'' = b . h . h' . h''$. Thus, $g \in a . h' . h'' \oplus_1 b . h . h' . h''$ and we have $gH \in aH \oplus' bH$. Hence, $\bar{f}$ is a d-isomorphism. Now, since G_1/H is d-isomorphic with the local d-group G_2, it is local and H is prime.

It should be observed that "$\ker f \neq \{1\}$" cannot be removed from 3.2. In fact, let $(G, \oplus)$ be a totally ordered local d-group such that there exists an element $a \in G$

with $a \notin a \oplus a$. Then the identity map $i : (G, \oplus) \to (G, \oplus_m)$ is a d-epimorphism but not a d-isomorphism.

The following simple lemma shows that in totally ordered local d-group only two multivalued additions are possible.

3.3. *Let $(G, \oplus)$ be a totally ordered local* d-*group. Then either $\oplus = \oplus_m$ or $\oplus = \oplus'_m$, where for $a, b \in G$, $a \neq b$, $a \oplus'_m b = a \oplus'_m b$, $a \oplus_m a = a \oplus_m a - \{a\}$.*

Proof. Let $a, b, c \in G$ be such that $a \in b \oplus_m c$, $a \notin b \oplus c$. Since the multivalued addition $\oplus$ is exact, it follows $a = b = c$, $a \notin a \oplus a$. Then for every $x \in G$ we have

$$x = xa^{-1}a \notin xa^{-1}(a \oplus a) = x \oplus x.$$

Then for every $x \in y \oplus'_m z$ we have $x = \min(y, z) < \max(y, z)$. Hence, $x \in \min(y, z) \oplus \max(y, z)$ and $x \in y \oplus z$. It follows $\oplus = \oplus'_m$.

Now, for a d-group G we set

$$T(G) = \{(G', \varepsilon') : G' \text{ is a totally ordered local d-group}; \\ \varepsilon' : G \to G' \text{ is a d-epimorphism}\},$$

and we set $(G_1, \varepsilon_1) = (G_2, \varepsilon_2)$ if there exists a d-isomorphism σ from G_1 onto G_2 such that $\sigma . \varepsilon_1 = \varepsilon_2$. Further, we set $(G_1, \varepsilon_1) \leqq (G_2, \varepsilon_2)$ if there exists a d-epimorphism σ from G_2 onto G_1 such that $\sigma . \varepsilon_2 = \varepsilon_1$.

3.4. *The ordered set $(T(G), \leqq)$ is an* inf-*semilattice.*

Proof. At first, we need to show that $(T(G), \leqq)$ is an ordered set. We suppose that $(G_1, \varepsilon_1) \leqq (G_2, \varepsilon_2) \leqq (G_1, \varepsilon_1)$ and let τ_1, τ_2 be the d-epimorphism such that $\tau_1\varepsilon_1 = \varepsilon_2$, $\tau_2\varepsilon_2 = \varepsilon_1$. Then τ_i is an o-isomorphism. For every $y \in G_2$, $x \in G_1$, such that $\tau_1(x) = y$ we have

$$x \oplus_1 x = \tau_2\tau_1(x \oplus_1 x) \subseteqq \tau_2(\tau_1(x) \oplus_2 \tau_1(x)) \subseteq x \oplus_1 x,$$
$$\tau_2(y \oplus_2 y) = \tau_2(\tau_1(x) \oplus_1 \tau_1(x)) = x \oplus_1 x = \tau_2(y) \oplus_1 \tau_2(y).$$

Hence, τ_2 is a d-isomorphism and $(G_1, \varepsilon_1) = (G_2, \varepsilon_2)$.

Now, let $(G_1, \varepsilon_1), (G_2, \varepsilon_2) \in T(G)$, $H_i = \ker \varepsilon_i$, $G' = G/H_1H_2$ and let $\sigma_i : G_i \to G'$ be the canonical map. Henceforth we shall assume that for the ordered group G_i, $G_i = G/H_i$. On G' we define a multivalued addition $\oplus'$ in the following way.

$$\oplus' = \begin{cases} \oplus'_m, & \text{when for every } x, y, z \in G_1,\ a, b, c \in G_2 \text{ such that } \sigma_1(x) = \sigma_1(y) = \\ & = \sigma_1(z),\ \sigma_2(a) = \sigma_2(b) = \sigma_2(c), \text{ we have } x \notin y \oplus_1 z,\ a \notin b \oplus_2 c; \\ \oplus_m, & \text{otherwise.} \end{cases}$$

Then $(G', \oplus')$ is a d-group and it is easy to see that σ_i is a d-epimorphism. Then for the canonical map $\varepsilon' : G \to G'$ we have $\varepsilon' = \sigma_i\varepsilon_i$ and $(G', \varepsilon') \in T(G)$. Let $(K, \varrho) \in T(G)$ be such that $(K, \varrho) \leqq (G_1, \varepsilon_1), (G_2, \varepsilon_2)$ and let τ_i be a d-epimorphism such that $\varrho = \tau_i \varepsilon_i$. Then a map σ defined by $\sigma(gH_1H_2) = \varrho(g)$ is an o-epimorphism and $\sigma\varepsilon' = \varrho$. We need to show that σ is a d-homomorphism. There are two cases to be considered.

(1) $\oplus' = \oplus'_m$. If $\oplus_K = \oplus_m$, σ is a d-homomorphism by 3.1. Let $\oplus_K = \oplus'_m$ and suppose that σ is not a d-homomorphism, i.e. there exist $aH_1H_2 > bH_1H_2$ such that $\sigma(aH_1H_2) = \sigma(bH_1H_2)$. Then $\tau_1(aH_1) = \sigma\sigma_1(aH_1) = \sigma(aH_1H_2) = \sigma(bH_1H_2) = \sigma\sigma_1(bH_1) = \tau_1(bH_1)$, $aH_1 > bH_1$ and from the facts that $bH_1 \in aH_1 \oplus_1 bH_1$ and τ_1 is a d-homomorphism it follows $\tau_1(bH_1) \in \tau_1(aH_1) \oplus'_m \tau_1(bH_1)$, a contradiction.

(2) $\oplus' = \oplus_m$. We may suppose that there exist $aH_1, bH_1, cH_1 \in G_1$ such that $\sigma_1(aH_1) = \sigma_1(bH_1) = \sigma_1(cH_1)$, $aH_1 \in bH_1 \oplus_1 cH_1$. Since τ_1 is a d-homomorphism, we have $\tau_1(aH_1) \in \tau_1(bH_1) \oplus_K \tau_1(cH_1)$, $\tau_1(aH_1) = \tau_1(bH_1) = \tau_1(cH_1)$ and it follows $\oplus_K = \oplus_m$. Thus, σ is a d-homomorphism by 3.1. Hence, $(K, \varrho) \leqq ((G', \oplus'), \varepsilon')$ and $(G', \varepsilon') = (G_1, \varepsilon_1) \wedge (G_2, \varepsilon_2)$ in $T(G)$.

Now, let G be a d-group, $(g_1, \ldots, g_n) \in G^n$. We say that $(g_1, \ldots, g_n)$ is *compatible* with respect to $((G_1, \varepsilon_1), \ldots, (G_n, \varepsilon_n)) \in T(G)^n$ if for every $1 \leqq i, j \leqq n$, $(G_{ij}, \varepsilon_{ij}) = (G_i, \varepsilon_i) \wedge (G_j, \varepsilon_j)$, we have

$$\varepsilon_{ij}(g_i) = \varepsilon_{ij}(g_j).$$

In what follows we denote by σ_{ij} the d-epimorphism such that $\varepsilon_{ij} = \sigma_{ij}\varepsilon_i$. Further, for $T' \subseteq T(G)$ we set

$$G(T') = \{g \in G : \varepsilon'(g) \geqq 1 \text{ for every } (G', \varepsilon') \in T'\}.$$

Then the following proposition is a simple modification of [4]; Proposition 5.

