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SUBGROUPS AND NORMAL SUBGROUPS AS SYSTEMS OF IDEALS

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In this paper we introduce the notion of an ideal operation. Aubert in [1] defined and studied a system of x-ideals which is in our conception an ideal system belonging to an ideal operator of a commutative semigroup.

General problem, if there exists an ideal operation on given closure space, is solved in Section 2 for the case of a system of all normal subgroups of a system of all normal subgroups of a given group.

In Section 3 an equivalent form of axioms of an ideal space in the case of system of all subgroups of a given group is shown.

1. IDEAL SPACE

In this section S will denote a non-empty set, the system of all subsets of the set S will be denoted by exp S.

1.1. Definition. *Let S be a set. A mapping $x : \exp S \to \exp S$ will be called a closure operator of S if it holds:*

A1: $A \subseteq S \Rightarrow A \subseteq A_x$,

A2: $A, B \subseteq S,\ A \subseteq B_x \Rightarrow A_x \subseteq B_x$.

A system $\Omega = \{A_x : A \subseteq S\}$ will be called a closure system (belonging to a closure operator x) and the pair (S, Ω) will be called a closure space.

1.2. Definition. *Let $(S, .)$ be a groupoid. A closure operator x of S will be called an ideal operator of S if it holds:*

A3: $A \subseteq S \Rightarrow SA_x \subseteq A_x$,

A4: $A, B \subseteq S \Rightarrow A_xB \subseteq (AB)_x$.

We say that $A \subseteq S$ is an ideal if $A = A_x$. A system $\Omega = \{A_x : A \subseteq S\}$ will be called an ideal system (belonging to an ideal operator x), the triad $(S, \Omega, .)$ will be called an ideal space.

1.3. Remark. If a non-empty system $\Omega \subseteq \exp S$ fulfils the conditions

1. $S \in \Omega$,

2. $A_i \in \Omega, i \in I \Rightarrow \bigcap A_i \ (i \in I) \in \Omega$,

then there exists a closure operator x of S with the property: Ω is the closure system belonging to the closure operator x.

Let us define operator x in the following way: For $A \subseteq S$ we put $A_x = \bigcap B(B \in \Omega, A \subseteq B)$. The converse is also true.

1.4. Definition. *Let* (S, Ω) *be a closure space. We say that an operation* . *on* S *is an ideal operation on* (S, Ω) *if* $((S, \Omega), .)$ *is an ideal space.*

1.5. Remark. Let $(S, .)$ be a groupoid. Let us put $A : B = \{c \in S : cB \subseteq A\}$ for non-empty sets $A, B \subseteq S$. In the case $B = \{b\}$ we write $A : b$.

1.6. Remark. It holds:

(a) $(A : B) . B \subseteq A$,

(b) $A : \bigcup B_i(i \in I) = \bigcap (A : B_i) \ (i \in I)$,

(c) $\bigcap A_i(i \in I) : B = \bigcap (A_i : B) \ (i \in I)$,

where $A, B, A_i, B_i \ (i \in I)$ are the subsets of groupoid S.

1.7. Theorem. *Let* $(S, .)$ *be a groupoid and* x *be a closure operator of* S. *Then the following statements are equivalent*:

(1) $A, B \subseteq S \Rightarrow A_xB \subseteq (AB)_x$,

(2) $A, B \subseteq S \Rightarrow A_x : B = (A_x : B)_x$,

(3) $A \subseteq S, b \in S \Rightarrow A_x : b = (A_x : b)_x$,

(4) $A, B \subseteq S \Rightarrow (A : B)_x \subseteq A_x : B$.

Proof. Let (1) hold. According to 1.6. (a) we can write $(A : B)_x B \subseteq ((A : B) B)_x \subseteq A_x$, hence $(A : B)_x \subseteq A_x : B$. Thus (1) implies (4). Let (4) hold. For $A \subseteq S, b \in S$ we have $(A_x : b)_x \subseteq (A_x)_x : b = A_x : b$, thus $(A_x : b)_x = A_x : b$. Therefore (4) $\Rightarrow$ (3). If (3) holds, then by 1.6. (b) we obtain $(A_x : B)_x = (A_x : \cup\{b\} \ (b \in B))_x = (\cap(A_x : b) \ (b \in B))_x = (\cap(A_x : b)_x \ (b \in B))_x = \cap(A_x : b) \ (b \in B) = A_x : \cup\{b\} \ (b \in B) = A_x : B$. Thus (3) implies (2).

From A1 it follows $AB \subseteq (AB)_x$ and therefore $A \subseteq (AB)_x : B$. If (2) holds, then $A_x \subseteq (AB)_x : B$, hence $A_xB \subseteq (AB)_x$. The proof is complete.

