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HOW TO DRAW TOLERANCE LATTICES OF FINITE CHAINS

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Let $\mathfrak{A} = (A, F)$ be an algebra. By a *tolerance* on $\mathfrak{A}$ is meant a reflexive and symmetric binary relation on A compatible with operations from F, i.e. it is a subalgebra of $\mathfrak{A} \times \mathfrak{A}$. The set of all tolerances on $\mathfrak{A}$ forms a lattice $LT(\mathfrak{A})$ with respect to the set inclusion, [2]. This lattice, called *tolerance lattice,* was studied from the point of view of its algebraic properties in the case of lattices, semilattices, semigroups etc. As our knowledge, there is no explicite method describing a construction of $LT(\mathfrak{A})$ for a given algebra $\mathfrak{A}$. In this paper, there is done a first attempt to solve this problem in the case of *finite chains* as the simplest kind of lattices.

Denote by $L(n) = \{x = [x_0, \dots, x_{n-1}] \in \underline{1} \times \dots \times \underline{n} \mid x_{i+1} \leqq x_i + 1\}$, where $\underline{n} = \{0, 1, \dots, n-1\}$ is an n-element chain considered as a lattice with binary operations min and max. $LT(\underline{n})$ is the tolerance lattice of $\underline{n}$ (see [1], [2]).

Lemma. *$L(n)$ forms a sublattice of $\underline{1} \times \dots \times \underline{n}$.*

Proof. Let $x = [x_0, \dots, x_{n-1}]$, $y = [y_0, \dots, y_{n-1}]$ be two elements of $L(n)$. Then

$$x \vee y = [\max(x_0, y_0), \dots, \max(x_{n-1}, y_{n-1})] \in L(n),$$
$$x \wedge y = [\min(x_0, y_0), \dots, \min(x_{n-1}, y_{n-1})] \in L(n),$$

because

$$\max(x_{i+1}, y_{i+1}) \leqq \max(x_i + 1, y_i + 1) = \max(x_i, y_i) + 1,$$
$$\min(x_{i+1}, y_{i+1}) \leqq \min(x_i + 1, y_i + 1) = \min(x_i, y_i) + 1. \ \square$$

The aim of this paper is to prove the following

Theorem. *$LT(\underline{n})$ is isomorphic to $L(n)$.*

Proof. Order homomorphisms $x : LT(\underline{n}) \to L(n)$ and $T : L(n) \to LT(\underline{n})$ will be constructed and their bijectivity proven.

Let $T \in LT(\underline{n})$. Put $x_i(T) = i - \min\{j \in \underline{n} \mid [i, j] \in T\}$. Clearly $x_i(T) \leqq i$, $x_{i+1}(T) = i + 1 - \min\{j \in \underline{n} \mid [i + 1, j] \in T\} = i - \min\{j \in \underline{n} \mid [i + 1, j] \in T\} + 1 \leqq i - \min\{j \in \underline{n} \mid [i, j] \in T\} + 1 = x_i(T) + 1$. Consequently, $x(T) = [x_0(T), \ldots, x_{n-1}(T)] \in L(n)$. Moreover, for $T \subseteq S$ holds

$$x_i(T) = i - \min\{j \in \underline{n} \mid [i, j] \in T\} \leqq i - \min\{j \in \underline{n} \mid [i, j] \in S\} = x_i(S).$$

An order homomorphism $x : LT(\underline{n}) \to L(n)$ was constructed.

Now, let $x = [x_0, \ldots, x_{n-1}] \in L(n)$. Define a binary relation $T(x)$ on $\underline{n}$ as follows: $[i, j] \in T(x) :\Leftrightarrow x_{\max(i,j)} \geqq |i - j|$. $T(x)$ is clearly a tolerance relation on $\underline{n}$. Its compatibility will be shown. Let $[i, j]$, $[i', j'] \in T(x)$, i.e. $|i - j| \leqq x_{\max(i,j)}$ and $|i' - j'| \leqq x_{\max(i',j')}$. Relationships

$$|\min(i, i') - \min(j, j')| \leqq x_{\max(\min(i, i'), \min(j, j'))}$$

and

$$|\max(i, i') - \max(j, j')| \leqq x_{\max(\max(i, i'), \max(j, j'))}$$

are to be proven. At least one of the following four cases will arise:

1. $i \leqq i'$ and $j \leqq j'$,
2. $i \leqq i'$ and $j' \leqq j$,
3. $i' \leqq i$ and $j \leqq j'$,
4. $i' \leqq i$ and $j' \leqq j$.

The third case is equivalent to the second one and the fourth case is equivalent to the first one.

