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ON CONNECTED UNARS WITH REGULAR ENDOMORPHISM MONOIDS

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A monounary algebra, i.e. a pair \((A, f)\), where \(A\) is a non-void set and \(f\) a self-map of the set \(A\), is briefly called a unar. This paper aims to give some conditions of the topological and algebraic character equivalent to the regularity of the endomorphism monoid of a connected unar. There are used the descriptions of unars with regular and inverse endomorphism monoids obtained by L. A. Skornjakov in [12] and results of papers [4], [6]. In the below stated characterizations we consider mostly endomorphism monoids which are not groups. For the characterization of unars whose endomorphism monoids are automorphism groups see [12] Theorem 3.

Fundamental used notions concerning monounary algebras can be found e.g. in papers [5], [8], [11], [12]. Let \((A, f)\) be a connected unar. The set of all cyclic elements of \((A, f)\) (i.e. such elements \(a \in A\) that \(f^n(a) = a\) for some integer \(n \geq 1\)) will be denoted in regard with [8] by \(A^{\omega_2}\) and further \(A^{\omega_1} = \{x \in A \setminus A^{\omega_2}: \text{there is a sequence } \{x_i\}_{i \in \omega} \text{ such that } x_0 = x \text{ and } f(x_{i+1}) = x_i \text{ for each } i \in \omega\}, A_0 = \{x \in A: f^{-1}(x) = \emptyset\}. A \text{ unar is called a cycle if } A = A^{\omega_1}. \text{The upper cone of an element } a, \text{i.e. the set } \{f^n(a): n = 0, 1, 2, \ldots\} \text{ will be denoted by } [a]_f, \text{the lower cone } \{x \in A: f^n(x) = a \text{ for some } n \in \omega\} \text{ by } (a)_f. \text{We agree on denoting the cardinality of a set } A \text{ by } |A|. A \text{ connected unar } (A, f) \text{ with } |A| = \aleph_0 \text{ and } f \text{ — a permutation of } A \text{ is called a line. A connected unar } (A, f) \text{ is said to be a cycle with short tails or a line with short tails if it contains a cycle or a line } C \text{ such that } f(x) \in C \text{ for every } x \in A. \text{If } |B^{\omega_2}| \leq 1 \text{ for each component } (B, f_b) \text{ of a unar } (A, f) \text{ we put } a \preceq b \text{ for } a, b \in A \text{ if there exists } n \in \omega \text{ with } f^n(a) = b \text{ and } a \prec b \text{ if } a \preceq b, a \neq b. \text{Further, we denote by } (A, f) \text{ the factor-unar (i.e. the factor-algebra of a monounary algebra } (A, f)) \text{ corresponding to the congruence } \equiv_f \text{ on } (A, f) \text{ defined by } a \equiv_f b \text{ if } a = b \text{ or } a, b \in A^{\omega_2}. \text{The monoid of all endomorphisms of } (A, f) \text{ is denoted by } E(A, f). \text{For the definition of a regular and inverse semigroup see [3] §1.9. A certain strengthening of the notion of a regular semigroup is the notion of an anti-regular semigroup (cf. [10]) called in [17] an anti-inverse semigroup. Let us recall the necessary definitions (see [1] and [10]): A semigroup } S \text{ is said to be anti-inverse if for each element } a \in S \text{ there is an element } b \in S \text{ such that } aba = b \text{ and } bab = a. \text{The elements } a \text{ and } b \text{ are then called anti-inverses.
A saturated topological space called also quasi-discrete ([2] 26A) is a topological space $(A, \tau)$ with the completely additive topological closure operation $\tau$ i.e. each point of this space possesses the minimum neighbourhood (cf. [9]). A discrete space of Alexandrov is a saturated $T_0$-space. Compactness is meant in the sense of [2] 41A, i.e. quasi-compactness considered in [9]. A continuous closed self-map of a topological space $(A, T)$ will be called as usual a closed deformation of $(A, \tau)$ and the monoid of all closed deformations of this space will be denoted by $S(A, \tau)$.

We say that a topological space $(A, \tau)$ has the fixed set property or briefly the FS-property (the fixed point property, briefly the FP-property) with respect to closed deformations if there exists a non-void proper subset $X \subset A$ (a point $x \in A$) with $f(X) = X(f(x) = x)$ for each $f \in S(A, \tau)$.

In what follows $\subseteq$ means the usual set inclusion and $A \subset B$ means $A \subseteq B$ $A \neq B$.

