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ON BILINEAR STRUCTURES ON DIFFERENTIABLE MANIFOLDS

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In this paper we consider the bilinear structure (M, ω) determined by an arbitrary bilinear form ω on a differentiable manifold M. We prolong this structure on the bilinear structure $(TM, \mathrm{d}\Omega)$ and study relations of $(TM, \mathrm{d}\Omega)$ to (M, ω). Our considerations are in the category C^∞.

1. Definition 1. Let M be a differentiable manifold, $n = \dim M$. Let ω be an arbitrary bilinear form on M. The couple (M, ω) will be called a bilinear structure.

Let (M, ω) be a bilinear structure. Let $X \in T_m M$. Denote by i_X the contraction of the tensor ω $(i_X\omega \in T_m^*M,\ i_X\omega(Y) = \omega(X, Y))$ and by $\bar{\omega}$ the linear morphism $TM \to T^*M$, $\bar{\omega}(X) = i_X\omega$.

Let us recall that there is a bijection $\varkappa$ of the set of all morphisms $f : TM \to T^*M$ to the set of all semi-basic Pfaff forms on TM. Let $\varkappa(f) = \varphi$. Then

$$\varphi(X) = \langle \pi_* X, fp(X) \rangle,$$

where $\pi : TM \to M$, $p : TTM \to TM$ are fibre projections.

In our case denote by Ω the semi-basic Pfaff form $\varkappa(\bar{\omega})$. Let d be the symbol of the exterior differentiation. Then $(TM, \mathrm{d}\Omega)$ is a bilinear structure which will be called the prolongation of (M, ω).

Let (x^i), or (x^i, y^i), or (x^i, z_i), be a local chart on M, or TM, or T^*M respectively. Let $\omega = a_{ij}(x^k)\, \mathrm{d}x^i \otimes \mathrm{d}x^j$. Then

$$\text{(1)} \qquad \bar{\omega} : \begin{cases} x^i = x^i, \\ z_j = a_{ij} y^i, \end{cases}$$

$$\Omega = a_{ij} y^i \mathrm{d}x^j,$$

$$\mathrm{d}\Omega = \frac{\partial a_{ij}}{\partial x^k} y^i\, \mathrm{d}x^k \wedge \mathrm{d}x^j + a_{ij}\, \mathrm{d}y^i \wedge \mathrm{d}x^j,$$

$$\overline{\mathrm{d}\Omega}\ ;\ Y \to \left[\left(\frac{\partial a_{ij}}{\partial x^k} - \frac{\partial a_{ik}}{\partial x^j} \right) a^k y^i + a_{ij} b^i \right] \mathrm{d}x^j - a_{ij} a^j\, \mathrm{d}y^i,$$

where $Y = a^i \dfrac{\partial}{\partial x^i} + b^i \dfrac{\partial}{\partial y^i} \in T(TM)$.

Remark 1. In the case of a symmetric form ω we have

$$\Omega = 1/2 d_v T,$$

where $T = \omega(X, X)$ is a function on TM determined by ω and d_v is vertical antidifferentiation on TM (see [2], p. 165).

Remark 2. A semibasic Pfaff form Ω on TM will be said to be $\mathscr{L}$-form if $\varkappa^{-1}(\Omega)$: $TM \to T^*M$ is a linear morphism. It is easy to see that there is a bijection $\bar{\varkappa}$ of the set of all $\mathscr{L}$-forms on TM to the set of all bilinear forms on M.

Denote by K_h the canonical identification $T_m M \equiv T_h(T_m M)$. Let X be a vector field on M. Let X_m mean the value of X at $m \in M$. Let $\tilde{X}_h = K_h(X_m)$. Then $\tilde{X} : h \mapsto \tilde{X}_h$ is a vector field in TM.

Proposition 1. *Let (M, ω) be a bilinear structure on M. Let $(TM, \mathrm{d}\Omega)$ be the prolongation of (M, ω). Let X be a vector field on M. Then*

$$\pi^*(i_X\omega) = i_{\tilde{X}}\, \mathrm{d}\Omega.$$

Proof. $X = a^i\partial/\partial x^i$, $\tilde{X} = a^i\partial/\partial y^i$, $i_X\omega = (a_{ij}a^i)\, \mathrm{d}x^j$, $i_{\tilde{X}}\, \mathrm{d}\Omega = (a_{ij}a^i)\, \mathrm{d}x^j$. This gives our assertion.