3.5. *Let G be a* d-*group and let $T' \subseteq T(G)$. Then the following conditions are equivalent.*

(1) *For any $N = ((G_1, \varepsilon_1), \ldots, (G_n, \varepsilon_n)) \in T'^n$ and every $(g_1, \ldots, g_n) \in G^n$ compatible with respect to N and such that $\varepsilon_i(g_i) \geqq 1$, $i = 1, \ldots, n$, there exists $g \in G(T')$ such that $\varepsilon_i(g) = \varepsilon_i(g_i)$, $i = 1, \ldots, n$.*

(2) *For every $(G_i, \varepsilon_i), (G_j, \varepsilon_j) \in T'$, $a \in G_j$, such that $\sigma_{ji}(a) = 1$ there exists $b \in G(T')$ such that $\varepsilon_i(b) = 1$, $\varepsilon_j(b) \geqq a$.*

Proof. (1) $\Rightarrow$ (2). Let $(G_i, \varepsilon_i), (G_j, \varepsilon_j) \in T'$, $a \in G_j$, $\sigma_{ji}(a) = 1$. Let $b' \in G$ be such that $\varepsilon_j(b') = a$. Then $\sigma_{ji}\varepsilon_j(b') = \varepsilon_{ij}(b') = \sigma_{ji}\varepsilon_i(b') = 1$ and $(1, b')$ is compatible with respect to $((G_i, \varepsilon_i), (G_j, \varepsilon_j))$. Then there exists $b \in G(T')$ with $\varepsilon_i(b) = \varepsilon_i(1) = 1$, $\varepsilon_j(b) = \varepsilon_j(b') = a$.

(2) $\Rightarrow$ (1). The proof is by induction on n. Let $(g_1, \ldots, g_n) \in G^n$ be compatible with respect to N, $\varepsilon_i(g_i) \geqq 1$. Then we may suppose that $(G_k, \varepsilon_k) \nleqq (G_j, \varepsilon_j)$ for $k \neq j$. Further, if we suppose that there exists j, $2 \leqq j \leqq n$, with $\ker \sigma_{j1} = \{1\}$, from the fact that $(g_1, \ldots, g_j, \ldots, g_n)$ is compatible with respect to $((G_1, \varepsilon_1), \ldots, (G_{1j}, \varepsilon'), \ldots, (G_n, \varepsilon_n))$, where $(G_{1j}, \varepsilon') = (G_j, \varepsilon_1) \wedge (G_j, \varepsilon_j)$, and by the induction it follows

that there exists $g \in G(T')$ such that $\varepsilon_t(g) = \varepsilon_t(g_t)$, $1 \leqq t \leqq n$, $t \neq j$. Since $(G_{1j}, \varepsilon') \leqq \leqq (G_j, \varepsilon_j)$ we have $\varepsilon'(g) = \varepsilon'(g_j)$. Then $\sigma_{j1}\varepsilon_j(g) = \varepsilon'(g) = \varepsilon'(g_j) = \sigma_{j1}\varepsilon_j(g_j)$, and $\varepsilon_j(g) = \varepsilon_j(g_j)$.

Thus, we may suppose that for every $j \geqq 2$ there exists $d_j \in G_j$, $d_j > 1$, $\sigma_{j1}(d_j) = 1$. Then it is possible to show for every i, $1 \leqq i \leqq n$, the existence of $a_i \in G(T')$ with

$$\varepsilon_i(a_i) = \varepsilon_i(g_i),$$
$$\varepsilon_k(a_i) > \varepsilon_k(g_k), \qquad k \neq i,$$

(see the proof of [11]; Prop. 5). Let

$$g \in a_1 \oplus \ldots \oplus a_n.$$

Since ε_i is a d-homomorphism, we have $\varepsilon_i(g) \geqq \min(\varepsilon_i(a_k)) = \varepsilon_i(a_i) = \varepsilon_i(g_i)$. If $\varepsilon_i(g) > \varepsilon_i(g_i)$, we have $\varepsilon_i(g_i) \in (\bigoplus_{k \neq i} \varepsilon_i(a_k)) \oplus \varepsilon_i(g)$ and since G_i is a local d-group, we obtain $\varepsilon_i(g_i) \in \bigoplus_{k \neq i} \varepsilon_i(a_k)$ and $\varepsilon_i(g_i) \geqq \min_{k \neq i}(\varepsilon_i(a_k)) > \varepsilon_i(a_i) = \varepsilon_i(g_i)$, a contradiction.

3.6. *Let G be a* d-*group and let $T' \subseteq T(G)$ be such that for every $(G', \varepsilon') \in T'$,* ker ε' *is a directed subgroup of G. Then the equivalent conditions of* 3.5. *are satisfied.*

Proof. Let (G_i, ε_i), $(G_j, \varepsilon_j) \in T'$, $a \in G_j$, be such that $\sigma_{ji}(a) = 1$. By the proof of 3.4. we may suppose that for ordered group G_{ij}, $G_{ij} = G/\ker \varepsilon_i \,.\, \ker \varepsilon_j$ and σ_{ji} is the canonical map from $G/\ker \varepsilon_j$ onto G_{ij}. Hence, there exist $b, a_i, a_j \in G$ such that $a_i \in \ker \varepsilon_i$, $a_j \in \ker \varepsilon_j$, $b = a_i a_j$, $\varepsilon_j(b) = a$. Since $\ker \varepsilon_i$ is directed, there exists $c \in \ker \varepsilon_i$ such that $c \geqq 1$, $c \geqq a_i$. Then $\varepsilon_i(c) = 1$, $\varepsilon_j(c) \geqq \varepsilon_j(a_i) = \varepsilon_j(b \,.\, a_j^{-1}) = \varepsilon_j(b) = a$.

Corollary. *Let G be a l-group, $\{H_i : i \in J\}$ be a set of prime l-ideals of G such that $\bigcap \{H_i : i \in J\} = \{1\}$. Let for $i_1, \ldots, i_n \in J$, $(g_1, \ldots, g_n) \in G^n$ be such that $g_k H_{i_k} H_{i_t} = g_t H_{i_k} H_{i_t}$, $g_k H_{i_k} \geqq H_{i_k}$, $k, t = 1, \ldots, n$. Then there exists $g \in G_+$ such that $g H_{i_k} = g_k H_{i_k}$, $k = 1, \ldots, n$.*

It should be observed that this corollary may be proved using the theorem of Krull, Kaplansky, Jaffard and Ohm which states, that for any l-group G there exists a Bezout domain A with G as its group of divisibility. In fact, in this case for every $i \in J$, the composition w_i of canonical maps

$$K^* \xrightarrow{v} G \xrightarrow{f_i} G/H_i$$

is a valuation on the quotient field K of A and by [4]; Prop. 5, $\{w_i : i \in J\}$ is a defining family for A satisfying the weak approximation theorem. Then for $i_1, \ldots, i_n \in J$, $(g_1, \ldots, g_n) \in G^n$ such that $g_k H_{i_k} H_{i_t} = g_t H_{i_k} H_{i_t}$, the family $(f_{i_1}(g_1), \ldots, f_{i_n}(g_n)) \in \in (G/H_{i_1})_+ \times \ldots \times (G/H_{i_n})_+$ is compatible and there exists $a \in A$ such that $w_{i_t}(a) = f_{i_t}(g_t)$, $t = 1, \ldots, n$. Hence, $v(a) H_{i_t} = g_t H_{i_t}$, $v(a) \in G_+$.

3.7. *Let G be a* d*-group and let $(G_i, \varepsilon_i) \in T(G)$, $i = 1, \dots, n$, $a \in G$. Then there exists $b \in G$ such that*

$$\varepsilon_i(b) = 1 \quad \text{for } \varepsilon_i(a) \geqq 1,$$
$$\varepsilon_i(b) \leqq \varepsilon_i(a) \quad \text{for } \varepsilon_i(a) < 1.$$

Proof. (Cf. [3]; 18.7) Let $\varepsilon_i(a) \geqq 1$ for $1 \leqq i \leqq k$, $\varepsilon_i(a) < 1$ for $k + 1 \leqq i \leqq n$. For $n \in Z_+$ and $c \in G$ we set

$$nc = c \oplus \dots \oplus c \qquad (n \text{ times}),$$

and for $1 \leqq i \leqq k$ we set

$$P_i = \{c \in G : \varepsilon_i(c) > 1\}.$$

Now, if there exist $n_{i,1}, \dots, n_{i,t_i} \in Z_+$ such that for some $c_i \in G$ we have

$$c_i \in (1 \oplus n_{i,1}a \oplus \dots \oplus n_{i,t_i}a^{t_i} \oplus a^{t_i+1}) \cap P_i,$$

we set β_i the first set in the intersection. Otherwise, we set $\beta_i = \{1\}$. Further, let

$$b \in 1 \oplus a^2 c_1 \dots c_k,$$

where $c_j = 1$ for $\beta_j = \{1\}$. If $\beta_i = \{1\}$ for every i, then $b \in 1 \oplus a^2$. Hence for $\varepsilon_i(a) \geqq 1$ we have $\varepsilon_i(b) \geqq 1$ and since $1 \oplus a^2 \cap P_i = \emptyset$, we obtain $\varepsilon_i(b) = 1$. If $\varepsilon_i(a) < 1$, then from the fact $\varepsilon_i(b) \in 1 \oplus_i \varepsilon_i(a^2)$ it follows $\varepsilon_i(b) = \varepsilon_i(a^2) \leqq \varepsilon_i(a)$.

