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ON CONJUGACY FUNCTIONS OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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W. Leighton defined in [2] a conjugacy function $\delta(x)$ relative to an ordinary differential equation $y'' + p(x)\,y = 0$ as the distance from a point x to its first conjugate point, larger than x. He treated there also the problem of sufficient conditions under which $\delta(x)$ is increasing or decreasing, as well as convex or concave.

The present paper generalizes the above investigations in using the methods and results of the dispersion theory established by O. Borůvka [1].

Let us consider the second-order linear differential equation in the Jacobian form

$$y'' = q(t)\,y \tag{q}$$

with q being negative and of class C^2 on the interval $\mathbf{R} = (-\infty, \infty)$. We assume that solutions of (q) are oscillatory towards both $-\infty$ and ∞. The trivial solution will be excluded from our consideration.

Definition. Let $t \in \mathbf{R}$ be an arbitrary point; let u and v be solutions of (q) such that $u(t) = v'(t) = 0$. Let φ_1 and ψ_1 denote the first zero of u and v' to the right from the zero t, respectively. We call φ_1 and ψ_1 the fundamental central dispersions of the first and 2nd kinds, respectively.

The conjugacy function δ defined by W. Leighton in [2] can be expressed with the aid of the fundamental central dispersion of the 1st kind φ_1 by the formula $\delta(t) = \varphi_1(t) - t$. From now on this conjugacy function will be denoted as Δ_φ.

Definition. Let φ_1 and ψ_1 be the fundamental central dispersions of the 1st and 2nd kinds of (q), respectively. The conjugacy functions of the 1st and 2nd kinds of (q) will be called the functions Δ_φ and Δ_ψ defined by the formulae

$$\Delta_\varphi(t) = \varphi_1(t) - t$$
$$\Delta_\psi(t) = \psi_1(t) - t, \qquad t \in \mathbf{R},$$

respectively.

The comparison theorem ([3], p. 277) yields the following.

Lemma 1. *For*

$$(\mathrm{Q}) \qquad y'' = Q(t)\, y,$$
$$(\bar{\mathrm{Q}}) \qquad y'' = \bar{Q}(t)\, y,$$

let $Q(t) \geqq \bar{Q}(t)$, $Q(t) \not\equiv \bar{Q}(t)$ *in the interval* $(t_0, \varphi_1(t_0))$. *Then* $\bar{\varphi}_1(t_0) < \varphi_1(t_0)$, *where* φ_1 *and* $\bar{\varphi}_1$ *are fundamental central dispersions of the 1st kind relative to the equations* (Q) *and* $(\bar{Q})$, *respectively.*

In accordance with O. Borůvka (cf. [1], p. 8 and onwards) we introduce the differential equation (q_1) associated to (q) as the equation

$$(\mathrm{q}_1) \qquad y'' = q_1(t)\, y,$$

where $q_1(t) = q(t) + \sqrt{|q(t)|}\,(1/\sqrt{|q(t)|})''$, $t \in \mathbf{R}$.

For each solution y_1 of the differential equation (q_1) the function $y_1 \sqrt{|q(t)|}$ represents the derivative y' of precisely one solution y of (q).

The definition of fundamental central dispersions of the 1st and 2nd kinds yields.

Lemma 2. *The fundamental central dispersion of the 2nd kind related to a differential equation* (q) *is the fundamental central dispersion of the 1st kind relative to the differential equation* (q_1) *associated to* (q).

In conformity with [2] we put

$$h(t) := \sqrt{-q^{-1}(t)}, \qquad t \in \mathbf{R}.$$

Then $-q_1(t) = -q(t) - \sqrt{|q(t)|}\,(1/\sqrt{|q(t)|})'' = h^{-2}(t) - (|h(t)|)''/|h(t)| = h^{-2}(t) - h''(t)/h(t)$, since $|h(t)| = \operatorname{sgn} h(t) \,.\, h(t)$ and $(|h(t)|)'' = (\operatorname{sgn} h(t) \,.\, h(t))'' = \operatorname{sgn} h \,.\, h''(t)$.

Lemma 3. *Given a differential equation* (q) *with* $\varphi_1(t_0) = c$, $\psi_1(t_0) = f$. *Let* $h = \sqrt{-q^{-1}(t)}$. *The following implications hold for* $t \in \mathbf{R}$:

$$h(t) \,.\, h''(t) < 0 \Rightarrow f < c,$$
$$h(t) \,.\, h''(t) > 0 \Rightarrow f > c,$$
$$h(t) \,.\, h''(t) \equiv 0 \Rightarrow f = c.$$

Proof. If $hh'' < 0$, then $q_1 - q = h''/h < 0$ and consequently $q_1 < q$. On applying Lemmas 1 and 2 we get from here that $f < c$. If $hh'' > 0$, then $q_1 - q = h''/h > 0$ and therefore $q_1 > q$. Making use of Lemmas 1 and 2 we get from here that $f > c$. If $h'' \equiv 0$ on $\mathbf{R}$, then $q_1 - q = h''/h \equiv 0$ and therefore $q_1 \equiv q$ so that $f = c$.