A tangent vector $X \in T_m M$, or a vector field X on M, is said to be associated at $m \in M$, or associated with (M, ω) respectively if $i_X\omega = 0$.

Corollary of Proposition 1. *A vector field X on M is associated with (M, ω) if and only if the field $\tilde{X}$ is associated with $(TM, \mathrm{d}\Omega)$. If a vertical tangent vector $Y \in T_h T_m M$ is associated with $(TM, \mathrm{d}\Omega)$ at h, then $K_h(Y)$ is associated with (M, ω) at $m \in M$.*

Let $X, Y \in T_m M$. The linear morphism $TM \xrightarrow{\bar{\omega}'} T^*M$ determined by $\bar{\omega}'(Y)(X) = \omega(X, Y)$ is called transposed to $\bar{\omega}$. Let Ω' be the semi-basic form on TM determined by $\bar{\omega}'$. The semi-bilinear structure $(M, d(\Omega'))$ is called τ-prolongation of (M, ω). Let us remark that if ω is symmetric, or antisymmetric, then $\bar{\omega}' = \bar{\omega}$, or $\bar{\omega}' = -\bar{\omega}$ respectively, and thus $\overline{\mathrm{d}(\Omega')} = -(\overline{\mathrm{d}\Omega})'$, or $\overline{\mathrm{d}(\Omega')} = (\overline{\mathrm{d}\Omega})'$ respectively. A tangent vector $X \in T_m M$ is said to be τ-associated with (M, ω) at $m \in M$ if $\bar{\omega}'(X) = 0 = \bar{\omega}(X)$. In the case of a symmetric, or antisymmetric form ω, any tangent vector associated with (M, ω) at $m \in M$ is τ-associated. There is such a nonsymmetric and nonantisymmetric form that there is a tangent vector associated and τ-associated with (M, ω).

A tangent vector $Y \in T_h TM$ is called v-conjugate, or v'-conjugate with (M, ω) at $h \in TM$ if $i_Y\, \mathrm{d}\Omega$ or $i_Y\, \mathrm{d}(\Omega')$ respectively is a semi-basic form on TM.

Proposition 2. *Let $Y \in T_h(TM)$, $\pi h = m \in M$. Then Y is v'-conjugate with (M, ω) at h if and only if $\pi_X Y$ is associated with (M, ω) at m.*

Proof. Let $Y = a^i\partial/\partial x^i + b^i\partial/\partial y^i$. Then

$$i_Y\, \mathrm{d}(\Omega') = c_j\, \mathrm{d}x^j - a_{ji}a^j\, \mathrm{d}y^i, \tag{2}$$

where c_j depends on (a^i), (b^i) and $h = (x^i, y^i)$. Comparing (2) with (1_1) we get our assertion.

Corollary. *A projectable vector field Y on TM is v'-conjugate with (M, ω) if and only if $\pi^* Y$ is associated with (M, ω).*

Let X be a vector field on M. Denote by X^1, or X^{*1}, the prolongation of X on TM, or T^*M respectively.

Proposition 3. *Let Y be a projectable vector field on TM which is v-conjugate with (M, ω). Then Y is associated with $(TM, \mathrm{d}\Omega)$ at $h \in TM$ if and only if*

$$\bar{\omega}_* Y_h = (\pi_* Y)^{*1}_{\bar{\omega}(h)}. \tag{3}$$

Proof. Let $a^i \partial/\partial x^i + b^i \partial/\partial y^i$ be v-conjugate with (M, ω). Then $a_{ij} a^j = 0$ and thus

$$a^j \frac{a_{ij}}{\partial x^k} + a_{ij} \frac{\partial a^j}{\partial x^k} = 0. \tag{4}$$

Since $\pi^* Y = a^i \partial/\partial x^i$ we have

$$(\pi_* Y)^{*1} = a^i \,\partial/\partial x^i - \frac{\partial a^i}{\partial x^j} z_i \,\partial/\partial z_j,$$

see [2], p. 134. Then

$$(\pi_* Y)^{*1}_{\bar{\omega}(h)} = a^i \,\partial/\partial x^i - \frac{\partial a^i}{\partial x^j} a_{ki} y^k \,\partial/\partial z_j.$$

Now the condition (3) has the following local form

$$\frac{\partial a_{ij}}{\partial x^k} a^k y^i - a_{ij} b^i = -\frac{\partial a^k}{\partial x^j} a_{ik} y^i. \tag{5}$$

The vector field Y (being v-conjugate with (M, ω)) is associated with $(TM, \mathrm{d}\Omega)$ if and only if

$$\frac{\partial a_{ij}}{\partial x^k} a^k y^i - \frac{\partial a_{ik}}{\partial x^j} a^k y^i + a_{ij} b^i = 0, \qquad \text{i.e.}$$

if and only if (5) (use the relations (4)) is true.

Proposition 4. *Let X be a vector field on M. Let X^1, or X^{*1}, be the prolongation of X on TM, or T^*M respectively. Then $\bar{\omega}_*(X^1_h) = X_{\bar{\omega}(h)}$ for every $h \in TM$ if and only if $L_X \omega = 0$, where L_X denotes the Lie differentiation by X.*

Proof. Let $X = a^i \partial/\partial x^i$, $\omega = a_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j$. Then

$$L_X \omega = \left(\frac{\partial a_{ij}}{\partial x^k} a^k + a_{kj} \frac{\partial a^k}{\partial x^i} + a_{ik} \frac{\partial a^k}{\partial x^j} \right) \mathrm{d}x^i \otimes \mathrm{d}x^j,$$

$$X^1 = a^i \,\partial/\partial x^i + \frac{\partial a^i}{\partial x^j} y^j \,\partial/\partial y^i,$$

$$\bar{\omega}_*(X^1_h) = a^i\,\partial/\partial x^i + \left(\frac{\partial a_{ij}}{\partial x^k}\,a^k + a_{kj}\frac{\partial a^k}{\partial x^i}\right)y^i\,\partial/\partial z^j,$$

$$X^{*1}_{\bar{\omega}(h)} = a^i\,\partial/\partial x^i - \frac{\partial a^k}{\partial x^j}\,a_{ik}y^i\,\partial/\partial z_j.$$

Comparing $L_X\omega$ with $\bar{\omega}_*(X^1_h) = X^{*1}_{\bar{\omega}(h)}$ we complete our proof.

Corollary. *Let X be a vector field on M. Let X be τ-associated with (M, ω). Then X^1 is associated with $(TM, \mathrm{d}\Omega)$ if and only if $L_X\omega = 0$.*

Lemma 1. *Let X be a vector field associated and τ-associated with (M, ω). Let f be an arbitrary real function on M. Then $L_{fX}\omega = fL_X\omega$.*

Proof. Let $X = a^i\partial/\partial x^i$, $a_{ij}a^i = 0$, $a_{ij}a^j = 0$. Then

$$L_X\omega = \left(\frac{\partial a_{ij}}{\partial x^k} - \frac{\partial a_{kj}}{\partial x^i} - \frac{\partial a_{ik}}{\partial x^j}\right)a^k\,\mathrm{d}x^i \otimes \mathrm{d}x^j.$$

Let X be a vector field on M. Denote by g_1, or g_2, the function $\Omega(X^1)$, or $\mathrm{d}\Omega(\tilde{X}, X^1)$ respectively.

Proposition 5. (i) *The form $\mathrm{d}g_1$ is a semibasic form on TM if and only if the field X is τ-associated with $M, \omega)$.*

(ii) $$g_2 = \pi^*(\omega(X, X)).$$

Proof. Let $X = a^i\partial/\partial x^i$. Then $g_1 = a_{ij}y^ia^j$ and thus $\mathrm{d}g_1 = D_i\,\mathrm{d}x^i + a_{ij}a^j\,\mathrm{d}y^i$. It gives (i).

(ii) We get directly $\mathrm{d}\Omega(\tilde{X}, X^1) = a_{ij}a^ia^j = \pi^*(\omega(X, X))$.

