Rudolf Oláh

Note on the oscillation of linear delay differential equations

Archivum Mathematicum, Vol. 16 (1980), No. 4, 213--216

Persistent URL: http://dml.cz/dmlcz/107076

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVI: 213—216, 1980

NOTE ON THE OSCILLATION OF LINEAR DELAY DIFFERENTIAL EQUATIONS

RUDOLF OLÁH, Žilina
(Received April 4, 1979)

1. INTRODUCTION

We assume that the functions p, q satisfy the next condition:

$$p, q \in C[[0, \infty), (0, \infty)].$$

Let G be the set to which g belongs if and only if g satisfies the conditions:

$$g \in C[[0, \infty), [0, \infty)], g(t) \leqq t, \lim_{t \to \infty} g(t) = \infty.$$

We consider the following linear delay differential equations

$$u^{(2n)}(t) + p(t)\, u(g(t)) = 0, \tag{1}$$

$$u^{(2n)}(t) + q(t)\, u(h(t)) = 0, \tag{2}$$

where $g, h \in G$.

Our purpose is a comparison of the oscillatory properties of (1) with the oscillatory properties of (2).

A solution $u(t)$ of the equation (1) is called oscillatory if the set of zeros of $u(t)$ is not bounded from the right. A solution $u(t)$ of the equation (1) is called nonoscillatory if it is eventually of constant sign. The equation (1) is called oscillatory if every solution of (1) is oscillatory.

The theorems of the section 2 are an extension of some second order results in [1]. Our primary sources for the comparison of the oscillatory properties of (1) and (2) are [1] and [2].

2. OSCILLATORY PROPERTIES

Theorem 1. Suppose that $h(t) \leqq g(t)$, $q(t) \leqq p(t)$ whenever $t \geqq t_0 \geqq 0$, and (1) has a nonoscillatory solution. Then

$$v^{(2n)}(t) + q(t)\, v(h(t)) = 0 \tag{3}$$

has a nonoscillatory solution.

Proof. Let $u(t)$ be a positive solution of (1). Let $t_1 \geqq t_0 \geqq 0$ be such that none of $u, u', \ldots, u^{(2n-1)}$ has a zero on $[t_1, \infty)$, and let j be the largest integer such that $u^{(i)} > 0$ on $[t_1, \infty)$ if $i \leqq j$. Choose $t_2 \geqq t_1$ such that if $t \geqq t_2$ then $h(t) \geqq t_1$.

An induction argument shows that if $t \geqq t_2$ and $1 \leqq k \leqq j - 1$, then

$$u(t) = u(t_2) + \sum_{i=1}^{k} \frac{(t - t_2)^i}{i!} u^{(i)}(t_2) + \frac{1}{k!} \int_{t_2}^{t} (t - s)^k u^{(k+1)}(s) \, ds.$$

If $k = j - 1$, we get

$$u(t) \geqq u(t_2) + \frac{1}{(j-1)!} \int_{t_2}^{t} (t - s)^{j-1} u^{(j)}(s) \, ds. \tag{4}$$

If $z \geqq t \geqq t_2$, then

$$u^{(j)}(t) = \sum_{i=0}^{2n-j-1} (-1)^i \frac{(z - t)^i}{i!} u^{(j+i)}(z) + \frac{1}{(2n - j - 1)!} \int_{t}^{z} (s - t)^{2n-j-1} p(s) \, u(g(s)) \, ds,$$

so

$$u^{(j)}(t) \geqq \frac{1}{(2n - j - 1)!} \int_{t}^{\infty} (s - t)^{2n-j-1} p(s) \, u(g(s)) \, ds. \tag{5}$$

Using (5) in (4) we get

$$u(t) \geqq u(t_2) + \frac{1}{(j-1)!\,(2n - j - 1)!} \int_{t_2}^{t} (t - s)^{j-1} \Big(\int_{s}^{\infty} (\xi - s)^{2n-j-1} p(\xi) u(g(\xi)) \, d\xi \Big) \, ds \geqq$$

$$\geqq u(t_2) + \frac{1}{(j-1)!\,(2n - j - 1)!} \int_{t_2}^{t} (t - s)^{j-1} \Big(\int_{s}^{\infty} (\xi - s)^{2n-j-1} q(\xi) \, u(h(\xi)) \, d\xi \Big) \, ds,$$

since $u(t)$ is increasing on $[t_1, \infty)$.

