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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF AN n-TH ORDER NONLINEAR DIFFERENTIAL EQUATION WITH DEVIATING ARGUMENT

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The present paper is devoted to the investigation of an n-th order nonlinear differential equation with deviating argument

$$(1) \qquad (r_{n-1}(t)\,(r_{n-2}(t)\,(\ldots(r_2(t)\,(r_1(t)\,y')')'\ldots)')')' + a(t)f(y(g(t))) = b(t),$$

where $a(t), b(t), g(t), r_1(t), \ldots, r_{n-1}(t)$ are continuous on $\langle t_0, \infty)$ and $f(y)$ on $(-\infty, \infty)$. In [1], sufficient conditions are given for any non-oscillatory solution $y(t)$ of (1) to converge to zero as $t \to \infty$ (Theorem 3). We shall demonstrate that it is possible to prove this theorem under weaker assumptions. In addition, there will be given further sufficient conditions for a non-oscillatory solution of (1) to converge to zero asymptotically as $t \to \infty$.

We shall assume throughout that the following conditions are satisfied:

(2) (a) $\lim\limits_{t\to\infty} g(t) = \infty$;

(b) $yf(y) > 0 \quad$ for $y \neq 0$;

(c) $a(t) \geqq 0,\ r_i(t) > 0 \quad$ for $i = 1, \ldots, n-1$.

Let us introduce the following notation:

(3) ( ) $\varrho_i(t) = \int\limits_t^\infty \frac{\varrho_{i-1}(s)}{r_i(s)}\,ds, \qquad i = 1, \ldots, n-1, (\varrho_0(t) \equiv 1)$:

(b) $\tau_j^{(i)}(t) = \int\limits_{t_0}^t \frac{\tau_{j-1}^{(i)}(s)}{r_{n-i-j+1}(s)}\,ds, \qquad i, j = 1, \ldots, n-1.$

$2 \leqq i + j \leqq n, \qquad (\tau_0^{(i)} \equiv 1)$;

(c) $G_0(t) = y(t),\ G_i(t) = r_i(t)\,G'_{i-1}(t),\ i = 1, \ldots, n-1.$

We shall consider solutions of (1) existing on $\langle t_0, \infty)$.

Theorem 1. *Let*

(4) $$\lim_{t\to\infty} \tau^{(i)}_{n-i}(t) < \infty \qquad \textit{for } i = 1, \ldots, n-1.$$

If

$$\left| \int_{t_0}^{\infty} b(t)\, dt \right| < \infty,$$

then every non-oscillatory solution $y(t)$ of (1) *is bounded on* $\langle t_0, \infty)$.

Proof. Let $y(t)$ be a non-oscillatory solution of (1). Suppose that $y(t) > 0$ for every $t \geqq t_1$. Because of (2a) there exists $t_2 \geqq t_1$ such that $g(t) \geqq t_1$ for $t \geqq t_2$. Thus $y(g(t)) > 0$ for every $t \geqq t_2$. Using (3c) and integrating (1) from t_2 to $t \geqq t_2$ we get

(5) $$G_{n-1}(t) - G_{n-1}(t_2) + \int_{t_2}^{t} a(s)\, f(y(g(s)))\, ds = \int_{t_2}^{t} b(s)\, ds.$$

Since – because of (2b) and (2c) the first integral of (5) is positive and the second one bounded, there exists a constant $K > 0$ such that

$$G_{n-1}(t) = r_{n-1}(t)\, G'_{n-2}(t) \leqq K \qquad \text{for every } t \geqq t_2.$$

Dividing the last inequality by $r_{n-1}(t)$ and integrating from t_2 to t, we get

$$G_{n-2}(t) \leqq G_{n-2}(t_2) + K \int_{t_2}^{t} \frac{1}{r_{n-1}(s)}\, ds \leqq G_{n-2}(t_2) + K\tau^{(1)}_1(t).$$

Dividing this by $r_{n-2}(t)$ and integrating from t_2 to t, we get – using (3b):

$$G_{n-3}(t) \leqq G_{n-3}(t_2) + G_{n-2}(t_2)\, \tau^{(2)}_1(t) + K\tau^{(1)}_2(t) \qquad \text{for } t \geqq t_2.$$

After $(n-3)$ successive applications of this method we get

$$G_0(t) = y(t) \leqq G_0(t_2) + G_1(t_2)\, \tau^{(n-1)}_1(t) + G_2(t_2)\, \tau^{(n-2)}_2(t) + \ldots +$$
$$+ G_{n-2}(t_2)\, \tau^{(2)}_{n-2}(t) + K\tau^{(1)}_{n-1}(t) \qquad \text{for every } t \geqq t_2.$$

Owing to the assumption (4) this means that $y(t)$ is bounded.

If $y(t) < 0$ for every $t \geqq t_1$, the proof is analogous. This completes the proof.

Theorem 2. *Let* $\lim_{t\to\infty} \varrho_i(t) = 0$, $i = 1, \ldots, n-1$, *moreover, let* (4) *and* (6) *hold,*

(6) $$\liminf_{|y|\to\infty} |f(y)| > 0.$$

If

$$\int_{t_0}^{\infty} \varrho_{n-1}(t)\, a(t)\, dt = \infty, \qquad \int_{t_0}^{\infty} |b(t)|\, dt < \infty,$$

then every non-oscillatory solution of (1) *converges to zero for* $t \to \infty$.

Proof. Since the hypotheses of Theorem 1 hold, every non-oscillatory solution of (1) is bounded on $\langle t_0, \infty)$. The proof can continue on the same lines as that of Theorem 3 of [1].