1.8. Theorem. *Let* $(S, .)$ *be a commutative groupoid and let* x *be a closure operator of* S. *Then the following statements are equivalent*:

(1) $A \subseteq S \Rightarrow SA_x \subseteq A_x$,

(2) $a, b \in S \Rightarrow (ab)_x \subseteq a_x \cap b_x$,

(3) $A, B \subseteq S \Rightarrow (AB)_x \subseteq A_x \cap B_x$,

(4) $A, B \subseteq S \Rightarrow (A_xB_x)_x \subseteq A_x \cap B_x$.

Proof. If (1) holds, then from properties of a closure operator it follows $(ab)_x \subseteq (a_x \cap b_x)_x = a_x \cap b_x$, $a, b \in S$, i.e. (2).

Let $A, B \subseteq S$ and let (2) hold. It is $(AB)_x = (\cup\{ab\} (a \in A, b \in B))_x \subseteq \subseteq (\cup\{a_x \cap b_x\} (a \in A, b \in B))_x = (\cup a_x(a \in A) \cap \cup b_x(b \in B))_x \subseteq (A_x \cap B_x)_x = A_x \cap B_x$. Thus (2) $\Rightarrow$ (3).

From (3) we obtain $(A_x B_x)_x \subseteq (A_x)_x \cap (B_x)_x = A_x \cap B_x$ for $A, B \subseteq S$. Therefore (3) implies (4).

Let (4) hold. If $A \subseteq S$, then $SA_x \subseteq (SA_x)_x = (S_x A_x)_x \subseteq S_x \cap A_x = A_x$. Hence $SA_x \subseteq A_x$ and the proof is complete.

2. NORMAL SUBGROUPS AS IDEALS

In what follows G will denote an additive group.

2.1. Remark. Let G be a group. If Ω is a system of all normal subgroups in G, then (G, Ω) is a closure space. Let $g, h, a \in G$. (g, h) will denote a commutator of pair g, h, i.e. $(g, h) = -g - h + g + h$. It follows directly:

(a) $(g + h, a) = -h + (g, a) + h + (h, a)$,

(b) $(g, h + a) = (g, a) - a + (g, h) + a$,

(c) $(g, h) = 0 \Leftrightarrow g + h = h + g$.

A set $c(g) = \{h \in G : (g, h) = 0\}$ (a centralizier of an element g) is a subgroup in G.

2.2. Lemma. *Let G be a group and let Ω be a system of all normal subgroups in G. Then for operation commutator there holds the axiom A3.*

Proof. For this case the axiom A3 is: $A \subseteq G \Rightarrow (G, A_x) \subseteq A_x$, where A_x is as follows a subgroup in G generated by A. The proof follows from properties of normal subgroups.

2.3. Lemma. *Let G be a group and let N be a normal subgroup of the group G. Then it holds:*

(a) $g, b \in G, (g, b) \in N \Rightarrow (-g, b) \in N$,

(b) $g, h, b \in G, (g, b), (h, b) \in N \Rightarrow (g + h, b) \in N$.

Proof. If $g, b \in G$, $(g, b) \in N$, then we have $g - (g, b) - g = g - b - g + b = (-g, b) \in N$. If moreover $h \in G$, $(h, b) \in N$, then we obtain $(g + h, b) = -h + (g, b) + h + (h, b) \in N$.

2.4. Remark. From 2.3 it follows: Let A_x be a normal subgroup of the group G generated by A for each $A \subseteq G$. Then for each $b \in G$, $A_x : b$ is a subgroup in G (division with respect to the operation commutator). If $A_x : b$ will be a normal subgroup in G, then 1.7. (3) implies that the operation commutator is an ideal operation.

2.5. Theorem. *Let G be a group and N be a normal subgroup in G. If $g, a, b \in G$, $(g, b) \in N$, then $(-a + g + a, b) \in N$ if and only if $((a, g), b) \in N$.*

Proof. Let us suppose that $g, a, b \in G$ and $(g, b) \in N$.

1. If $((a, g), b) \in N$, then from 2.3. we obtain $(g - (a, g), b) = (-a - g + a, b) \in N$. Hence also $(-a + g + a, b) \in N$.

2. Let us assume that $(-a + g + a, b) \in N$. This implies $(-a - g + a, b) \in N$, which with $(g, b) \in N$ give $(-a - g + a + g, b) = ((a, g), b) \in N$.

2.6. Lemma. *Let N be a normal subgroup of a group G. If $a \in A \in G/N$, $b \in B \in G/N$, then $(A, B) = (a, b) + N$.*

Proof. Follows from properties of normal subgroups.

2.7. Theorem. *Let G be a group and let Ω be a system of all normal subgroups in G. Then the following conditions are equivalent:*

(a) *The operation commutator is an ideal operation for a closure space (G, Ω).*

(b) *For an arbitrary $N \in \Omega$ it holds: If $Y \in G/N$, then $c(Y)$ is a normal subgroup in G/N.*

Proof. 1. First we suppose that the condition (b) holds. Let $N \in \Omega$, $y, b \in G$, $(y, b) \in N$ exist. If we put $Y = y + N$, $B = b + N$, then Y, B are elements of G/N fulfilling $(Y, B) = (y, b) + N = N$. Thus (Y, B) is a zero in the factorgroup G/N and therefore $Y \in c(B)$. With regard to (b) $c(B)$ is a normal subgroup in G/N. We put $P = p + N$ for arbitrary $p \in G$. According to (b) we have $(-P + Y + P, B) = N$ (zero in G/N). By 2.6. we have $N = (-P + Y + P, B) = (-p + y + p, b) + N$ and from this $(-p + y + p, b) \in N$. Thus (a) holds (see 1,7, (3)).

2. Conversely, let (a) hold. Suppose $N \in \Omega$, $Y \in G/N$. If $B \in c(Y)$, then (B, Y) is a zero in G/N and therefore $(B, Y) = (b, y) + N = N$, i.e. $(b, y) \in N$. Let $P \in G/N$ be an arbitrary element. We can write $P = p + N$ for $p \in P$. According to (a) and 2.5. we have $((p, b), y) \in N$ that implies $((P, B), Y) = ((p, b), y) + N = N$. Thus $((P, B), Y)$ is a zero in G/N. Hence $(P, B) \in c(Y)$, where $c(Y)$ is a subgroup in G/N. From here we obtain $B - (P, B: = -P + B + P \in c(Y)$. Therefore $c(Y)$ is a normal subgroup in G/N and (b) holds.

2.8. Corollary. *Let G be a group and Ω be a system of all normal subgroups in G. If the commutator is an ideal operation on (G, Ω), then the centralizier of an arbitrary element of G is a normal subgroup in G.*

Proof. The proof follows from 2.7. (b). We put $N = \{0\}$.

2.9. Example. Let M be a multiplicative group of regular matrices of the type *2/2* with integer elements. Let us find the centralizier of element $Y \in M(Y = (y_{ij}), y_{11} = y_{12} = y_{22} = 1, y_{21} = 0)$.

A solution of the equation $AY = YA$, $A \in M$, $A = (a_{ij})$ is $a_{11} = a_{22}$, $a_{21} = 0$, a_{12} is arbitrary. Thus $c(Y)$ consists of all matrices $A \in M$, where $a_{21} = 0$, $a_{11} = a_{22} \neq 0$ are integers. It is easy to verify that for $B \in M(B = (b_{ij}), b_{11} = b_{21} = b_{22} = 1, b_{12} = 2)$ it does not hold $BYB^{-1} \in c(Y)$. According to 2.8. the commutator is not an ideal operation for (M, Ω) (Ω is a system of all normal subgroups in M).

2.10. Remark. A commutator is in general not an ideal operation on a closure space (G, Ω) (Ω is a system of all normal subgroups of given group G). The necessary and sufficient condition is (b) in 2.7.

3. SUBGROUPS AS IDEALS

A greatest common divisior of the integers m, n will be denoted by $(m; n)$. $Z(Z_n)$ will denote integers (modulo n).

3.1. Theorem. *Let G be a group and let (G, Ω) be a closure space such that A_x is a subgroup in G generated by A for $A \subseteq G$. A commutative operation $*$ is an ideal operation on (G, Ω) if and only if it holds:*

(1) *For every $g, h \in G$ there exist integers m, n (depending on g, h) such that $g * h = mg = nh$.*

(2) *If $A \subseteq G$, $g, h, b \in G$, $g * b = mg = nb \in A_x$, $h * b = kh = lb \in A_x$, $(g - h) * b = p(g - h) = sb$, where $s \neq 0, n, m, k, l$ are integers, then $(ml + nk; s)\, b \in A_x$.*

Proof. Let us prove that (1) is equivalent to the axiom A3 and (2) is equivalent to the axiom A4.

1. We assume that (1) holds. If $g, h \in G$, then there exist integers m, n fulfilling $g * h = mg = nh$. Therefore $(g * h)_x \subseteq g_x \cap h_x$ and by 1.8. A3 holds.

Let us suppose, conversely, that A3 holds. If $g, h \in G$ then according to 1.8. (2) we obtain $g * h \in g_x \cap h_x$. In other words, there exist the integers m, n such that $g * h = mg = nh$.