Proposition 6. *Let (M, ω) be a bilinear structure. Let X be a vector field on M. Then*

$$\bar{\varkappa}(L_X\omega) = L_{X^1}\bar{\varkappa}(\omega).$$

Proof. Let $a^i\partial/\partial x^i = X$. Then

$$L_{X^1}(\bar{\varkappa}(\omega)) = \left(\frac{\partial a_{ij}}{\partial x^k}\,a^k + a_{kj}\frac{\partial a^k}{\partial x^i} + a_{ik}\frac{\partial a^k}{\partial x^j}\right)y^i\,\mathrm{d}x^j = \bar{\varkappa}(L_X\omega).$$

Corollary. *The form $\bar{\varkappa}(\omega)$ is invariant by X^1 if and only if the form ω is invariant by X.*

Let X be a vector field on M and ε be an arbitrary p-form on M. Let us recall that $L_X = \mathrm{d}i_X + i_Xd$. Therefore

(6) $$d(L_X\varepsilon) = \mathrm{d}i_X\,\mathrm{d}\varepsilon.$$

Definition 2. *Let X be a vector field on M. Let (M, ω) be a bilinear structure. Then X will be said to be the dynamic system of (M, ω) if the form $i_X\omega$ is closed.*

Let $X = a^i \partial/\partial x^i$, $\omega = a_{ij}\, dx^i \otimes dx^j$. By the direct evaluation we get

$$(7) \qquad d(i_{X^1} d\Omega) = A_{ij}\, dx^i \wedge dx^j + \left(\frac{\partial a_{ij}}{\partial x^k} a^k + a_{kj} \frac{\partial a^k}{\partial x^i} + a_{ik} \frac{\partial a^k}{\partial x^j}\right) dy^i \wedge dx^j.$$

where A_{ij} are functions (local) on TM. The relation 7 immediately yields that the form $d(i_{X^1} d\Omega)$ is semibasic if and only if $L_X \omega = 0$.

Proposition 7. *Let X be a vector field on M. Let (M, ω) be a bilinear structure. Then X^1 is a dynamic system of $(TM, d\Omega)$ if and only if ω is invariant by X.*

Proof. If $i_{X^1} d\Omega$ is closed then $d i_{X^1} d\Omega = 0$ is semibasic and thus $L_X \omega = 0$. Conversely, if $L_X \omega = 0$, then by Proposition 6 $L_{X^1} \Omega = 0$. Then $0 = dL_{X^1}\Omega = d i_{X^1} d\Omega$.

Corollary. *The form $d i_{X^1} d\Omega$ is semibasic if and only if it* is null, i.e. if $i_{X^1 n} d\Omega$ is closed. As $L_X d\Omega = d i_X d\Omega$, the form $d\Omega$ is invariant by X^1 if and only if ω is invariant by X.

Lemma 2. *Let ω be an 2-form on M. Let X be a vector field on M. If $i_X \omega$ is closed, then it is invariant by X.*

Proof is obvious because $L_X i_X \omega = i_X\, d i_X \omega$.

Proposition 8. *Let X be a vector field on M. Let (M, ω) be a bilinear structure where ω is a closed 2-form. Then X^1 is a dynamic system of $(TM, d\Omega)$ if and only if X is a dynamic system of (M, ω).*

Proof. By Proposition 7 $i_{X^1} d\Omega$ is closed if and only if $L_X \omega = 0$. In the case of a closed form $L_X \omega = d i_X \omega$.

Proposition 9. *Let X be a vector field on M. Let ω be a closed 2-form on M.* Then $\varpi_*(X^1_h) = X^{*1}_{\varpi(h)}$ *for any $h \in M$ if and only if $i_X \omega$ is closed.*

Proof. Since $L_X \omega = d i_X \omega$. Proposition 4 completes our proof.

Further, let us suppose that the form ω determining the bilinear structure (M, ω) is a form of a constant rank, i.e.

Proposition 10. *The distribution* Ker ϖ *is integrable if and only if every subfield Y of* Ker ϖ *is associated with $(M, L_X \omega)$, where X is arbitrary subfield of* Ker ϖ.

Proof. Let X, Y be vector fields associated with (M, ω). Then $i_{[XY]}\omega = L_X i_Y \omega - i_Y L_X \omega = -i_Y L_X \omega$. It gives our assertion.

Lemma 3. *Let ω be an 2-form. Then the distribution* Ker ϖ *is integrable if and only if $i_Y i_X\, d\omega = 0$ for any vector subfields X, Y of* Ker ϖ.

It is true because $i_Y L_X \omega = i_Y(i_X\, d\omega + d i_X \omega) = i_Y i_X\, d\omega$.

Corollary. *If ω is a closed 2-form, then the distribution* Ker ϖ *is integrable. Hence the distribution* Ker $\overline{d\Omega}$ *is integrable.*

It is obvious that dim Ker $d\overline{\Omega} \geqq$ dim Ker ϖ. The relations ($^1_4$) directly yield that the distribution Ker $d\overline{\Omega}$ is null if and only if Ker ϖ is null. Let us recall that the symplectic structure is a bilinear structure (M, ω), where dim $M = 2n$, ω is a closed 2-form and the distribution Ker ϖ is null. Let (M, ω) be a bilinear structure. Then $(TM, d\Omega)$ is a symplectic structure if and only if the distribution Ker ϖ is null.

2. Examples. a. Let (M, ω) be a quasi-Riemannian space, i.e. ω be a symmetric and regular form of the second order on M.

Lemma 4. *Let Γ be a linear connection on TM. Let ∇ be the covariant derivation determined by Γ. Let X, Y be vector fields on M and ω be an arbitrary form on M. Then*

$$\nabla_Y i_X \omega = i_{\nabla_Y X}\omega + i_X \nabla_Y \omega \tag{8}$$

the mapping $m \mapsto$ Ker ϖ_m is a distribution on M.

Proof. $\nabla_Y(X \otimes \omega) = \nabla_Y X \otimes \omega + X \otimes \nabla_Y \omega$,

$$C^1_1(\nabla_Y(X \otimes \omega)) = C^1_1(\nabla_Y X \otimes \omega) + C^1_1(X \otimes \nabla_Y \omega),$$

where C^1_1 denotes the contraction of $Z \otimes \omega$. As $C^1_1 \nabla_Y = \nabla_Y C^1_1$, the relation (8) is true.

Let us recall that every quasi-Riemannian structure (M, ω) determines on TM the unique linear connection (the quasi-Riemannian connection), the covariant derivation of which satisfies

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{9}$$

$$\nabla_Y \omega = 0 \qquad \text{for any } Z. \tag{10}$$

Locally, let $\omega = a_{ij}\, dx^i \otimes dx_j$, $a_{ij} = a_{ji}$ and let

$$\nabla_m Y = \left(\frac{\partial b^i}{\partial x^j} a^j + \Gamma^i_{jk} a^j b^k\right) \partial/\partial x^i, \qquad \text{see [3],} \tag{11}$$

where $Y = b^i\, \partial/\partial x^i$, $X = a^i\, \partial/\partial x^i$. Then ∇ is quasi-Riemannian if and only if

$$\Gamma^i_{jk} = \Gamma^i_{jk},$$

$$\frac{\partial a_{ij}}{\partial x^k} = a_{sj}\Gamma^s_{ki} + a_{is}\Gamma^s_{kj}.$$

The local rule

$$(x^i, y^i) \mapsto (x^i, y^i, y^i_j = -\Gamma^i_{jk}(x)\, y^k), \tag{12}$$

for the distribution $T: TM \to J^1TM$ of the horizontal tangent subspaces follows directly from (11). Every distribution $T: TM \to J^1TM$ determines on TM the differential equation P of the second order which is only in the case of linear connection a spray on TM. In our case, (12) yields

$$P = y^i \,\partial/\partial x^i - \Gamma^i_{jk} y^j y^k \,\partial/\partial y^i.$$

Sternberg, [4], proves that the spray P in the case of a Riemannian connection is the geodesic spray (Euler vector field) of the Lagrange function $T = 1/2 a_{ij} y^i y_j$. One can easy observe that it is also true in the case of a quasi-Riemannian connection. It immediately gives

Assertion. *Let (M, ω) be a quasi-Riemannian structure. Then the spray P of the quasi-Riemannian connection on TM determined by (M, ω) is a dynamic system of the symplectic structure $(TM, \mathrm{d}\Omega)$.*

Let X be a vector field on M. Denote by $\bar{X}$ the Γ-lift of X in the case of a quasi-Riemannian connection Γ. By (12)

$$\bar{X} = a^i \,\partial/\partial x^i - \Gamma^i_{jk} a^j y^k \,\partial/\partial y^i,$$

for $X = a^i \,\partial/\partial x^i$. Using (9′) and (10′) we obtain by direct evaluation

$$L_{\bar{X}}\, \mathrm{d}\Omega = B_{k_j}\, \mathrm{d}x^k \wedge \mathrm{d}x^j + a_{is}\left(\Gamma^s_{kj} a^k + \frac{\partial a^s}{\partial x^j}\right) \mathrm{d}y^i \wedge \mathrm{d}x^j, \tag{13}$$

where $B^k{}_j$ are some local function on TM. (13) immediately yields: If $L_{\bar{X}}\,\mathrm{d}\Omega$ is semibasic at $h_0 \in T_m M$, then it is semibasic at every $h \in T_m M$.

Lemma 5. *The form $L_{\bar{X}}\,\mathrm{d}\Omega$ is semibasic at $h_0 \in T_m M$ if and only if $\nabla_Y(i_X \omega) = 0$ for every $Y \in T_m M$.*

Proof. In the case of the quasi-Riemannian structure (M, ω) the relation (8) gives

$$\nabla_Y(i_X \omega) = i_{\nabla_Y X}\omega.$$

But $ie_{_yX}\omega$ is null if and only if $\nabla_Y X = 0$. Since ω is regular, the comparison of (11) with (13) verifies our assertion.

Let Γ be a linear connection on TM. Let Γ' be transposed to Γ and ∇' be the covariant derivation determined by Γ'. In the paper [1] we have shown that

$$\nabla'_Y X = K_Y(X^1 - \bar{X})_Y,$$

where K_Y denotes the canonical identification $T_m M = T_Y(T_m M)$, $\pi Y = m$ and X^1 is the prolongation of X on TM. Let us recall that in the case of a quasi-Riemannian connection $\Gamma = \Gamma'$. Therefore, if Γ is quasi-Riemannian then $\nabla_Y X$ is null if and only if $X^1_Y = \bar{X}_Y$, $Y \in T_m M$. Hence the form $L_{\bar{X}}\,\mathrm{d}\Omega$ is semibasic at $h_0 \in T_m M$ if and only if $X^1_h = \bar{X}_h$ for every $h \in T_m M$. Then $L_{\bar{X}}\,\mathrm{d}\Omega$ is semibasic on TM if and only if $X^1 = \bar{X}$. But $L_{\bar{X}}\,\mathrm{d}\Omega = \mathrm{d}i_{\bar{X}}\,\mathrm{d}\Omega$ and by Corollary of Proposition 7 the form $\mathrm{d}i_{X^1}\,\mathrm{d}\Omega$ is semibasic if and only if is null. We summarize our result in theorem form

Proposition 11. *Let (M, ω) be a quasi-Riemannian structure. Let X be a vector field on M and $\bar{X}$ be its Γ-lift by the quasi-Riemannian connection Γ. Then $\bar{X}$ is a dynamic system of the symplectic structure $(TM, \mathrm{d}\Omega)$ if and only if $\bar{X} = X^1$.*

Corollary. *By Corollary of Proposition 7, the form $\mathrm{d}\Omega$ is invariant by X^1 if and only if the form ω is invariant by X. Hence if $\bar{X}$ is a dynamic system of the prolongation $(TM, \mathrm{d}\Omega)$ of a quasi-Riemannian structure, then $L_X \mathrm{d}\Omega = 0$.*

b. Let (M, ω) be a symplectic structure. Then its prolongation $(TM, \mathrm{d}\Omega)$ is also symplectic. Proposition 8 yields.