Remark. One of the consequences of Theorem 2 is that the condition

$$\int_{t_0}^{\infty} \frac{dt}{r_i(t)} < \infty \qquad \text{for } i = 1, \dots, n-1$$

in Theorem 3 of [1] can be replaced by a more general condition $\lim_{t\to\infty} \varrho_i(t) = 0$, $\lim_{t\to\infty} \tau_{n-i}^{(i)}(t) < \infty$ for $i = 1, \dots, n-1$.

Example 1. Consider the equations

(7) $$(2\sqrt{t}(t(t^2y')')')' + t^5\sqrt{t}y^3(t) = \frac{61\sqrt{t}}{t^4}, \qquad t > 0,$$

and

(8) $$(2\sqrt{t}(t(t^2y')')')' + \frac{1}{\sqrt{t}}y^7(\beta t) = \beta^{-1} \cdot \frac{\sqrt{t}}{t^8}, \qquad t > 0,$$

where β is a positive constant. In this case

$$\varrho_1(t) = \frac{1}{t}, \quad \varrho_2(t) = \frac{1}{t}, \quad \varrho_3(t) = \frac{1}{\sqrt{t}}, \quad \tau_1^{(3)}(t) = -\frac{1}{t} + \frac{1}{t_0},$$

$$\tau_2^{(2)}(t) = -\frac{\ln t}{t} + \frac{\ln t_0}{t} - \frac{1}{t} + \frac{1}{t_0},$$

$$\tau_3^{(1)}(t) = -\frac{4}{\sqrt{t}} + \sqrt{t_0}\frac{\ln t}{t} - \frac{\sqrt{t_0}}{t} + \frac{\sqrt{t_0}\ln t_0}{t} + \frac{2}{\sqrt{t_0}} + \frac{\ln t_0}{\sqrt{t_0}} - \ln\sqrt{t_0} - 1.$$

Since the hypotheses of Theorem 2 are satisfied, every non-oscillatory solution of (7) and (8) converges to zero as $t \to \infty$. The equations do have non-oscillatory solutions: $y(t) = t^{-3}$ for (7), $y(t) = t^{-2}$ for (8).

Theorem 3. *Suppose that* (6) *holds and that in addition*

(9) $$\lim_{t\to\infty} \tau_j^{(1)}(t) = \infty \qquad \text{for } j = 1, \dots, n-1.$$

If

$$\int_{t_0}^{\infty} a(t)\,dt = \infty, \qquad \left|\int_{t_0}^{\infty} b(t)\,dt\right| < \infty,$$

then, for every non-oscillatory solution of (1),

$$\liminf_{t\to\infty} |y(t)| = 0.$$

Proof. Let $y(t)$ be a non-oscillatory solution of (1). Suppose e.g. that $y(g(t)) > 0$ for $t > t_1$, and that $\liminf_{t\to\infty} y(t) = c > 0$. Then there exists $t_2 \geqq t_1$ such that $y(g(t)) > \frac{c}{2}$ for every $t \geqq t_2$. Since $f(y)$ is continuous on $(-\infty, \infty)$ and (2b) and (6) hold, there exists a constant $K > 0$ such that $f(y(g(t))) > K$ for every $t \geqq t_2$. For every $t \geqq t_2$, (1) yields

$$G'_{n-1}(t) \leqq b(t) - Ka(t).$$

Integrating this from t_2 to $t \geqq t_2$, we get

$$G_{n-1}(t) \leqq G_{n-1}(t_2) + \int_{t_2}^{t} b(s)\,ds - K \int_{t_2}^{t} a(s)\,ds.$$

By hypothesis there exists positive constant A_1 such that, for every $t \geqq t_3 \geqq t_2$,

$$G'_{n-2}(t) \leqq +A_1 \frac{1}{r_{n-1}(t)}.$$

Integrating this from t_3 to $t \geqq t_3$, we obtain the relation

$$G_{n-2}(t) \leqq G_{n-2}(t_3) - A_1 \int_{t_3}^{t} \frac{1}{r_{n-1}(s)}\,ds =$$
$$= G_{n-2}(t_3) + A_1 \int_{t_0}^{t_3} \frac{1}{r_{n-1}(s)}\,ds - A_1\tau_1^{(1)}(t).$$

Owing to (9), there exists a positive constant A_2 such that for every $t \geqq t_4 \geqq t_3$,

$$G'_{n-3}(t) \leqq -A_2 \frac{\tau_1^{(1)}(t)}{r_{n-2}(t)}.$$

By successive integrations (and using (9)), we finally obtain

$$G_0(t) = y(t) < -A_n\tau_{n-1}^{(1)}(t) \qquad \text{for every } t \geqq t_{n+1}.$$

It follows that $y(t) \to -\infty$ as $t \to \infty -$, a contradiction. Thus necessarily $\liminf_{t\to\infty} y(t) = 0$.

For $y(t) < 0$ the proof is analogous.

This completes the proof.

Example 2. Consider the equation

$$(10) \qquad (\sqrt{t}(t\sqrt{t}y'')')' + ty^3(t) = \frac{1}{2t^2}, \qquad t > 0.$$

In this case $\tau_1^{(1)}(t) = 2\sqrt{t} - 2\sqrt{t_0}$,

$$\tau_2^{(1)}(t) = 2\ln t + \frac{4\sqrt{t_0}}{\sqrt{t}} - 2\ln t_0 - 4,$$

$$\tau_3^{(1)}(t) = \int_{t_0}^{t} \frac{\tau_2^{(1)}(s)}{r_1(s)}\,ds = \int_{t_0}^{t} \tau_2^{(1)}(s)\,ds.$$

Thus the assumption (9) is satisfied as well as the other hypotheses of Theorem 3. Thus $\liminf_{t \to \infty} |y(t)| = 0$ for every non-oscillatory solution of (10), which does have a non-oscillatory solution, namely $y(t) = t^{-1}$.

REFERENCES

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