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How to draw tolerance lattices of finite chains

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Let \( \mathcal{A} = (A, F) \) be an algebra. By a *tolerance* on \( \mathcal{A} \) is meant a reflexive and symmetric binary relation on \( A \) compatible with operations from \( F \), i.e. it is a subalgebra of \( \mathcal{A} \times \mathcal{A} \). The set of all tolerances on \( \mathcal{A} \) forms a lattice \( LT(\mathcal{A}) \) with respect to the set inclusion, [2]. This lattice, called *tolerance lattice*, was studied from the point of view of its algebraic properties in the case of lattices, semilattices, semigroups etc. As our knowledge, there is no explicite method describing a construction of \( LT(\mathcal{A}) \) for a given algebra \( \mathcal{A} \). In this paper, there is done a first attempt to solve this problem in the case of *finite chains* as the simplest kind of lattices.

Denote by \( L(n) = \{ x = [x_0, \ldots, x_{n-1}] \in 1 \times \ldots \times n \mid x_{i+1} \leq x_i + 1 \} \), where \( n = \{0, 1, \ldots, n - 1\} \) is an \( n \)-element chain considered as a lattice with binary operations min and max. \( LT(n) \) is the tolerance lattice of \( n \) (see [1], [2]).

**Lemma.** \( L(n) \) forms a sublattice of \( 1 \times \ldots \times n \).

**Proof.** Let \( x = [x_0, \ldots, x_{n-1}], y = [y_0, \ldots, y_{n-1}] \) be two elements of \( L(n) \). Then

\[
x \vee y = [\max(x_0, y_0), \ldots, \max(x_{n-1}, y_{n-1})] \in L(n),
\]

\[
x \wedge y = [\min(x_0, y_0), \ldots, \min(x_{n-1}, y_{n-1})] \in L(n),
\]

because

\[
\max(x_{i+1}, y_{i+1}) \leq \max(x_i + 1, y_i + 1) = \max(x_i, y_i) + 1,
\]

\[
\min(x_{i+1}, y_{i+1}) \leq \min(x_i + 1, y_i + 1) = \min(x_i, y_i) + 1. \quad \square
\]

The aim of this paper is to prove the following

**Theorem.** \( LT(n) \) is isomorphic to \( L(n) \).

**Proof.** Order homomorphisms \( x : LT(n) \rightarrow L(n) \) and \( T : L(n) \rightarrow LT(n) \) will be constructed and their bijectivity proven.
Let $T \in LT(n)$. Put $x_i(T) = i - \min \{j \in n \mid [i, j] \in T\}$. Clearly $x_i(T) \leq i$, $x_{i+1}(T) = i + 1 - \min \{j \in n \mid [i + 1, j] \in T\} = i - \min \{j \in n \mid [i + 1, j] \in T\} + 1 \leq i - \min \{j \in n \mid [i, j] \in T\} + 1 = x_i(T) + 1$. Consequently, $x(T) = \{x_0(T), \ldots, x_{n-1}(T)\} \in L(n)$. Moreover, for $T \subseteq S$ holds

$$x_i(T) = i - \min \{j \in n \mid [i, j] \in T\} \leq i - \min \{j \in n \mid [i, j] \in S\} = x_i(S).$$

An order homomorphism $x : LT(n) \to L(n)$ was constructed.

Now, let $x = [x_0, \ldots, x_{n-1}] \in L(n)$. Define a binary relation $T(x)$ on $n$ as follows: $[i, j] \in T(x) :\iff x_{\text{max}}(i, j) \geq |i - j|$. $T(x)$ is clearly a tolerance relation on $n$. Its compatibility will be shown. Let $[i, j], [i', j'] \in T(x)$, i.e. $|i - j| \leq x_{\text{max}}(i, j)$ and $|i' - j'| \leq x_{\text{max}}(i', j')$. Relationships

$$|\min (i, i') - \min (j, j')| \leq x_{\text{max}}(\min(i, i'), \min(j, j'))$$

and

$$|\max (i, i') - \max (j, j')| \leq x_{\text{max}}(\max(i, i'), \max(j, j'))$$

are to be proven. At least one of the following four cases will arise:

1. $i \leq i'$ and $j \leq j'$,
2. $i \leq i'$ and $j' \leq j$,
3. $i' \leq i$ and $j \leq j'$,
4. $i' \leq i$ and $j' \leq j$.

The third case is equivalent to the second one and the fourth case is equivalent to the first one.