**Theorem 1.** Let $(A, f)$ be a connected unar whose endomorphism monoid is not a group. Then $E(A, f)$ is regular if and only if there exists a discrete topology of Alexandrov $\tau$ on the set $A$ such that $E(A, f) = S(A, \tau)$ and the space $(A, \tau)$ has the FS-property with respect to closed deformations.

**Proof.** Let $(A, f)$ be a connected unar satisfying the assumption of the theorem. Since $(A, f)$ contains at most one cyclic element, by Theorem 3.3 [4] there exists a discrete topology of Alexandrov $\tau$ with $E(A, f) = S(A, \tau)$ if and only if the unar $(A, f)$ has one of the following forms:

(i) $f^2 = f$,

(ii) $A = A^{\omega_1} \cup A^0$, where either $A^0 = \emptyset$ or $(A^{\omega_1}, \leq_f)$ is a chain of the type $\omega^* \oplus \omega$ and $A^0 = \emptyset$ (i.e. $(A, f)$ is a line with short tails),

(iii) $A = A^0 \cup A_1$, where $(A_1, \leq_f)$ is a chain of the type $\omega$ with the first element $c$ and $f(a) = c$ for each $a \in A^0$.

Suppose $A = A^{\omega_1}$ and simultaneously $(A^{\omega_1}, f)$ is not a line. Admit there exists a non-void set $B \subset A$ with $g(B) = B$ for each $g \in E(A, f)$. Since $f^k \in E(A, f)$ for every $k \in \omega$, the ordered set $(B, \leq_f)$ does not contain any minimal and maximal element and $[b]_f \subseteq B$ for each $b \in B$. There exists a pair of elements $a, b \in A$ such that $a \in A \setminus B, b \in B$ and $f^n(a) = f^n(b)$ for some $n \in \omega$. Since elements $a, b$ form a pair of h-elements in the sense of [8] Definition 1.22 and xii [8] there exists $g \in E(A, f)$ such that $g(b) = a$. We get a contradiction, hence in the considered case for every non-void subset $B \subset A$ there exists an endomorphism $g$ of $(A, f)$ with $g(B) \neq B$. Consequently $(A^{\omega_1}, f)$ is a line in the considered case. Since the existence of a non-void subset $B \subseteq A$ with the property $g(B) = B$ for each $g \in E(A, f)$ implies the inclusion $B \subseteq A^{\omega_1} \cup A^0$, we have that the case (iii) is eliminated. On the other hand if $(A, f)$ is a connected unar with $f^2 = f$ and $|A| \geq 2$ or $(A, f)$ is a line with short tails then $A$ contains an $E(A, f)$-invariant non-void proper subset. (A singleton formed by the cyclic element in the first case and the carrier of the line in the second one). Therefore
there exists a discrete topology of Alexandrov $\tau$ on $A$ with $S(A, \tau) = E(A, f)$ and the space $(A, \tau)$ has the FS-property with respect to closed deformations if and only if $(A, f)$ is either a cycle with short tails or a line with short tails. Now, from Theorem 1 [12] there follows the assertion, q.e.d.

In the following proposition $LT(A)$ means the left zeros subsemigroup of the full transformation monoid $T(A)$ on the set $A$. Recall that a unar is said to be nested if the system of all its subunars ordered by means of set inclusion forms a chain.

**Proposition 1.** Let $(A, f)$ be a connected unar. The following conditions are equivalent:

1. $E(A, f)$ is regular and $LT(A) \cap E(A, f^k) \neq \emptyset$ for some $k \in \omega$.
2. There exists a compact saturated topology $\tau$ on $A$ with the property $E(A, f) = S(A, \tau)$.
3. There exists a saturated topology $\tau$ on $A$ with $E(A, f) = S(A, T)$ and the space $(A, T)$ has the FP-property with respect to closed deformations.

**Proof.** 1. $\Rightarrow$ 2. Since for some positive integer $k \in \omega$ there exists a constant self-map $g$ of $A$ with $g \in E(A, f^k)$ we have by [12] Theorem 1 $(A, f)$ is a cycle with short tails (or without tails). If we define a topology $\tau$ on the set $A$ by putting a $T$-cofinal subset of a subset $X \subseteq A$ as $\tau X = X \cup f(X)$, Condition 2 is satisfied.