Proposition 12. *Let X be a vector field on M and X^1 be its prolongation on TM. Let (M, ω) be a symplectic structure. Then X^1 is a dynamic system of $(TM, \mathrm{d}\Omega)$ if and only if X is a dynamic system of (M, ω), i.e. if and only if ω is invariant by X.*

c. Let (M, α) be a contact structure, $\dim M = 2n + 1$, α is a Pfaff form on M. Then $(M, \mathrm{d}\alpha)$ is a bilinear structure. Let us recall that there is the unique tangent vector field Y on M (dynamic system of the contact structure (M, α)) for which $\alpha(Y) = 1$, $\mathrm{d}\alpha(Y) = 0$. Then Y is associated with $(M, \mathrm{d}\alpha)$. Locally (see for example [2]).

$$\alpha = \mathrm{d}x^1 + \sum_{i=2}^{2n} x^i \, \mathrm{d}x^{i+1}, \tag{14}$$

$$\omega = \mathrm{d}\alpha = \sum_{i=2}^{2b} \mathrm{d}x^i \wedge \mathrm{d}x^{i+1},$$

$$\Omega = \sum_{i=2}^{2n} y^i \, \mathrm{d}x^{i+1} - \sum_{i=2}^{2n} y^{i+1} \, \mathrm{d}x^i,$$

$$\mathrm{d}\Omega = \sum_{i=2}^{2n} \mathrm{d}y^i \wedge \mathrm{d}x^{i+1} - \sum_{i=2}^{2n} \mathrm{d}y^{i+1} \wedge \mathrm{d}x^i.$$

Hence $Y = \partial/\partial x^1$ is the dynamic system of (M, α). By Corollary of Proposition 1 the vector field $\bar{Y} = \partial/\partial y^1$ is associated with the bilinear structure $(TM, \mathrm{d}\Omega)$.

Lemma 6. *Let Y be the dynamic system of a contact structure (M, α). Then $\mathrm{d}\alpha$ is invariant by Y.*

Proof. $L_Y \mathrm{d}\alpha = i_Y d(\mathrm{d}\alpha) + \mathrm{d}i_Y \mathrm{d}\alpha = 0$.

Proposition 13. Let Y^1 be the prolongation of the dynamic system of a contact structure (M, α). Then Y^1 is associated with the prolongation of the bilinear structure $(M, \omega = \mathrm{d}\alpha)$.

Our assertion follows from (14_4).

Remark. Proposition 13 also follows from Lemma 6 and from Corollary of Proposition 4 because the dynamic system of (M, α) is associated and τ-associated with $(M, \mathrm{d}\alpha)$.

Proposition 14. *Let Y^1 be the prolongation of the dynamic system of (M, α). Then*

$$\bar{\omega}_* Y^1_h = Y^{*1}\, \bar{\omega}(h).$$

It follows from Lemma 6 and Proposition 4.

Remark. The relation (14_4) immediately yields that the distribution of the tangent subspaces Ker $\mathrm{d}\Omega$ is generated by vector fields Y^1 and $\tilde{Y}$.

3. Let ω be an arbitrary bilinear form on M. Let us recall that there is such a unique antisymetric form ω^- that

$$\omega = \omega^+ + \omega^-.$$

Denote by $(TM, \mathrm{d}\Omega^+)$ the prolongation of (M, ω^+).

Lemma 7. *Let (M, ω) be a bilinear structure. Then the symmetry of ω is a necessary condition for $(TM, \mathrm{d}\Omega)$ to have a dynamic system being a differential equation of the second order.*

Proof. Let $\omega = a_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j$. Let $Y = y^i\, \partial/\partial x^i + c^i(x_j, y^k)\, \partial/\partial y^i$ be a differential equation of the second order. Then our assertion follows from

$$L_Y\, \mathrm{d}\Omega = A_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j + B_{ij}\, \mathrm{d}y^i \wedge \mathrm{d}x^j + a_{ij}\, \mathrm{d}y^i \wedge \mathrm{d}y^j.$$

Corollary. Let (M, ω) be a bilinear structure. Let (M, ω^+) be a quasi-Riemannian structure. Then the spray P of (M, ω^+) is a dynamic system of (M, ω) if and only if (M, ω) is also a quasi-Riemannian structure.

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