In the first case,

$$|\min (i, i') - \min (j, j')| = |i - j| \leq x_{\text{max}}(i, j) = x_{\text{max}}(\min(i, i'), \min(j, j')),$$

$$|\max (i, i') - \max (j, j')| = |i' - j'| \leq x_{\text{max}}(i', j') = x_{\text{max}}(\max(i, i'), \max(j, j')).$$

In the second case, four subcases are to be distinguished:

2.1. $i \leq j'$ and $j \leq i'$,
2.2. $i \leq j'$ and $i' \leq j$,
2.3. $j' \leq i$ and $j \leq i'$,
2.4. $j' \leq i$ and $i' \leq j$.

In the first and the second subcases,

$$|\min (i, i') - \min (j, j')| = |i - j'| =$$

$$= |i - j| - (j - j') \leq x_j - (j - j') \leq x_{i'} = x_{\text{max}}(\min(i, i'), \min(j, j')).$$

In the third and the fourth subcases,

$$|\min (i, i') - \min (j, j')| = |i - j'| =$$

$$= |i' - j'| - (i' - i) \leq x_{i'} - (i' - i) \leq x_i = x_{\text{max}}(\min(i, i'), \min(j, j')).$$
In the first and the third subcases, 

\[ | \max(i, i') - \max(j, j') | = | i' - j | \leq | i' - j' | \leq x_{j'} = x_{\max(x_{\max(i, i'), \max(j, j')})}. \]

In the second and the fourth subcases, 

\[ | \max(i, i') - \max(j, j') | = | i' - j | \leq | i - j | \leq x_j = x_{\max(x_{\max(i, i'), \max(j, j')})}. \]

The compatibility of \( T(x) \) was proven. Clearly \( T : L(n) \to LT(n) \) is an order homomorphism, because \( x \leq y \) implies 

\[ [i, j] \in T(x) \iff | i - j | \leq x_{\max(i, j)}, \quad | i - j | \leq y_{\max(i, j)} \iff [i, j] \in T(y). \]

Bijectivity will be shown by verifying \( x . T = id, \quad T . x = id \)

\[ x_i(T(x)) = i - \min \{ j \in n \mid [i, j] \in T(x) \} = i - \min \{ j \in n \mid i - j \leq x_{\max(i, j)} \} = \]

\[ = i - \min \{ j \in n \mid i - j \leq x_i, \quad j \leq i \} = i - (i - x_i) = x_i. \]

Conversely, \([i, j] \in T(x(T)) \iff | i - j | \leq x_{\max(i, j)}(T) \iff | i - j | \leq \max(i, j) - \min \{ k \in n \mid \max(i, j), k \in T \} \iff [i, j] \in T. \]

The Theorem enables us to draw tolerance lattices of finite chains. Denote \( V_i(n) = \{ x \in L(n) \mid x_{n-1} = i \} \) and call \( V_i(n) \) the \( i \)-th layer of \( L(n) \). Then we can draw \( L(n) \) as follows:

\( L(1) \) is the one-element lattice consisting of one layer. For constructing \( L(n) \) we take \( n \) copies of \( L(n - 1) \), draw \( L(n - 1) \times n \) and construct \( V_j(n) = \{ [x, k] \in L(n - 1) \times n \mid k = i \) and \( x \in V_j(n - 1) \) for some \( j \geq i - 1 \} \) for \( i = 0, \ldots, n - 1 \)
For the number of elements of $L(n)$, it holds the formula

$$|L(0)| = 1, \quad |L(n)| = \sum_{k=0}^{n-1} |L(k)| \cdot |L(n-k-1)| \quad \text{for } n \geq 1.$$ 

Proof can be easily done by the reader.

**Example.**

- $|L(1)| = |L(0)| \cdot |L(0)| = 1 \cdot 1 = 1$,
- $|L(2)| = |L(0)| \cdot |L(1)| + |L(1)| \cdot |L(0)| = 1 \cdot 1 + 1 \cdot 1 = 2$,
- $|L(3)| = 5$, $|L(4)| = 14$, $|L(5)| = 42$, $|L(6)| = 132$, $|L(7)| = 429$ ...

The diagrams of the lattices $LT(n)$ for $n = 1, 2, 3, 4, 5$ are visualized in Fig. 1, Fig. 2, Fig. 3, Fig. 4, Fig. 5, respectively.

Added in proof. Tolerance lattices of finite distributive lattices are characterized in [3].
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