2. $\Rightarrow$ 3. Let $\tau$ be a compact saturated topology on the set $A$ such that $E(A, f) = S(A, \tau)$. By [4] Theorem 3.3 the unar $(A, f)$ has one of the forms (i)-(iii) listed in the proof of Theorem 1. For each $a \in A$ there exists a nested subunar $(B, f_B)$ of $(A, f)$, an element $b \in B$ and a surjective homomorphism $g : (A, f) \to (B, f_B)$ such that $g(a) = b$ and the equality $f^m(a) = f^n(b)$ with integers $m, n$ minimal with respect to this property implies $m = n$. Since $f^n \in S(A, \tau)$ for each $n \in \omega$ we have that for each $a \in A$ the closure $\tau \{a\}$ is a right cofinal subset of $[a]_f$ and has the following property: If $x, y, z \in \tau \{a\}$, $x <_f y <_f z$ then from $f^n(x) = y$, $f^n(y) = z$ with minimal $m, n$ it follows either $n = m$ or $z = f(y)$. Then the least $\tau$-neighbourhood of $a$ (i.e. the closure of $\{a\}$ in the saturated topology dual to $\tau$) is a left cofinal subset of $[a]_f$. Since the space $(A, \tau)$ is compact by [9] Proposition 1 the unar $(A, f)$ contains a cyclic element, say $e$. Hence $f^2 = f$ and $g(e) = e$ for each $g \in S(A, \tau)$.

3. $\Rightarrow$ 1. Since $f \in S(A, \tau)$ and the unar $(A, f)$ is connected there exists exactly one element $e \in A$ with $f(e) = e$. By [4] Theorem 3.3 $f^2 = f$. Condition 1 follows easy with respect to [12] Theorem 1, q.e.d.

**Corollary.** Let $(A, f)$ be a connected unar. The following conditions are equivalent:

1. Each element of $E(A, f)$ has a unique anti-inverse element in $E(A, f)$.
2. The set $A$ is either a singleton or $(A, \tau)$ is the Sierpinski-space for each topology $\tau$ on $A$ with the property $S(A, \tau) = E(A, f)$.
3. There exists a discrete topology of Alexandrov $\tau$ on $A$ such that $(A, \tau)$ is a tower space and $E(A, f) = S(A, \tau)$.
4. There exists a saturated tower topology $\tau$ on $A$ such that the space $(A, \tau)$ has the FP-property with respect to closed deformations and $E(A, f) = S(A, \tau)$. 

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The other characterization is expressed in terms of the groupoid theory. Similarly to [5] § 1 we associate a groupoid with every connected unar \((A, f)\). For \(a, b \in A\), denote by \(m, n\) the smallest non-negative integers such that \(f^m(a) \in [b]_f, f^n(b) \in [a]_f\). We put \(\delta(a, b) = m - n\). Evidently \(\delta(a, b) + \delta(b, a) = 0\) for each pair \(a, b \in A\) and \(\delta(a, b) = \delta(b, a) = 0\) for each pair \(a, b \in A^\text{x2}\). Further we put \(ae_f b = f(b)\) if \(\delta(a, b) \geq 0\) and \(ae_f b = f(a)\) if \(\delta(a, b) < 0\). It is to be noted that the groupoid \((A, e_f)\) associated in this way with a unar \((A, f)\) is neither associative nor commutative in general. In papers [5], [6] the binary operation \(e_f\) is denoted by \(\nabla_f\).

The following statement is contained in [5] Proposition 1.2.

**Proposition 2.** Let \((A, f)\) be a connected unar such that either \(A^{\times 2} = \emptyset\) or \(f^2 = f\). Then \(E(A, f) = E(A, e_f)\).

**Proposition 3.** Let \((A, f)\) be a connected unar with the regular endomorphism monoid \(E(A, f)\). Then there exists a commutative binary operation \(\circ\) on the set \(A\) such that \(E(A, f) = E(A, \circ)\).

**Proof.** According to [12] Theorem 1 the unar \((A, f)\) has one of the following forms:

1. it is trivial (i.e. \(|A| = 1\)),
2. \(f^2 = f\),
3. \((A, f)\) is a line,
4. \((A, f)\) is a line with short tails.

If one of cases (3), (4) occurs, then \(A = A, f = f\). Putting for each pair \(a, b \in A\) \(a \circ b = ae_f b\), we get evidently a commutative groupoid \((A, \circ)\) satisfying the condition \(E(A, f) = E(A, \circ)\) with respect to Proposition 2, q.e.d.