Let $\beta_i \neq \{1\}$ for $i = 1, \dots, p$, $1 \leqq p \leqq k$. We set

$$A = \bigoplus_{j = s_1 + \dots + s_p} (n_{1,s_1} \dots n_{p,s_p} a^j) =$$
$$= 1 \oplus n_1 a \oplus \dots \oplus n_{t-1} a^{t-1} \oplus a^t,$$

where $t = t_1 + \dots + t_p + p$, $n_{i,0} = n_{i,t_i+1} = 1$, $0 \leqq s_i \leqq t_i + 1$. Then

$$b \in 1 \oplus a^2 c_1 \dots c_p \subseteq 1 \oplus a^2 \beta_1 \dots \beta_p \subseteq 1 \oplus a^2 . A.$$

Let $1 \leqq i \leqq k$. Then $\varepsilon_i(a) \geqq 1$. If $\beta_i = \{1\}$, we have $1 \oplus a^2 A \cap P_i \neq \emptyset$ and $\varepsilon_i(b) = 1$. If $\beta_i \neq \{1\}$, we have $\varepsilon_i(a^2 c_1 \dots c_p) \geqq \varepsilon_i(c_i) > 1$ and it follows $\varepsilon_i(b) = 1$. Let $k + 1 \leqq$ $\leqq i \leqq n$, Since

$$b \in 1 \oplus a^2 \oplus n_1 a^3 \oplus \dots \oplus n_{t-1} a^{t+1} \oplus a^{t+2},$$

there exist $x_1 \in n_1 a^3, \dots, x_{t-1} \in n_{t-1} a^{t+1}$ such that

$$b \in 1 \oplus a^2 \oplus x_1 \oplus \dots \oplus x_{t-1} \oplus a^{t+2}.$$

Since $\varepsilon_i(x_j) \in n_j \varepsilon_i(a^{j+2})$, $j = 1, \dots, t - 1$, $\varepsilon_i(a^{j+2}) > \varepsilon_i(a^{t+2})$, we have

$$\varepsilon_i(x_j) > \varepsilon_i(a^{t+2}), \qquad j = 1, \dots, t - 1$$

and

$$\varepsilon_i(a^{t+2}) < \varepsilon_i(x_j),\ 1,\ \varepsilon_i(a^2), \qquad j = 1, \ldots, t-1.$$

Therefore $\varepsilon_i(b) = \varepsilon_i(a^{t+2}) < \varepsilon_i(a)$.

3.8. *Let G be a* d-*group*, $(G_i, \varepsilon_i) \in T(G)$, $i = 1, \ldots, n$, $g_i \in \varepsilon_i^{-1}(G_i^+)$. *Then the following conditions are equivalent.*

(1) *There exists* $g \in G$ *such that* $\varepsilon_i(g) = \varepsilon_i(g_i)$, $1 \leqq i \leqq n$.

(2) $(g_1, \ldots, g_n)$ *is compatible with respect to*

$$((G_1, \varepsilon_1), \ldots, (G_n, \varepsilon_n)).$$

Proof. (2) ⇒ (1). Let $G'_+ = \bigcap_{i=1}^{n} \varepsilon_i^{-1}(G_i^+)$ and let G' be a d-group relative to G'_+. Hence, $G' = \{gU : g \in U\}$, where $U = \{h \in G : \varepsilon_i(h) = 1, i = 1, \ldots, n\}$. For every i we define

$$\varepsilon'_i(gU) = \varepsilon_i(g), \qquad gU \in G'.$$

It is easy to see that $(G_i, \varepsilon'_i) \in T(G')$. Further, $\ker \varepsilon'_i$ is a directed subgroup of G'. For, by 3.7. for every $gU \in \ker \varepsilon'_i$ there exists $h \in G$ such that $\varepsilon_j(h) = 1$ for $\varepsilon_j(g) \geqq 1$ and $\varepsilon_k(h) \leqq \varepsilon_k(g)$ for $\varepsilon_k(g) < 1$. Then $hU \in \ker \varepsilon'_i$ and $hU \leqq U$, gU in G'. We set

$$T' = \{(G_1, \varepsilon'_1), \ldots, (G_n, \varepsilon'_n)\} \subseteq T(G').$$

Let $(g_1, \ldots, g_n) \in G^n$ be compatible with respect to $((G_1, \varepsilon_1), \ldots, (G_n, \varepsilon_n))$ and let for $1 \leqq i, j \leqq n$, $i \neq j$, $(G_{ij}, \varepsilon) = (G_i, \varepsilon_i) \wedge (G_j, \varepsilon_j)$ in $T(G)$. Then for the canonical d-epimorphism ε' from G' onto G_{ij} we have

$$(G_{ij}, \varepsilon') = (G_i, \varepsilon'_i) \wedge (G_j, \varepsilon'_j)$$

in $T(G')$. Since $\varepsilon(g_i) = \varepsilon(g_j)$, we obtain $\varepsilon'(g_iU) = \varepsilon'(g_jU)$. We obtain the proposition by 3.5., 3.6.

(1) ⇒ (2). Trivial.

3.9. *Let G be a* d-*group*, $H, H_1, \ldots, H_n \in \mathfrak{M}(G)$, $H_1^+ \cap \ldots \cap H_n^+ \subseteq H^+$. *Then there exists* i, $1 \leqq i \leqq n$, *such that* $H_i \subseteq H$.

Proof. The proof is by induction on n. Let $n = 2$, $H_1^+ \cap H_2^+ \subseteq H^+$ and suppose that $H_i^+ \nsubseteq H^+$. Hence there exist $x \in H_1^+ - H$, $y \in H_2^+ - H$. Since H is prime, we have by [11]; Lemma 6, $x \oplus y \cap H = \emptyset$ and $x \oplus y \cap (H_1^+ \cap H_2^+) = \emptyset$, Let $z \in x \oplus y$. Then $x \in z \oplus y \cap H_1$, $y \notin H_1$ and we have $z \in H_1^+$. Analogously, $z \in H_2^+$ and $z \in x \oplus y \cap (H_1^+ \cap H_2^+)$, a contradiction. Let $n \geqq 3$. If for some i, $1 \leqq i \leqq n$, $\bigcap_{j \neq i} (H^+ \cup H_j^+) \subseteq H^+ \cup H_i^+$, then $\bigcap_{j \neq i} H_j^+ \subseteq H^+$ and the induction hypothesis implies that $H_k \subseteq H$ for some k. Now, suppose that for every i, $1 \leqq i \leqq n$, there exists $z_i \in \bigcap_{j \neq i} (H^+ \cup H_j^+) - (H^+ \cup H_i^+)$. Let $z \in z_1 \oplus z_2 \ldots z_n$. Since H is convex and prime, we obtain $z \notin H$. Hence, there exists i such that $z \notin H_i$. Since H_i is convex and prime, it is easy to see that we obtain a contradiction, and the conclusion of 3.9. follows by the case previously considered and by induction.

3.10. *Let G be a* d*-group, $(G_i, \varepsilon_i) \in T(G)$, $i = 1, \ldots, n$, and let G' be a* d*-group relative to $\bigcap \{\varepsilon_i^{-1}(G_i^+): i = 1, \ldots, n\}$. Then G' is a Prüfer* d*-group.*

Proof. Let H_i be the quotient subgroup of a semigroup $G'_+ - \{gU \in G'_+ : \varepsilon_i(g) > 1\}$. By [8]; Lemma 2.1, $H_i \in \mathfrak{M}(G')$ for $i = 1, \ldots, n$. Then G'/H_i is o-isomorphic with G_i. In fact, in the proof of 3.8. it has been shown that ε'_i is an o-epimorphism from G' onto G_i and $\ker \varepsilon'_i$ is a directed subgroup of G'. Then we have

$$(\ker \varepsilon'_i)_+ = \ker \varepsilon'_i \cap G'_+ = H_i \cap G'_+ = H_i^+$$

and $\ker \varepsilon'_i = H_i$. Let $H \in \mathfrak{M}(G)$ be a minimal element of $\mathfrak{M}(G')$. Then $\{U\} = \bigcap_{i=1}^{n} H_i^+ \subseteq H$ and by 3.9. we have $H_i = H$ for some i. Hence, G/H is totally ordered and G' is Prüfer.

Problem. Let G be a l-group, then G is a d-group with respect to the addition $\oplus_m$. Let $(G_i, \varepsilon_i) \in T(G)$, $1 \leqq i \leqq n$, and let G' be the same as in 3.10. It is easy to see that G' is a l-group. The problem we want presented here is the following: Does for the multivalued addition $\oplus$ in G', $\oplus = \oplus_m$ hold? It is easy to see that this is equivalent to the following:

For every prime l-ideals $H_1, \ldots, H_n$ of G, $x, y, z \in G$, $a, b \in \bigcap_{i=1}^{n} H_i$ such that

$$x \wedge y = a(x \wedge y) = b(y \wedge z),$$

there exist $c, d \in \bigcap_{i=1}^{n} H_i$ such that

$$cx \wedge dy = z \wedge cx = z \wedge dy.$$

3.11. *Let B be a m-ring, $J_1, \ldots, J_n$ m-ideals of B such that for every i, j, $i \neq j$, $J_i \oplus J_j = \{z \in B : \exists\, a \in J_i,\ b \in J_j \text{ with } z \in a \oplus b\} = B$ holds. Then the canonical map*

$$B \to B/J_1 \times \ldots \times B/J_n$$

is a surjection.

The proof is a straightforward modification of [1]; Ch. 2, § 1, Prop. 5, and will be omited.

Let (G_1, ε_1), $(G_2, \varepsilon_2) \in T(G)$. We say that these elements are *independent* provided that $(G', \varepsilon') = (G_1, \varepsilon_1) \wedge (G_2, \varepsilon_2)$ is a trivial pair, i.e. $G' - \{\infty\}$ is a trivial group.

3.12. *Let G be a d-group, $(G_1, \varepsilon_1), \ldots, (G_n, \varepsilon_n) \in T(G)$ be such that they are pairwise independent and let $g_1, \ldots, g_n, b_1, \ldots, b_n \in G$. Then there exist $x, y_1, \ldots, y_n \in G$ such* *that*

$$y_i \in x \oplus b_i,$$
$$\varepsilon_i(y_i) \geqq \varepsilon_i(g_i), \qquad i = 1, \ldots, n.$$

Proof. (Cf. [1]; Ch. 6, § 7, Th. 1) Let G' be a d-group relative to $\bigcap_{i=1}^{n} \varepsilon_i^{-1}(G_i^+)$. There is no loss of generality in assuming $b_i U \in G'_+$ and $\varepsilon_i(g_i) > 1$. We set

$$J'_i = \{gU \in G'_+ : \varepsilon_i(g) \geqq \varepsilon_i(g_i)\}.$$

It is easy to see that J'_i is a m-ideal of G'_+. We admit that for some $i, j, i \neq j, J'_i \oplus' J'_j \neq G'_+$ holds, where $\oplus'$ is the addition on G'. Then there exists a maximal m-ideal P of G'_+ such that $J'_i \oplus' J'_j \subseteq P$. Then $\psi^{-1}(P)$ is a minimal prime d-convex subgroup of G' and by the proof of 3.10., there is an index k, $1 \leqq k \leqq n$, such that $\psi^{-1}(P) = H_k$. Let for every i, $1 \leqq i \leqq n$, T_i be the quotient subgroup of a semigroup

$$\{gU \in G'_+ : \varepsilon_i(g^n) < \varepsilon_i(g_i) \text{ for any } n \in Z_+\}.$$

Then by [8]; Lemma 2.1., $T_i \in \mathfrak{M}(G')$. If we suppose that there are $H_{i_1}, H_{i_2} \in \{H_i\}$ such that $H_{i_1}, H_{i_2} \subseteq T_i$, then the canonical map $\varepsilon' : G' \to G'/T_i$ is a d-epimorphism. Since $g_i U \notin T_i$, we obtain that G'/T_i is not trivial. In this case we have $(G_{i_1}, \varepsilon_{i_1})$, $(G_{i_2}, \varepsilon_{i_2}) \geqq (G'/T_i, \varphi_{i_1}\varepsilon_{i_1})$, where φ_{i_1} is the canonical d-epimorphism from G_{i_1} onto G'/T_i, a contradiction. Hence,