We shall prove that there is a continuous function $v(t)$ on $[t_0, \infty)$ such that $u(t_2) \leqq v(t) \leqq u(t)$ if $t \geqq t_2$ and $v(t)$ is a solution of (3). We define a sequence of continuous functions on $[t_0, \infty)$ as follows:

$$v_1(t) = u(t), \; t \geqq t_0,$$

$$v_{m+1}(t) = u(t), \; t_0 \leqq t < t_2, \qquad m = 1, 2, \ldots,$$

$$v_{m+1}(t) = u(t_2) + \frac{1}{(j-1)!\,(2n - j - 1)!} \int_{t_2}^{t} (t - s)^{j-1} \Big(\int_{s}^{\infty} (\xi - s)^{2n-j-1} q(\xi) \, v_m(h(\xi)) \, d\xi \Big) \, ds,$$

for $t \geqq t_2, m = 1, 2, \ldots$

Then we have

$$v_2(t) = u(t_2) + \frac{1}{(j-1)!\,(2n - j - 1)!} \int_{t_2}^{t} (t - s)^{j-1} \Big(\int_{s}^{\infty} (\xi - s)^{2n-j-1} q(\xi) \, u(h(\xi)) \, d\xi \Big) \, ds \leqq$$

$$\leqq u(t), \; t \geqq t_2,$$

so

$$v_2(t) \leqq v_1(t), t \geqq t_2.$$

It follows by induction that:

$$u(t_2) \leqq v_{m+1}(t) \leqq v_m(t), \qquad \text{for } t \geqq t_2, m = 1, 2, \ldots$$

We conclude that $v_m(t)$ tends to a limit function $v(t)$ such that $u(t_2) \leqq v(t) \leqq u(t)$ if $t \geqq t_2$ and by Lebesgue's theorem we have

(6) $$v(t) =$$

$$= u(t_2) + \frac{1}{(j-1)!\,(2n-j-1)!} \int_{t_2}^{t} (t-s)^{j-1} \left(\int_{s}^{\infty} (\xi - s)^{2n-j-1}\, q(\xi)\, v(h(\xi))\, d\xi\right) ds$$

if $t \geqq t_2$. Differentiation of (6) says that $v(t)$ is a solution of (3) and clearly $v(t)$ is nonoscillatory, so the proof is complete.

Theorem 2. Suppose that $h(t) \leqq g(t)$ whenever $t \geqq t_0 \geqq 0$ and $g(t) - h(t)$ is bounded on $[t_0, \infty)$. Then (1) is oscillatory if and only if

(7) $$v^{(2n)}(t) + p(t)\, v(h(t)) = 0$$

is oscillatory.

Proof. Let $u(t)$ be a nonoscillatory solution of (1). Then applying the Theorem 1 we conclude that (7) has a nonoscillatory solution $v(t)$.

Now let $v(t)$ be a positive solution of (7). Let $t_1 \geqq t_0 \geqq 0$ be such that none of $v, v', \ldots, v^{(2n-1)}$ has a zero on $[t_1, \infty)$, and let j be the largest integer such that $v^{(i)} > 0$ on $[t_1, \infty)$ if $i \leqq j$. Let $g(t) - h(t) \leqq K$ for $t \geqq t_0$. Choose $t_2 \geqq t_1$ such that if $t \geqq t_2$ then $g(t) - K \geqq t_1$. We put $y(t) = v(t - K)$. Then

$$y(g(t)) = v(g(t) - K) \leqq v(h(t)), t \geqq t_2.$$

With regard to (5) and (4) for $t \geqq t_2$ we have

$$v^{(j)}(t + K) \geqq \frac{1}{(2n-j-1)!} \int_{t+K}^{\infty} (s - t - K)^{2n-j-1}\, p(s)\, v(h(s))\, ds,$$

$$v(t) \geqq v(t_2) + \frac{1}{(j-1)!} \int_{t_2}^{t} (t-s)^{j-1}\, v^{(j)}(s)\, ds \geqq$$

$$\geqq v(t_2) + \frac{1}{(j-1)!} \int_{t_2}^{t} (t-s)^{j-1}\, v^{(j)}(s + K)\, ds,$$

since $v^{(j)}(t)$ is decreasing on $[t_1, \infty)$.

Then

$$v(t) \geqq v(t_2) +$$

$$+ \frac{1}{(j-1)!\,(2n-j-1)!} \int_{t_2}^{t} (t-s)^{j-1} \left(\int_{s+K}^{\infty} (\xi - s - K)^{2n-j-1}\, p(\xi)\, v(h(\xi))\, d\xi\right) ds,$$

$$y(t) = v(t-K) \geqq v(t_2) +$$

$$+ \frac{1}{(j-1)!\,(2n-j-1)!} \int_{t_2}^{t-K} (t-K-s)^{j-1} \Big(\int_{s+K}^{\infty} (\xi - s - K)^{2n-j-1}\, p(\xi)\, y(g(\xi))\, d\xi \Big)\, ds.$$

Now we define a sequence of continuous functions on $[t_0, \infty)$ as follows:

$$u_1(t) = y(t), \quad t \geqq t_0,$$

$$u_{m+1}(t) = y(t),\ t_0 \leqq t < t_2 + K, \quad m = 1, 2, \ldots,$$

$$u_{m+1}(t) = v(t_2) +$$

$$+ \frac{1}{(j-j)!\,(2n-j-1)!} \int_{t_2}^{t-K} (t-K-s)^{j-1} \Big(\int_{s+K}^{\infty} (\xi - s - K)^{2n-j-1}\, p(\xi)\, u_m(g(\xi))\, d\xi \Big)\, ds,$$

for $t \geqq t_2 + K, \ m = 1, 2, \ldots$

Then there is a continuous function $u(t)$ on $[t_0, \infty)$ such that $v(t_2) \leqq u(t) \leqq y(t)$ if $t \geqq t_2 + K$ and such that

$$u(t) = v(t_2) +$$

$$+ \frac{1}{(j-1)!\,(2n-j-1)!} \int_{t_2}^{t-K} (t-K-s)^{j-1} \Big(\int_{s+K}^{\infty} (\xi - s - K)^{2n-j-1}\, p(\xi)\, u(g(\xi))\, d\xi \Big)\, ds,$$

if $t \geqq t_2 + K$. Differentiation of the last equation says that $u(t)$ is a nonoscillatory solution of (1), so the proof is complete.

Corollary. Suppose that $t - g(t)$ is bounded on $[t_0, \infty)$. Then (1) is oscillatory if and only if

$$v^{(2n)}(t) + p(t)\, v(t) = 0$$

is oscillatory.

REFERENCES

[1] **Brands, J. J. A. M.:** *Oscillation Theorems for Second-Order Functional Differential Equations,* J. Math. Appl. 63 (1978), 54—64.

[2] **Lovelady, D. L.:** *Oscillation and a class of linear delay differential equations,* Trans. Amer. Math. Soc. 226 (1977), 345—364.

[3] **Lovelady, D. L.:** *An asymptotic analysis of an odd order linear differential equation,* Pacific J. Math. 57 (1975), 475—480.

R. Oláh
010 88 Žilina, Marxa-Engelsa 25
Czechoslovakia