By an ideal of a groupoid \((A, \cdot)\) we mean a both-side ideal, i.e. a non-empty subset \(J \subseteq A\) such that \(a \in J, b \in A\) implies \(a \cdot b, b \cdot a \in J\). The principal ideal generated by an element \(a\) is denoted by \(J(a)\). An ideal \(J\) is said to be trivial if \(|J| = 1\). If \((A, \cdot)\) is a groupoid and \(J\) an ideal of this groupoid then the corresponding Rees factor-groupoid is denoted by \((A/J, \cdot_J)\); cf. [3] and [7]. A groupoid \((A, \cdot)\) is called distributive if for each triad \(a, b, c \in A\) equalities \(a.(b.c) = (a.b) \cdot (a.c), (a.b) \cdot c = = (a.c) \cdot (b.c)\) hold and it is called a BD-groupoid (in accordance with [7]) if it satisfies one of the following equivalent conditions (see [7] Proposition 1.2):

(i) \((A, \cdot)\) is distributive and the set of all its idempotents contains just one element,
(ii) there is an element \(e \in A\) with \(a.e = e = e.a\) (a zero of \((A, \cdot)\)) and \(a.(b.c) = = e = (a.b).c\) for all \(a, b, c \in A\).

**Theorem 2.** Let \((A, f)\) be a connected unar such that the endomorphism monoid \(E(A, f)\) is not a group. \(E(A, f)\) is regular if and only if \((A, e_f)\) is a commutative groupoid containing the least proper ideal \(J\) with the following properties:
(i) The factor-groupoid \((A / J, e_J)\) is a BD-groupoid.

(ii) If \(J\) is principal or contains an idempotent then it is trivial.

Proof. Let \((A, f)\) be a connected unar with the regular endomorphism monoid \(E(A, f)\) being not a group. With respect to [12] Theorem 1 and the definition of the binary operation \(e_f\) we have that \((A, e_f)\) is a commutative groupoid. If \(f^2 = f\) we put \(J = \{e\}\), where \(e = A^{\omega^1}\). If \((A, f)\) is a line with short tails, i.e. \(A = A^{\omega^1} \cup A^0\) with \((A^{\omega^1}, f)\) a line—we put \(J = A^{\omega^1}\). It can be easily shown (see the third part of the proof of Theorem 3.8 [6]) that in this case \(J\) is the least proper ideal of the groupoid \((A, e_f)\). Since the Rees factor-groupoid \((A/J, e_J)\) of the groupoid \((A, e_f)\) is associated with a connected idempotent unar \((A/J, F)\) (which is a factor unar of \((A, f)\)) we have by [5] Lemma 1.3 that \((A/J, e_J)\) is a BD-groupoid. The ideal \(J\) is principal if it contains an idempotent of \((A, e_f)\), i.e. if \(J = \{e\}\), where \(e\) is the only cyclic element of \((A, f)\).

Suppose \((A, f)\) is a connected unar such that \(E(A, f)\) is not a group and such that \((A, e_f)\) is a commutative groupoid the least proper ideal \(J\) of which satisfies the above assumptions. From the commutativity of \(e_f\) it follows that for each pair \(a, b \in A\), the equality \(\delta(a, b) = 0\) implies \(f(a) = f(b)\). Since \(E(A, f)\) is not a group, \((A, f)\) is neither a cycle nor a line. If \(A^{\omega^1} \neq \emptyset\) then it can be easily verified (in the same way as in the proof of Theorem 3.8 [6] p. 150) that the least ideal of \((A, e_f)\) coincides with the least subunar of \((A, f)\) containing the set \(A^{\omega^1}(= A^{\omega^1})\). This ideal is non-trivial hence \((A, f)\) is a line with short tails. If \(A^{\omega^1} = \emptyset\) then there exists an element \(a \in A\) with \(\delta(a, x) \leq 0\) for each \(x \in A\). Then \(\{f^k(a) : k = 1, 2, \ldots\}\) is the least ideal of \((A, e_f)\) and since it is principal we have \(f^2 = f\). Hence \((A, f)\) is a cycle with short tails. Applying [12] Theorem 1 we get \(E(A, f)\) is regular, q.e.d.

In [12] Theorem 2 there are given necessary and sufficient conditions under which the endomorphism monoid of a unar is an inverse semigroup. In fact these conditions strengthen those which are necessary and sufficient for the regularity of \(E(A, f)\). In the case of a connected unar \(E(A, f)\) is an inverse semigroup if and only if \(|f^{-1}(a)| \leq 2\) for each \(a \in A\) and \((A, f)\) is either a cycle with short tails or a line with short tails (cf. [12] Theorem 2). From this result, using the binary operation \(e_f\), we get the below stated characterization analogical to Theorem 3.9 [6].