Let us recall the Theorem of [1] p. 120: The derivatives of the central dispersions may be expressed in terms of a solution u of (q) and of its derivatives. In the case of the central dispersion of the 1st kind we can write

$$\varphi_1'(t) = \begin{cases} u^2[\varphi_1(t)]/u^2(t) & \text{for } u(t) \neq 0 \\ u'^2(t)/u'^2[\varphi_1(t)] & \text{for } u(t) = 0. \end{cases}$$

Theorem. *Let* $q(t) < 0$, $q_1(t) := q(t) + \sqrt{|q(t)|}\,(1/\sqrt{|q(t)|})'' < 0$ *and let* h, h_1 *be defined by the formulae* $h(t) := \sqrt{-q^{-1}(t)}$, $h_1(t) = \sqrt{-q_1(t)}$. *The following implications hold for* $t \in \mathbf{R}$*:*

(a_1) $h(t) \,.\, h''(t) < 0 \Rightarrow \Delta_\varphi(t)$ *is concave,*

(b_1) $h(t) \,.\, h''(t) > 0 \Rightarrow \Delta_\varphi(t)$ *is convex,*

(a_2) $h_1(t) \,.\, h_1''(t) < 0 \Rightarrow \Delta_\psi(t)$ *is concave,*

(b_2) $h_1(t) \,.\, h_1''(t) > 0 \Rightarrow \Delta_\psi(t)$ *is convex.*

Proof. The cases (a_1) and (b_1). Since $\Delta_\varphi(t) = \varphi_1(t) - t$, we have $\Delta_\varphi(t_0) = \varphi_1(t_0) - t_0 = c - t_0$. Then $\Delta_\varphi'(t) = \varphi_1'(t) - 1 = [u^2[\varphi_1(t)]/u^2(t)] - 1$ under the assumption that $u(t_0) \neq 0$. Further $\Delta_\varphi''(t) = \varphi_1''(t) = \{2u[\varphi_1(t)]\, u'[\varphi_1(t)]\, \varphi_1'(t)\, u^2(t) - u^2[\varphi_1(t)]\, 2u(t) \,.\, u'(t)\}/u^4(t) = \{2u^2[\varphi_1(t)]/u^2(t)\} \,.\, \{u[\varphi_1(t)]\, u'[\varphi_1(t)] - u(t) \,.\, u'(t)\}$ $u^2(t)$.

Let t_0 be an arbitrary point of $\mathbf{R}$ and let u be such a solution that $u'(t_0) = 0$. Then obviously $u(t_0) \neq 0$ and we have $\Delta_\varphi''(t_0) = \varphi_1''(t_0) = [2u^2(c)/u^2(t_0)] \,.\, [u(c)u'(c)/u^2(t_0)]$.

For $u(t_0) \gtrless 0$ we have $u[\varphi_1(t_0)] = u(c) \lessgtr 0$. If $f < c$, then $u'(c) \gtrless 0$ and if $f > c$, then $u'(c) \lessgtr 0$.

For $h \,.\, h'' < 0$ we have $f < c$ by Lemma 3. It follows that in this case $u(c) \,.\, u'(c) < 0$ and consequently $\Delta_\varphi''(t_0) < 0$. The function $\Delta_\varphi(t)$ is concave.

For $h \,.\, h'' > 0$ we have $f > c$. Therefore $u(c) \,.\, u'(c) > 0$ and also $\Delta_\varphi''(t_0) > 0$. The function $\Delta_\varphi(t)$ is convex.

The cases (a_2) and (b_2).

On applying the assertions (a_1), (b_1) to the differential equation (q_1) associated to (q), we obtain with respect to Lemma 2 the implications given in (a_2), (b_2).

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REFERENCES

[1] O. Borůvka: *Linear Differential Transformations of the Second Order.* The English University Press, London 1971.

[2] W. Leighton: *The Conjugacy Function.* Proceedings of the American Mathematical Society Vol. 24, No. 4, April, 1970, pp. 820—823.

[3] V. V. Stěpanov: *Kurs diferenciálních rovnic.* Přírodovědecké nakladatelství, Praha 1952.

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