In the first case,

$$|\min(i, i') - \min(j, j')| = |i - j| \leqq x_{\max(i,j)} = x_{\max(\min(i, i'), \min(j, j'))},$$
$$|\max(i, i') - \max(j, j')| = |i' - j'| \leqq x_{\max(i',j')} = x_{\max(\max(i, i'), \max(j, j'))}.$$

In the second case, four subcases are to be distinguished:

2.1. $i \leqq j'$ and $j \leqq i'$,
2.2. $i \leqq j'$ and $i' \leqq j$,
2.3. $j' \leqq i$ and $j \leqq i'$,
2.4. $j' \leqq i$ and $i' \leqq j$.

In the first and the second subcases,

$$|\min(i, i') - \min(j, j')| = |i - j'| =$$
$$= |i - j| - (j - j') \leqq x_j - (j - j') \leqq x_{j'} = x_{\max(\min(i, i'), \min(j, j'))}.$$

In the third and the fourth subcases,

$$|\min(i, i') - \min(j, j')| = |i - j'| =$$
$$= |i' - j'| - (i' - i) \leqq x_{i'} - (i' - i) \leqq x_i = x_{\max(\min(i, i'), \min(j, j'))}.$$

In the first and the third subcases,

$$|\max(i,i') - \max(j,j')| = |i' - j| \leqq |i' - j'| \leqq x_{i'} = x_{\max(\max(i,i'),\max(j,j'))}.$$

In the second and the fourth subcases,

$$|\max(i,i') - \max(j,j')| = |i' - j| \leqq |i - j| \leqq x_j = x_{\max(\max(i,i'),\max(j,j'))}.$$

The compatibility of $T(x)$ was proven. Clearly $T : L(n) \to LT(\underline{n})$ is an order homomorphism, because $x \leqq y$ implies

$$[i,j] \in T(x) \Leftrightarrow |i - j| \leqq x_{\max(i,j)} \Rightarrow |i - j| \leqq y_{\max(i,j)} \Leftrightarrow [i,j] \in T(y).$$

Bijectivity will be shown by verifying $x \,.\, T = id$, $T \,.\, x = id$

$$x_i(T(x)) = i - \min\{j \in \underline{n} \mid [i,j] \in T(x)\} = i - \min\{j \in \underline{n} \mid |i - j| \leqq x_{\max(i,j)}\} =$$
$$= i - \min\{j \in \underline{n} \mid |i - j| \leqq x_i,\ j \leqq i\} = i - (i - x_i) = x_i.$$

Conversely, $[i,j] \in T(x(T)) \Leftrightarrow |i - j| \leqq x_{\max(i,j)}(T) \Leftrightarrow |i - j| \leqq \max(i,j) - \min\{k \in \underline{n} \mid [\max(i,j), k] \in T\} \Leftrightarrow [i,j] \in T$. □

The Theorem enables us to draw tolerance lattices of finite chains. Denote $V_i(n) = \{x \in L(n) \mid x_{n-1} = i\}$ and call $V_i(n)$ the *i-th layer* of $L(n)$. Then we can draw $L(n)$ as follows:

$L(1)$ is the one-element lattice consisting of one layer. For constructing $L(n)$ we take n copies of $L(n-1)$, draw $L(n-1) \times \underline{n}$ and construct $V_i(n) = \{[x, k] \in L(n-1) \times \underline{n} \mid k = i$ and $x \in V_j(n-1)$ for some $j \geqq i - 1\}$ for $i = 0, \ldots, n - 1$.

Fig. 1

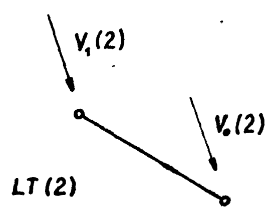

Fig. 2

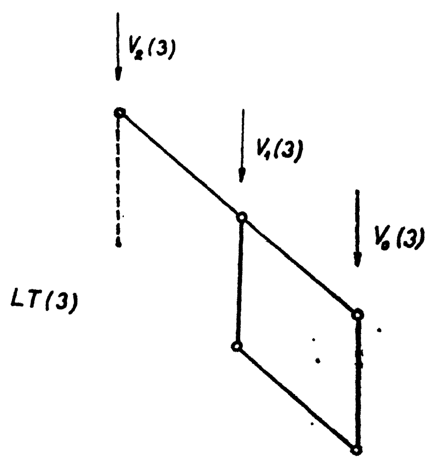

Fig. 3

For the number of elements of $L(n)$, it holds the formel

$$|L(0)| = 1, \qquad |L(n)| = \sum_{k=0}^{n-1} |L(k)| \,.\, |L(n-k-1)| \qquad \text{for } n \geqq 1.$$

Proof can be easily done by the reader.