For each element \(a\) of a groupoid \((A, \cdot)\) we put \(\sqrt{a} = \{x \in A : x \cdot x = a\}\). Every element \(b \in \sqrt{a}\) is called a square root of the element \(a\) in the groupoid \((A, \cdot)\). If \(|\sqrt{a}| = 1\) we say the element \(a\) possesses the unique square root in \((A, \cdot)\). Especially, \(\sqrt{a} = f^{-1}(a)\) for each element \(a\) of the groupoid \((A, e_f)\) and thus evidently \(E(A, f)\) is a group (in the case of connected \((A, f)\)) if and only if each element of \((A, e_f)\) possesses the unique square root.

Proposition 4. Let \((A, f)\) be a connected unar. \(E(A, f)\) is an inverse semigroup if and only if either \(|\sqrt{a}| = 1\) holds for each element \(a\) of \((A, e_f)\) or \(|\sqrt{a}| \leq 2\) for
every \( a \in (A, e_f) \) and \((A, s_f)\) contains the least ideal \( J \) each element of which possesses the unique square root in \((J, e_f)\).

Proof. Let \((A, f)\) be a connected unar. For \( E(A, f) \) being a group the assertion is evident. Thus we assume \( E(A, f) \) is not a group. If \( E(A, f) \) is an inverse semigroup then by \([12]\) Theorem 2 for each \( a \in A \) we have \(|f^{-1}(a)| \leq 2\) and either \( A = A^0 \cup A^{\infty_1} \) (where \((A^{\infty_1}, f)\) is a line) or \( A = A^0 \cup A^{\infty_2} \). Putting \( J = A^{\infty_1} \) in the first case and \( J = A^{\infty_2} \) in the second one, we obtain the assertion with respect to \( A^0 \neq \emptyset \).

Assume the groupoid \((A, e_f)\) is satisfying conditions from the above proposition. Since each element of \( J \) possesses the unique square root in \((J, e_f)\) and \( \sqrt{a} = f^{-1}(a) \) for each \( a \in A \), the subunar \((J, f_J)\) is either a cycle or a line. Since \( J \) is the least ideal of \((A, s_f)\) we have \( A \setminus J = A^0 \). Thus \((A, f)\) is either a cycle with short tails or a line with short tails and \(|f^{-1}(a)| = |\sqrt{a}| \leq 2\) for each \( a \in A \). Consequently \( E(A, f) \) is an inverse semigroup, q.e.d.

The requirement of the anti-inversibility of \( E(A, f) \) enforced a very simple structure of the unar \((A, f)\).

**Proposition 5.** Let \((A, f)\) be a connected unar. \( E(A, f) \) is an anti-inverse semigroup if and only if \((A, f)\) is a cycle of the cardinality 1 or 2 with at most one short tail.

Proof. Suppose \((A, f)\) has one of the required form. If \( E(A, f) \) is non-trivial, then either \( E(A, f) = \{\text{id}_A, f\} \) or \( E(A, f) = \{\text{id}_A, f, f^2\} \). Since \( E(A, f) \) is commutative, by \([10]\) Theorem 4 (i) and (ii) it is anti-inverse. It is to be noted that as the multiplicative table for \( E(A, f) \setminus \{\text{id}_A\} = \{f, f^2\} \) can serve the table 3) from \([1]\) Example 2.1.

Let \((A, f)\) be a connected unar such that \( E(A, f) \) is an anti-inverse semigroup. Since \( E(A, f) \) is regular by \([10]\) Theorem 1 or \([1]\) Corollary 2.1 (i), we have in virtue of \([12]\) Theorem 1 and \([1]\) Theorem 2.1 \((A, f)\) is a cycle of the cardinality at most 4 (except 3) with at most short tails. Admit \(|A^0| \geq 2\). Assume \( a, b \in A^0, a \neq b \). Since there exists \( g \in E(A, f) \) such that \( g(a) = b, g(b) \in A^{\infty_2} \), we have \( g^3 \neq g \) thus in regard with \([1]\) Theorem 2.1 \( E(A, f) \) is not anti-inverse. Hence \((A, f)\) is a cycle with at most one short tail. Then \( E(A, f) \) is a commutative monoid. Admitting \(|A^{\infty_2}| = 4\), we have \( f^3 \neq f \), which is a contradiction in virtue of \([10]\) Theorem 4. Consequently \(|A^{\infty_2}| \leq 2\), q.e.d.

Remark. It is easy to verify that \( E(A, f) \) is anti-inverse for a connected unar \((A, f)\) with \(|A| > 1\) if and only if the groupoid \((A, e_f)\) has one of the following multiplicative table (or the other formed by a permutation of elements):

<table>
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