$$H_i \subseteq T_i, \qquad H_j \nsubseteq T_i, \qquad j \neq i.$$

Let $gU \in H_k^+$. If $gU \notin T_i^+$, there exists $n \in Z_+$ such that $\varepsilon_i(g^n) \geqq \varepsilon_i(g_i)$, $g^n U \in J'_i \subseteq P$ and since P is prime, $gU \in P$, a contradiction. Thus, $H_k \subseteq T_i$ and analogously we can see that $H_k \subseteq T_j$, a contradiction. Therefore, $J'_i \oplus' J'_j = G'_+$. Then it is easy to see that $J_i = \{g \in G : gU \in J'_i\}$ is a m-ideal of $B = \{g \in G : gU \in G'_+\}$ and $J_i \oplus J_k = B$, $i \neq k$. By 3.11. there exist $x, y_1, \ldots, y_n \in G$ such that $y_i \in x \oplus b_i \cap J_i$ which completes the proof.

Corollary. *Let G be a l-group, $H_1, \ldots, H_n$ prime l-ideals of G such that $H_iH_j = G$, $i \neq j$, and let $g_1, \ldots, g_n, b_1, \ldots, b_n \in G$. Then there exist $x, y_1, \ldots, y_n \in G$ such that*

$$x \wedge y_i = y_i \wedge b_i = x \wedge b_i,$$
$$y_i H_i \geqq g_i H_i, \qquad i = 1, \ldots, n.$$

Again, it should be observed that this corollary may be proved using the notion of a group of divisibility. In fact, let A be a Bezout domain such that its group of divisibility is G and let w_i be the composition of canonical maps as it was mentioned before. Then w_i are pairwise independent valuations on the quotient field K of A. Let $\beta_1, \ldots, \beta_n \in K$ be such that $v(\beta_i) = b_i$, where v is the canonical map from K^*

onto G. By the approximation theorem for independent valuations there exists $\alpha \in K$ such that

$$w_i(\alpha - \beta_i) \geqq g_i H_i, \qquad i = 1, \ldots, n.$$

We set $x = v(\alpha)$, $y_i = v(\alpha - \beta_i)$. Since

$$(\alpha, \alpha - \beta_i) = (\alpha - \beta_i, \beta_i) = (\alpha, \beta_i),$$

where (α, β) is an ideal of A generated by α, β, we have

$$x \wedge y_i = y_i \wedge b_i = x \wedge b_i$$

and $y_i H_i = w_i(\alpha - \beta_i) \geqq g_i H_i$.

REFERENCES

[1] N. Bourbaki, *Algèbre commutative*, Hermann, Paris, 1961.

[2] M. Fukawa, *On the theory of valuations*, J. Fac. Sci. Univ. Tokyo, Sec. I, 12 (1965), 57—79.

[3] R. Gilmer, *Multiplicative Ideal Theory*, Queens' Papers on Pure and Applied Mathematics, Kingston, 1968.

[4] M. Griffin, *Rings of Krull type*, J. Reine Angew. Math. 229 (1968), 1—27.

[5] P. Jaffard, *Théorie arithmètique des anneaux du type de Dedekind*, I, Bul. Soc. Math. France 80 (1952), 61—100.

[6] P. Jaffard, *Solution d'un probléme de Krull*, Bull. Soc. Math. 2[e] série, 85 (1961), 127—135.

[7] J. Močkoř, *Realization of d-groups*, Czech. Math. J. 27 (102) (1977), 296—312.

[8] J. Močkoř, *Prüfer d-groups*, to appear in Czech. Math. J.

[9] J. Močkoř, *A realization of groups of divisibility*, to appear in Com. Math. Univ. St. Pauli, Tokyo.

[10] D. Müller, *Verbandsgruppen und Durchschnitte endlich vieler Bewertungsringe*, Math. Z. 77 (1961), 45—62.

[11] T. Nakano, *Rings and partly ordered systems*, Math. Z. 99 (1967), 355—376.

[12] T. Nakano, *A theorem on lattice ordered groups and its applications to the valuation theory*, Math. Z. 83 (1964), 140—146.

[13] P. Ribenboim, *Le thèoréme d'approximation pour les valuations de Krull*, Math. Z. 68 (1957), 1—18.

J. Močkoř
708 33 Ostrava 4, tř. Vítězného února
Czechoslovakia