Archivum Mathematicum

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Archivum Mathematicum, Vol. 17 (1981), No. 2, 81--86

Persistent URL: `http://dml.cz/dmlcz/107095`

ARCH. MATH. 2, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XVII: 81—86, 1981

STACKBASES IN POWER SETS OF NEIGHBOURHOOD SPACES PRESERVING THE CONTINUITY OF MAPPINGS

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(Received September 29, 1980)

A generalized filter base in a set M is defined in [7] as a nonempty family of nonempty subsets of M. The generalized filter base is called a proper family in [4] and a stackbase in [3]. We shall use the last term. If (M, c_1), (M, c_2) are topological spaces determined by Kuratowski closure operations c_1, c_2 then there exist stackbases σ_1, σ_2 in M such that for each continuous mapping $f : (M, c_1) \to (M, c_2)$ whenever $X \in \sigma_2$ then $f^{-1}(X) \in \sigma_1$ or $f^{-1}(X) = \emptyset$. Further, for each $x \in M$ and each $X_2 \in \sigma_2 \cap [f(x))$ (where $[x) = \{X \subset M : x \in X\}$) there exists a set $X_1 \in \sigma_1 \cap [x)$ with $f(X_1) \subset X_2$. Moreover, the assignment $c \to \sigma$ is one-to-one. Indeed, assigning to a Kuratowski closure operation c on M the system σ of all nonempty open subsets of the topological space (M, c) we obtain that $\sigma \cap [x)$ is a neighbourhood base at the point x and the above statements follow e.g. from [6] Theorem 1.4.6. The just formulated continuity condition at $x \in M$ can be written in the form $\sigma_2 \cap [f(x)) \prec f(\sigma_1 \cap [x))$, where $\lambda_1 \prec \lambda_2$ for $\lambda_1, \lambda_2 \subset \exp M$ means, by [4], that for each $X \in \lambda_1$ there exists $Y \in \lambda_2$ with $Y \subset X$.

This note aims to show that the above described assertion does not hold in the case of neighbourhood spaces ([5], [7]) which are not topological, i.e. corresponding closure operations are the so called Fréchet–Čech closure operations ([1], [5] – satisfying the following three axioms only: 1° $c\emptyset = \emptyset$, 2° $X \subset cX$, 3° $X \subset Y$ implies $cX \subset cY$). Further we shall prove the existence of an assignment of a stackbase $\mathscr{S}(t)$ in $\exp M$ to an arbitrary Fréchet–Čech closure operation t on M with the following properties: If $t_1 \neq t_2$ then $\mathscr{S}(t_1) \neq \mathscr{S}(t_2)$ and for every continuous mapping f of the neighbourhood space (M, t_1) into the neighbourhood space (M, t_2) the corresponding self-map $\hat{f}$ of $\exp M$ satisfies the condition $\hat{f}^{-1}(X) \in \mathscr{S}(t_1) \cup \{\emptyset\}$ for each $X \in \mathscr{S}(t_2)$, consequently $\mathscr{S}(t_2) \cap [f(X)) \prec \hat{f}(\mathscr{S}(t_1) \cap [X))$ for each $X \in \exp M$, where $[X) = \{\lambda \subset \exp M : X \in \lambda\}$. In what follows we denote by $\exp' M$ the system of all non-void subsets of M (including M). The system of all Fréchet–Čech closure operations on a set M will be denoted by $\mathfrak{C}(M)$.

Definition. Suppose $\lambda_1, \lambda_2 \subset \exp M$. A self-map f of the set M is said to be (λ_1, λ_2)-continuous if for every $X \in \lambda_2$ we have $f^{-1}(X) \in \lambda_1 \cup \{\emptyset\}$.

It is to be noted that it is inessential to use in our considerations only one fixed set M instead of two sets M_1, M_2 (or spaces (M_1, t_1), (M_2, t_2)) and mappings between them. The corresponding changes are only of a formal character. Then the below constructed mapping $\mathcal{S}$ becoms in fact an object function of a functor from the category of all neighbourhood spaces and continuous mappings into the category of sets endowed with stackbases with morphisms—mappings compatible in the sense of the above definition. Hence from the main theorem there follows the proof of Proposition 5 from [2].

Proposition 1. *Let M be a set of the cardinality at least 3, $F : \mathfrak{C}(M) \to \exp' \exp' M$ a mapping such that for each pair $t_1, t_2 \in \mathfrak{C}(M)$, every cotinuous mapping $f : (M, t_1) \to (M, t_2)$ is $(F(t_1), F(t_2))$-continuous. Then F is not injective.*

Proof. Let M be a set with card $M \geqq 3$. Denote by M_1 an arbitrary three element subset of M containing elements x_1, x_2, x_3 and put $M_2 = M \setminus M_1$. Let u_1, u_2 be Fréchet–Čech closure operations on M_1 defined by $u_1\{x_i\} = \{x_i, x_{i+1}\}$, $u_2\{x_i\} = \{x_i, x_{i+2}\}$ where addition of indices is modulo 3 and $i = 1, 2, 3$. Denote by t_* the discrete topology on M_2 and put $(M, t_i) = (M_1, u_i) + (M_2, t_*)$, $i = 1, 2$ [i.e. (M, t_1) is the disjoint sum of neighbourhood spaces (M_1, u_1), (M_2, t_*) and similarly for (M, t_2)]. Evidently $t_1 \neq t_2$. Define permutations $f_1, f_2, f_3 : M \to M$ as follows: For $x \in M_2$ we put $f_i(x) = x$ and $f_i(x_i) = x_i$, if $i \neq j$ then $f_i(x_j) = x_k$, where $k \in \{1, 2, 3\}$, $i \neq k \neq j$, $i, j \in \{1, 2, 3\}$. It is easy to verify that every f_i, $i = 1, 2, 3$ is a homeomorphism of the space (M, t_1) onto (M, t_2) and also (M, t_2) onto (M, t_1) for $f_i = f_i^{-1}$, $i = 1, 2, 3$. Then these mappings are $(F(t_1), F(t_2))$-continuous as well as $(F(t_2), F(t_1))$-continuous. We are going to show $F(t_1) = F(t_2)$. Suppose $X \in F(t_2)$. If $M_1 \subset X$ or $X \subset M_2$ we have $f_i(X) = X = f_i^{-1}(X)$ for $i = 1, 2, 3$ thus $X = f_i^{-1}(X) \in F(t_1)$. Suppose card $(X \cap M_1) = 1$ and $X \cap M_1 = \{x_{j_0}\}$, $j_0 \in \{1, 2, 3\}$. Consider a mapping f_{j_0}, i.e. the permutation of M with the fixed point x_{j_0}. Then $f_{j_0}(X) = X = f_{j_0}^{-1}(X)$ hence $X \in F(t_1)$. If card $(X \cap M_1) = 2$, say $X \cap M_1 = \{x_j, x_k\}$, $j, k \in \{1, 2, 3\}$, we use the homeomorphism f_i where $i \in \{1, 2, 3\}$, $j \neq i \neq k$. Then $X = f_i^{-1}(X) \in F(t_1)$, consequently we have $F(t_2) \subset F(t_1)$. Using the equality $f_i^{-1} = f_i$ for each $i \in \{1, 2, 3\}$ we get in the same way as above $F(t_1) \subset F(t_2)$, i.e. $F(t_1) = F(t_2)$. This completes the proof.

Consider $\sigma_1, \sigma_2 \in \exp' \exp M$. In regard with [4] 1.1 we put $\sigma_1(\cup)\sigma_2 = \{S_1 \cup S_2 : S_1 \in \sigma_1, S_2 \in \sigma_2\}$. Let (M, t) be a neighbourhood space with $M \neq \emptyset$, $X \in \exp' M$. We put $\mathscr{C}_t(X) = \{\exp M\} \in \exp' \exp M$ if the set X is dense in the space (M, t), i.e. $tX = M$ and $\mathscr{C}_t(X) = \{(\sigma(\cup) \exp' X) \cup \exp' X : \sigma \in \exp' \exp (M \setminus tX)$ otherwise. Further we put

$$\mathcal{S}_M(t) = \bigcup_{X \in \exp' M} \mathscr{C}_t(X).$$

From the definition of $\mathcal{S}_M(t)$ it follows immediately $\emptyset \in \mathcal{S}_M(t)$ (for $M \neq \emptyset$), $\exp M \in \in \mathcal{S}_M(t)$ hence $\mathcal{S}_M(t) \neq \emptyset$, i.e. $\mathcal{S}_M(t)$ is a stackbase in $\exp M$.

Lemma 1. *Let t_1, t_2 be arbitrary different Fréchet–Čech closure operations on a non-void set M. Then $\mathcal{S}_M(t_1) \neq \mathcal{S}_M(t_2)$.*

Proof. There exists a non-void set $X_0 \subset M$, such that $t_1X_0 \neq t_2X_0$. Exactly one of the following cases is possible: (i) $t_1X_0 \subsetneqq t_2X_0$, (ii) $t_2X_0 \subsetneqq t_1X_0$, (iii) $t_1X_0 \parallel t_2X_0$ (i.e. t_1X_0, t_2X_0 are incomparable with respect to the set inclusion).

In the case (i) we put

$$\sigma = (\exp'(M \setminus t_1X_0)(\cup)\exp' X_0) \cup \exp' X_0 .$$

There is $\sigma \in \mathcal{S}_M(t_1)$ and $\sigma \notin \mathcal{C}_{t_2}(X_0)$. Admit $\sigma \in \mathcal{S}_M(t_2)$. Since $\emptyset \neq \sigma \neq \exp M$, there exists a non-void set $X_1 \subset M$ such that $\sigma = (\lambda(\cup)\exp' X_1) \cup \exp' X_1$, where $\emptyset \neq \neq \lambda \subset \exp(M \setminus t_2X_1)$. Since $\exp' X_1 \subset \sigma$, we have $X_1 \cap X_0 \neq \emptyset$. In the opposite case we would have $X_1 \notin \exp' X_0$ and simultaneously $X_1 \notin \exp'(M \setminus t_1X_0)(\cup)\exp' X_0$. Assume $X_1 \not\subset X_0$. Then $X_1 \cap (M \setminus X_0) \neq \emptyset$ thus there exists a nonempty set $S \subset X_1 \cap \cap (M \setminus X_0)$ with $S \notin \exp' X_0$, $S \notin \exp'(M \setminus t_1X_0)(\cup)\exp' X_0$ for $S \cap X_0 = \emptyset$. Then $S \notin \sigma$ which contradicts $S \in \exp' X_1 \subset \sigma$. Hence $X_1 \subset X_0$. Simultaneously $\exp' X_0 \subset (\lambda(\cup)\exp' X_1) \cup \exp' X_1$. If $\exp' X_0 \subset \lambda(\cup)\exp' X_1$ then for every point $x \in X_0$ we have $\{x\} = X \cup Y$ where $X \in \lambda$, $Y \subset X_1$ and thus X, Y are disjoint, $Y \neq \emptyset$. Then $x \in Y \subset X_1$, i.e. $X_0 \subset X_1$. (The same follows also from the inclusion $\exp' X_0 \subset \exp' X_1$). Hence $X_1 = X_0$, which means $\sigma \in \mathcal{C}_{t_2}(X_0)$ – a contradiction. Consequently $\sigma \notin \mathcal{S}_M(t_2)$ in the considered case. Since the case (ii) is analogous to (i) we consider (iii). Put again $\sigma = (\exp'(M \setminus t_1X_0)(\cup)\exp' X_0) \cup \exp' X_0$. Since by the above consideration the assumption $\sigma \in \mathcal{S}_M(t_2)$ implies $\sigma \in \mathcal{C}_{t_2}(X_0)$ which is impossible for $M \setminus t_1X_0 \notin \exp'(M \setminus t_2X_0)$, we have $\sigma \notin \mathcal{S}_M(t_2)$. Consequently $\mathcal{S}_M(t_1) \neq \mathcal{S}_M(t_2)$.

Lemma 2. *Let f be a continuous mapping of the neighbourhood space (M, t_1) into the neighbourhood space (M, t_2). The induced mapping $\hat{f}: \exp M \to \exp M$ is $(\mathcal{S}_M(t_1), \mathcal{S}(t_2))$-continous.*

Proof. Assume $\sigma \in \mathcal{S}_M(t_2)$. We are going to show $\hat{f}^{-1}(\sigma) \in \mathcal{S}_M(t_1)$ whenever $\hat{f}^{-1}(\sigma) \neq \emptyset$. If $\sigma = \exp M$ then $\hat{f}^{-1}(\sigma) = \exp M \in \mathcal{S}_M(t_1)$. Let $X \subset M$ be a non-empty set which is not dense in the space (M, t_2). Suppose $\sigma = (\lambda(\cup)\exp' X) \cup \cup \exp' X$, $\emptyset \neq \lambda \subset \exp(M \setminus t_2X)$. It holds $\hat{f}^{-1}(\sigma) = \{S \subset M : \hat{f}(S) = f(S) \in \sigma\}$. We show that if $\hat{f}^{-1}(\sigma) \neq \emptyset$ then there exists a non-void set $Y \subset M$ such that

(i) $\hat{f}^{-1}(\sigma) \subset (\exp(M \setminus t_1Y)(\cup)\exp' Y) \cup \exp' Y$,

(ii) if $S \in \hat{f}^{-1}(\sigma) \setminus \exp' Y$ and $T \subset Y$ is an arbitrary non-void subset then $(S \cap (M \setminus t_1Y) \cup T \in \hat{f}^{-1}(\sigma) \setminus \exp' Y$.

(iii) $\exp' Y \subset \hat{f}^{-1}(\sigma)$.

Assume $\hat{f}^{-1}(\sigma) \neq \emptyset$ and put $Y = f^{-1}(X)$, admit $Y = \emptyset$. There is a non-void set $S \in \hat{f}^{-1}(\sigma)$, thus either $f(S) \subset Y$ or;there exist non-void disjoint subsets $Q_1, Q_2 \subset M$,

$Q_1 \in \exp(M \setminus t_2X)$, $Q_2 \subset X$ such that $f(S) = Q_1 \cup Q_2$. Thus $f(S) \cap X \neq \emptyset$. On the other hand $f(S) \subset f(M)$ and $f^{-1}(X) = \emptyset$, i.e. $f(M) \cap X = \emptyset$ hence $f(S) \cap X = \emptyset$ which is a contradiction. Consequently $Y = f^{-1}(X) \neq \emptyset$. We shall prove (i). Suppose $S \in \hat{f}^{-1}(\sigma)$. Thus $f(S) \in \sigma$, i.e. either $f(S) \subset X$ or $f(S) = Q_1 \cup Q_2$, where Q_1, Q_2 are suitable subsets of M with the above mentioned properties. If $f(S) \subset X$ we have $S \subset f^{-1}f(S) \subset f^{-1}(X) = Y$, thus $S \in \exp' Y$. In the second case $S \subset f^{-1}f(S) = f^{-1}(Q_1) \cup f^{-1}(Q_2)$ and there exist non-void sets $P_i \subset f^{-1}(Q_i)$, $i = 1, 2$ such that $P_1 \cup P_2 = S$. (Indeed, since $Q_1 \cap Q_2 = \emptyset$, $Q_1 \neq \emptyset \neq Q_2$ we have $f^{-1}(Q_1) \cap f^{-1}(Q_2) = \emptyset$. $f^{-1}(Q_1) \neq \emptyset \neq f^{-1}(Q_2)$ and putting $P_i = S \cap f^{-1}(Q_i)$, $i = 1, 2$ we get the just used statement). Since the mapping $f : (M, t_1) \to (M, t_2)$ is continuous we have $f(t_1 Y) \subset t_2 f(Y) = t_2 f f^{-1}(X) = t_2(X \cap f(M)) \subset t_2 X$. Then $t_1 Y \subset f^{-1}f(t_1 Y) \subset f^{-1}(t_2 X)$, thus $M \setminus f^{-1}(t_2X) \subset M \setminus t_1 Y$. Further, with respect to the inclusion $Q_1 \subset M \setminus t_2 X$, we have $P_1 \subset f^{-1}(Q_1) \subset f^{-1}(M \setminus t_2 X) = f^{-1}(M) \setminus f^{-1}(t_2 X) = M \setminus f^{-1}(t_2 X) \subset M \setminus t_1 X$ and $P_2 \subset f^{-1}(Q_2) \subset f^{-1}(X) = Y$ which implies $S = P_1 \cup P_2 \in \exp(M \setminus t_1 Y)(\cup) \exp' Y$. Hence $S \in (\exp(M \setminus t_1 Y)(\cup) \exp' Y) \cup \exp' Y$, therefore (i) holds. We shall prove (ii). If $S \in f^{-1}(\sigma) \setminus \exp' Y$ then by (i) S belongs to $\exp(M \setminus t_1 Y)(\cup) \exp' Y$ and by the above considerations there exist non-void disjoint sets P_1, P_2 with $P_1 \subset f^{-1}(M \setminus t_2 X)$, $P_2 \subset Y$. Since $P_1 \subset M \setminus t_1 Y$ we have $S \cap (M \setminus t_1 Y) = (P_1 \cup P_2) \cap (M \setminus t_1 Y) = [P_1 \cap (M \setminus t_1 Y)] \cup [P_2 \cap (M \setminus t_1 Y)] = P_1$. