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GENERALIZED LOGARITHMIC MEANS OF AN ENTIRE DIRICHLET SERIES (I)

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1. INTRODUCTION

Consider a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, $(s = \sigma + it$, $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geqq 0$, $\lambda_n \to \infty$ with $n)$, which we assume to be absolutely convergent for all finite s, and hence it defines an entire function. The logarithmic mean of $f(s)$ is defined [1, p. 13] as:

$$L(\sigma) = \lim_{T\to\infty} \left\{ \frac{1}{2T} \int_{-T}^{T} \log |f(\sigma + it)| \, dt \right\}. \tag{1.1}$$

For any $\delta > 0$, we define the generalized logarithmic means of $f(s)$ as:

$$L_\delta(\sigma) = e^{-\delta\sigma} \int_0^\sigma e^{\delta x} L(x) \, dx, \tag{1.2}$$

and

$$L_\delta^*(\sigma) = \sigma^{-\delta-1} \int_0^\sigma x^\delta L(x) \, dx. \tag{1.3}$$

In this paper we have investigated a few properties of $L(\sigma)$, $L_\delta(\sigma)$ and $L_\delta^*(\sigma)$.

2. LEMMAS

In this section we state and prove certain lemmas which are of fundamental importance in the proofs of the results of this paper.

Lemma 1. $\sigma^{\delta+1}L(\sigma)$ is an increasing convex function of $\sigma^{\delta+1}L_\delta(\sigma)$.

Proof. Since $\log L(\sigma)$ is an increasing convex function of σ [1, p. 13], we may write

$$\log L(\sigma) = \log L(\sigma_0) + \int_{\sigma_0}^{\sigma} \eta(x) \, dx, \qquad \sigma \geqq \sigma_0, \tag{2.1}$$

where $\eta(x)$ is a non-decreasing function of x and tends to infinity with x (see [3], equation (4), p. 73). Also

$$\frac{d[\sigma^{\delta+1}L(\sigma)]}{d[\sigma^{\delta+1}L_\delta^*(\sigma)]} = \frac{\frac{d}{d\sigma}[\sigma^{\delta+1}L(\sigma)]}{\frac{d}{d\sigma}[\sigma^{\delta+1}L_\delta^*(\sigma)]} =$$

$$= \frac{(\delta+1)\,\sigma^\delta L(\sigma) + \sigma^{\delta+1}\frac{d}{d\sigma}[L(\sigma)]}{\sigma^\delta L(\sigma)} =$$

$$= (\delta+1) + \sigma\left\{\frac{\frac{d}{d\sigma}(L(\sigma))}{L(\sigma)}\right\} = \delta + 1 + \sigma\eta(\sigma).$$

Therefore

$$\sigma^{\delta+1}L(\sigma) = O(1) + \int_{\sigma_0}^{\sigma} \eta^*(x)\, d\big(x^{\delta+1}L_\delta^*(x)\big),$$

where $\eta^*(x) = \delta + 1 + x\eta(x)$.

Hence the lemma follows.

Lemma 2. $\log L_\delta^*(\sigma)$ is an increasing convex function of $\log\sigma$.

Proof. We have

$$\frac{d[\log L_\delta^*(\sigma)]}{d[\log\sigma]} = \frac{\frac{d}{d\sigma}[\log L_\delta^*(\sigma)]}{\frac{d}{d\sigma}[\log\sigma]} =$$

$$= \frac{L(\sigma) - (\delta+1)\,\sigma^{-\delta-1}\int_0^{\sigma} x^\delta L(x)\,dx}{L_\delta^*(\sigma)} = \frac{L(\sigma)}{L_\delta^*(\sigma)} - (\delta+1),$$

which increases with σ, by virtue of lemma 1. Hence, we have

$$\frac{d^2[\log L_\delta^*(\sigma)]}{d[\log\sigma]^2} > 0, \qquad \sigma \geqq \sigma_0,$$

and lemma 2 follows.

Since $\log L_\delta^*(\sigma)$ is an increasing convex function of $\log\sigma$, we have

(2.2) $$\log L_\delta^*(\sigma) = \log L_\delta^*(\sigma_0) + \int_{\sigma_0}^{\sigma} \frac{U(x)}{x}\,dx, \qquad \sigma \geqq \sigma_0,$$

where $U(x)$ is a positive real valued indefinitely increasing function of x.