Example. $|L(1)| = |L(0)| \,.\, |L(0)| = 1 \,.\, 1 = 1,$
$|L(2)| = |L(0)| \,.\, |L(1)| + |L(1)| \,.\, |L(0)| = 1 \,.\, 1 + 1 \,.\, 1 = 2,$
$|L(3)| = 5, |L(4)| = 14, |L(5)| = 42, |L(6)| = 132, |L(7)| = 429 \ldots$

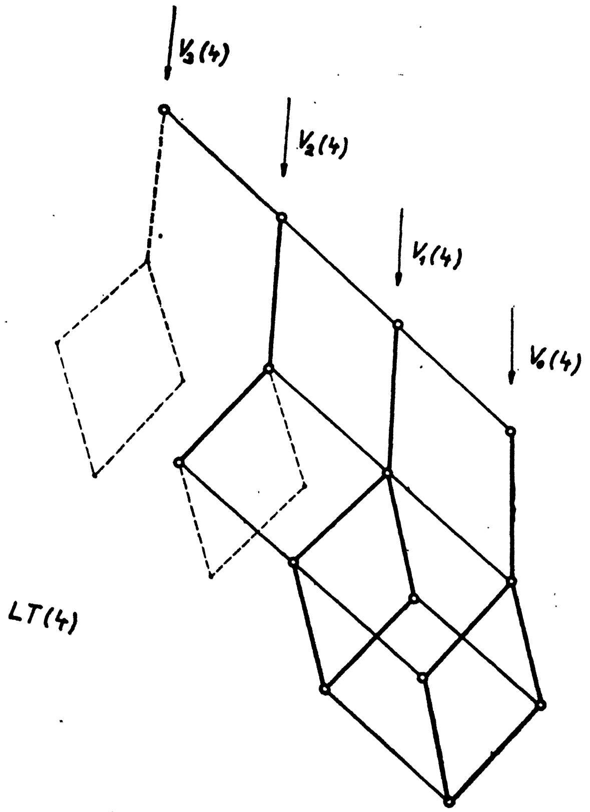

Fig. 4

The diagrams of the lattices $LT(n)$ for $n = 1, 2, 3, 4, 5$ are visualized in Fig. 1, Fig. 2, Fig. 3, Fig. 4, Fig. 5, respectively.

Added in proof. Tolerance lattices of finite distributive lattices are characterized in [3].

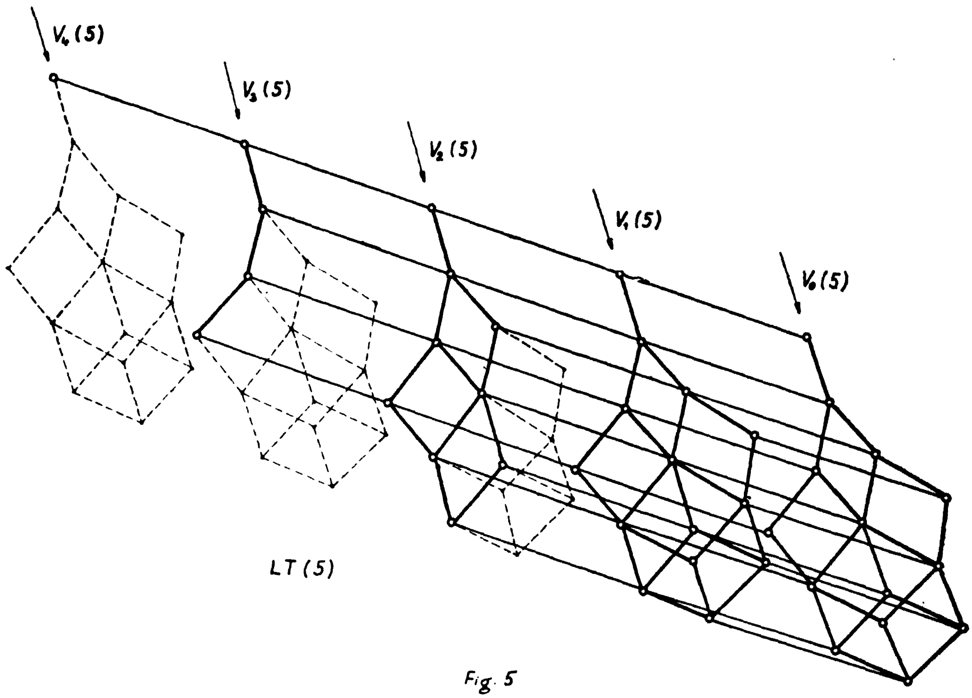

Fig. 5

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