2. Let there hold $A \subseteq G$, $g, h, b \in G$, $g * b = mg = nh \in A_x$, $h * b = kh = lb \in A_x$, $(g - h) * b = p(g - h) = sb(s \neq 0,\ m,\ n,\ k,\ l,\ p \in Z)$. If (2) holds, then $(ml + nk; s)\, b \in A_x$. We can write $g, h \in A_x : b$ (according to the operation $*$). From the definition of the greatest common divisior it follows: There exists an integer q fulfilling $s = q(ml + nk; s)$. With respect to $(ml + nk;\ s)\, b \in A_x$ we have $sb = q(ml + nk; s)\, b \in A_x$. Hence $(g - h) \in A_x : b$ and by 1.7. (3) the axiom A4 is valid. Now let A4 hold. Let us suppose that $(ml + nk;\ s)\, b \notin A_x$. By 1.7. (3), $A_x : b$ is a subgroup and hence $g - h \in A_x$. Therefore $sb \in A_x$. We denote $s_0 = \min\{|s| : s \neq 0,\ s \in Z,\ sb \in A_x\}$. There exist s', $s_1 \in Z$, $|s_1| < s_0$ such that $s = s's_0 + s_1$. From here we obtain $s_1 b = sb - s's_0 b \in A_x$ which implies $s_1 = 0$. Thus s_0 is a divisior of s. In the same way we can prove that s_0 is a divisior of $ml + nk$. It is in a contradiction with $(ml + nk; s)\, b \notin A_x$. Thus it is verified A4 $\Rightarrow$ (2).

The assertion is proved.

3.2. Corollary. *Let G be a group and Ω be a system of all subgroups in G. If $*$ is a commutative ideal operation on (G, Ω), then it holds:*

(1) *An element $g(h)$ of the group G has an infinity order if and only if there exists exactly one integer $m(n)$ fulfilling $g * h = mg(g * h = nh)$.*

(2) *If $g = 0$ or $h = 0$ $(g, h \in G)$, then $g * h = 0$.*

Proof. (1) An element $g \in G$ has an infinity order if and only if $m, k \in Z$, $mg = kg \Rightarrow m = k$.

(2) It follows from 3.1. (1).

3.3. Lemma. *Let G be a group, Ω a system of all subgroups in G. Let $*$ be a commutative ideal operation on (G, Ω) with the property $g, a, h \in G \Rightarrow (g + h) * a = (g * a) + (h * a)$. Then it holds:*

If $k \in Z$, then the relationship $g, h \in G \Rightarrow g \,.\, h = k(g * h)$ defines an ideal operation on (G, Ω).

Proof. Let $s \in G$, $a \in A_x (A \subseteq G)$ be. From $s * a \in A_x$ we have $k(s * a) = s \,.\, a \in A_x$. Thus the axiom A3 holds. Assume $g, h, b \in G$, $A \subseteq G$, $g \,.\, b \in A_x$, $h \,.\, b \in A_x$. Then $(g - h) \,.\, b = k((g - h) * b) = k((g * b) + ((-h) * b)) = k(g * b) + k((-h) * b) \in A_x$. A4 holds by 1.7. (3).

Now, we shall use our results to cyclic groups.

3.4. Theorem. *Let (Z, Ω) be a closure space such that A_x is a subgroup in Z generated by A for $A \subseteq Z$. Let $k \in Z$. Then the implication $g, h \in Z \Rightarrow g * h = kgh$ defines the ideal operation on (Z, Ω).*

Proof. Let us show that $(Z, \Omega, .)$ is an ideal space. Axiom A3 holds evidently. If $A \subseteq Z$, $g, h, b \in Z$, $gb, hb \in A_x$, then there exist integers a, z_1, z_2 such that $A_x = aZ$, $gb = az_1$, $hb = az_2$. From here $(g - h)\, b = gb - hb \in A_x$ holds. According to 1.7. (3) the axiom A4 is verified. Now the assertion follows directly from 3.3.

3.5. Theorem. *Let (Z_n, Ω) be a closure space such that A_x is a subgroup in Z_n generated by A for $A \subseteq Z_n$. Let $k \in Z$. Then the implication $g, h \in Z_n \Rightarrow g * h = k(g \,.\, h)$ (multiplication . is modulo n) defines an ideal operation on (Z_n, Ω).*

Proof. Analogously as in the proof of 3.4.

3.6. Remark. It remains a question, if all ideal operations on (Z, Ω) $((Z_n, \Omega))$ are in the form used in 3.4. (3.5.). The next examples give the negative answer.

3.7. Examples.

(a) We put $a * b = |ab|$ for $a, b \in Z$.

(b) On Z_4 we define the operation $*$ as follows:

$$\{g, h\} \cap \{0\} = \emptyset \Rightarrow g * h = 2,$$
$$\{g, h\} \cap \{0\} \neq \emptyset \Rightarrow g * h = 0, \qquad (g, h \in Z_4).$$

It is easy to verify that the operation in (a) (in (b)) is an ideal operation on (Z, Ω) $((Z_4, \Omega))$.

REFERENCES

[1] Aubert, K. E.: *Theory of x-ideals*, Acta Math., vol. 107 (1962), 1—52.

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