Lemma 3. For $\Delta > \sigma$, we find

$$(2.3)\qquad L_\delta(\sigma) \leqq \frac{L(\sigma)}{\delta} \leqq \frac{e^{\delta\Delta}}{e^{\delta\Delta} - e^{\delta\sigma}} L_\delta(\Delta)$$

and

$$(2.4)\qquad L_\delta^*(\sigma) \leqq \frac{L(\sigma)}{\delta + 1} \leqq \frac{\Delta^{\delta+1}}{\Delta^{\delta+1} - \sigma^{\delta+1}} L_\delta^*(\Delta).$$

Proof. Since $L(\sigma)$ is a steadily increasing function of σ [1, p. 13], therefore

$$(2.5)\qquad L_\delta(\sigma) = e^{-\delta\sigma} \int_0^\sigma e^{\delta x} L(x)\,dx \leqq L(\sigma)\left\{\frac{1 - e^{-\delta\sigma}}{\delta}\right\} \leqq \frac{L(\sigma)}{\delta}.$$

Also, we have

$$L_\delta(\Delta) = e^{-\delta\Delta} \int_0^\Delta e^{\delta x} L(x)\,dx \geqq e^{-\delta\Delta} \int_\sigma^\Delta e^{\delta x} L(x)\,dx \geqq$$
$$\geqq e^{-\delta\Delta} L(\sigma)\left\{\frac{e^{\delta\Delta} - e^{\delta\sigma}}{\delta}\right\}.$$

Therefore

$$(2.6)\qquad \frac{L(\sigma)}{\delta} \leqq \frac{e^{\delta\Delta}}{e^{\delta\Delta} - e^{\delta\sigma}} L_\delta(\Delta).$$

(2.5) and (2.6) complete the proof of (2.3). (2.4) follows exactly on the lines of the proof of (2.3).

Lemma 4. $e^{\delta\sigma}L(\sigma)$ is an increasing convex function of $e^{\delta\sigma}L_\delta(\sigma)$.
The proof of this lemma is given by Gupta and Bala [2, p. 809].

3. THEOREMS

Theorem 1. If $L_\delta(\sigma)$ is the generalized logarithmic mean of $f(s)$ of Ritt order p. lower order λ; type T and lower type t then

$$(3.1)\qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log L_\delta^*(\sigma)}{\sigma} \leqq \frac{\varrho}{\lambda}, \qquad 0 \leqq \lambda \leqq \varrho \leqq \infty.$$

But if $f(s)$ is of nonzero finite Ritt order ϱ, then

$$(3.2)\qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{L_\delta^*(\sigma)}{e^{\varrho\sigma}} \leqq \frac{T/(\delta + 1)}{t/(\delta + 1)}, \qquad 0 \leqq t \leqq T \leqq \infty.$$

Proof. From (1.1) and (1.3), we have

$$L_\delta^*(\sigma) = \lim_{T\to\infty} \left\{\frac{\sigma^{-\delta-1}}{2T} \int_0^\sigma \int_{-T}^T x^\delta \log|f(x + it)|\,dx\,dt\right\} \leqq \frac{\log M(\sigma)}{\delta + 1},$$

where $M(\sigma) = \sup\{|f(\sigma + it)| : -\infty < t < \infty\}$.

Therefore

$$(3.3)\qquad \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log L_\delta^*(\sigma)}{\sigma} \leq \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log M(\sigma)}{\sigma}$$

and

$$(3.4)\qquad \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{L_\delta^*(\sigma)}{e^{\varrho\sigma}} \leq \frac{1}{\delta+1}\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log M(\sigma)}{e^{\varrho\sigma}}.$$

The result in (3.1) and (3.2) now follow from (3.3) and (3.4), respectively, since [4, p. 77]:

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log M(\sigma)}{\sigma} = \frac{\varrho}{\lambda},$$

and

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log M(\sigma)}{e^{\varrho\sigma}} = \frac{T}{t}.$$

Theorem 2. If $L(\sigma)$ and $L_\delta^*(\sigma)$ are, respectively, the logarithmic and the generalized logarithmic means of $f(s)$, then, for $0 < \sigma_1 < \sigma_2$,

$$L(\sigma_1) \leq (\delta+1)\frac{\sigma_2^{\delta+1}L_\delta^*(\sigma_2) - \sigma_1^{\delta+1}L_\delta^*(\sigma_1)}{\sigma_2^{\delta+1} - \sigma_1^{\delta+1}} \leq L(\sigma_2).$$

Proof. From (1.3), we have

$$L_\delta^*(\sigma) = \sigma^{-\delta-1}\int_0^\sigma x^\delta L(x)\,\mathrm{d}x.$$

Therefore

$$(3.5)\qquad L_\delta^*(\sigma_1) = \sigma_1^{-\delta-1}\int_0^{\sigma_1} x^\delta L(x)\,\mathrm{d}x,$$

and

$$(3.6)\qquad L_\delta^*(\sigma_2) = \sigma_2^{-\delta-1}\int_0^{\sigma_2} x^\delta L(x)\,\mathrm{d}x.$$

From (3.5) and (3.6), we find

$$(3.7)\quad \sigma_2^{\delta+1}L_\delta^*(\sigma_2) - \sigma_1^{\delta+1}L_\delta^*(\sigma_1) = \int_{\sigma_1}^{\sigma_2} x^\delta L(x)\,\mathrm{d}x \leq \frac{1}{\delta+1}L(\sigma_2)(\sigma_2^{\delta+1} - \sigma_1^{\delta+1}),$$

and

$$(3.8)\quad \sigma_2^{\delta+1}L_\delta^*(\sigma_2) - \sigma_1^{\delta+1}L_\delta^*(\sigma_1) = \int_{\sigma_1}^{\sigma_2} x^\delta L(x)\,\mathrm{d}x \geq \frac{1}{\delta+1}L(\sigma_1)(\sigma_2^{\delta+1} - \sigma_1^{\delta+1}).$$

(3.7) and (3.8) complete the proof of theorem 2.

Theorem 3. If

$$(3.9)\qquad \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log L_\delta^*(\sigma)}{\sigma} = \frac{p}{q},\qquad 0 \leq q \leq p \leq \infty,$$

then

$$(3.10)\qquad \liminf_{\sigma\to\infty}\frac{\sigma\log L_s^*(\sigma)}{U(\sigma)}\leqq\frac{1}{p}\leqq\frac{1}{q}\leqq\limsup_{\sigma\to\infty}\frac{\sigma\log L_s^*(\sigma)}{U(\sigma)},$$

where $U(\sigma)$ is given by (2.2).

Proof. We let

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{U(\sigma)}{\sigma\log L_s^*(\sigma)}=\frac{c}{d},\qquad 0\leqq d\leqq c\leqq\infty.$$

We first suppose that $0<d\leqq c<\infty$. Then, for any $\varepsilon>0$ and sufficiently large σ, we have

$$(3.11)\qquad (d-\varepsilon)\,\sigma\log L_s^*(\sigma)<U(\sigma)<(c+\varepsilon)\,\sigma\log L_s^*(\sigma).$$

Differentiating both sides of (2.2), we get, for almost sufficiently large σ,

$$(3.12)\qquad \frac{L_s^{*\prime}(\sigma)}{L_s^*(\sigma)}=\frac{U(\sigma)}{\sigma},$$

where $L_s^{*\prime}(\sigma)$ is the derivative of $L_s^*(\sigma)$ with respect to σ. From (3.11) and (3.12), for any $\varepsilon>0$ and sufficiently large σ, we find that

$$d-\varepsilon<\frac{L_s^{*\prime}(\sigma)}{L_s^*(\sigma)\log L_s^*(\sigma)}<c+\varepsilon.$$

Integrating the above inequalities between suitable limits, we get

$$(d-\varepsilon)\,\sigma-O(1)<\log\log L_s^*(\sigma)-O(1)<(c+\varepsilon)\,\sigma-O(1).$$

Dividing throughout by σ, proceeding to limits, and making use of (3.9), we get

$$(3.13)\qquad d\leqq q\leqq p\leqq c,$$

which also holds when $d=0$ or $c=\infty$. If $d=\infty$, then so is c and $\lim_{\sigma\to\infty}\frac{U(\sigma)}{\sigma\log L_s^*(\sigma)}=$ $=\infty$. So taking an arbitrarily large number M in place of $d-\varepsilon$ and proceeding as above, we obtain $p=q=\infty$. Similarly, if $c=0$, it can easily be shown that $p=$ $=q=0$. Hence, in each case, (3.13) implies (3.10).

Theorem 4. If $L(\sigma)$ and $L_s^*(\sigma)$ are, respectively, the logarithmic and the generalized logarithmic means of $f(s)$ of order ϱ and lower order λ, then

$$(3.14)\qquad \limsup_{\sigma\to\infty}\frac{L(\sigma)}{\sigma L_s^*(\sigma)}\leqq e\varrho,$$

and

$$(3.15)\qquad \liminf_{\sigma\to\infty}\frac{L(\sigma)}{\sigma L_s^*(\sigma)}\leqq e\lambda.$$

Proof. It is readily seen that

$$\frac{d}{dx}\{(\delta+1)\log x+\log L_\delta^*(x)\}=\frac{1}{x}\left\{\frac{L(x)}{L_\delta^*(x)}\right\},$$

so that

$$(\delta+1)\log\frac{\sigma}{\sigma_0}=\log L_\delta^*(\sigma)-\log L_\delta^*(\sigma_0)=\int_{\sigma_0}^{\sigma}\frac{L(x)}{L_\delta^*(x)}\frac{dx}{x},$$

that is

$$\log L_\delta^*(\sigma)=\log L_\delta^*(\sigma_0)+\int_{\sigma_0}^{\sigma}\frac{m(x)}{x}dx, \tag{3.16}$$

where

$$m(x)=\left\{\frac{L(x)}{L_\delta^*(x)}-(\delta+1)\right\} \tag{3.17}$$

increases with x, by virtue of lemma 1. Thus, for $\sigma\geqq\sigma_0$ and $\mu>1$, (3.16) gives

$$\log L_\delta^*(\mu\sigma)\geqq\int_{\sigma}^{\mu\sigma}\frac{m(x)}{x}dx\geqq m(\sigma)\log\mu. \tag{3.18}$$

Hence

$$\limsup_{\sigma\to\infty}\frac{m(\sigma)\log\mu}{\sigma}\leqq\limsup_{\sigma\to\infty}\frac{\mu\log L_\delta^*(\mu\sigma)}{\mu\sigma}$$

and

$$\liminf_{\sigma\to\infty}\frac{m(\sigma)\log\mu}{\sigma}\leqq\liminf_{\sigma\to\infty}\frac{\mu\log L_\delta^*(\mu\sigma)}{\mu\sigma},$$

which give the desired results in view of (3.1) and (3.17), on taking $\mu=e$.

Theorem 5. We find

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log L_\delta(\sigma)}{\sigma}=\frac{\alpha}{\beta},\qquad 0\leqq\beta\leqq\alpha\leqq\infty, \tag{3.19}$$

where the quantities α and β are given by

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log L(\sigma)}{\sigma}=\frac{\alpha}{\beta}. \tag{3.20}$$

Proof. Putting $\Delta=\sigma+1$ in (2.3), the theorem follows in view of (3.20). The details are omitted.

Lastly, we prove:

Theorem 6. For a class of entire functions for which

$$\liminf_{\sigma\to\infty} \frac{\log L_\delta(\sigma)}{\sigma} = \infty,$$

we find

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\log L_\delta(\sigma)}{\log\sigma} = \frac{L+1}{l-1},$$

where

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \left\{\frac{L(\sigma)}{L_\delta(\sigma)}\right\}^{1/\log\sigma} = \frac{e^L}{e^l}.$$

Proof. We have

$$\log(e^{\delta\sigma} L_\delta(\sigma)) = \int_0^\sigma \frac{e^{\delta x} L(x)}{e^{\delta x} L_\delta(x)}\, dx,$$

since numerator on the right – hand side is the differential coefficient of the denominator. This gives

$$\log(e^{\delta\sigma} L_\delta(\sigma)) < O(1) + \int_{\sigma_0}^{\sigma} x^{L+\varepsilon}\, dx,$$

for any $\varepsilon > 0$ and $\sigma \geqq \sigma_0$. Therefore,

$$\log(e^{\delta\sigma} L_\delta(\sigma)) < O(1) + \frac{\sigma^{L+\varepsilon+1}}{L+\varepsilon+1}(1 - O(1)).$$

Proceeding to limits, we get

$$\limsup_{\sigma\to\infty} \frac{\log\log L_\delta(\sigma)}{\log\sigma} \leqq L + 1, \tag{3.21}$$

since

$$\liminf_{\sigma\to\infty} \frac{\log L_\delta(\sigma)}{\sigma} = \infty.$$

Further, using lemma 4, we have

$$\log(e^{1\delta\sigma} L_\delta(2\sigma)) \geqq \int_\sigma^{2\sigma} \frac{L(x)}{L_\delta(x)}\, dx \geqq \frac{L(\sigma)}{L_\delta(\sigma)}\,\sigma > \sigma^{L-\varepsilon+1},$$

for a sequence of values of σ tending to infinity. Consequently,

$$\limsup_{\sigma\to\infty} \frac{\log\log L_\delta(\sigma)}{\log\sigma} \geqq L + 1. \tag{3.22}$$

From (3.21) and (3.22), we find

$$\limsup_{\sigma\to\infty} \frac{\log\log L_\delta(\sigma)}{\log\sigma} = L + 1.$$

Similarly, it can easily be seen that

$$\lim_{\sigma\to\infty} \inf \frac{\log \log L_s(\sigma)}{\log \sigma} = l + 1.$$

This completes the proof of theorem 6.

REFERENCES

[1] Gupta, J. S. and Manjit Singh, *Logarithmic mean function of entire functions defined by Dirichlet series*, Pub. Math. Debrecen, *20* (1973), 13—19.

[2] Gupta, J. S. and Shakti Bala, *Generalized logarithmic mean function of entire functions defined* by *Dirichlet series*, Rendi. di Mat., *6* (1973), 803—810.

[3] Kamthan, P. K., *Convex functions and their applications*, Rev. de la Fac. des Sci. de l'Univ. d'Istanbul, Sér A, *28* (1963), 71—78.

[4] Ritt, J. F., *On certain points in the theory of Dirichlet series*, Amer. J. Math., *50* (1928), 73—86.

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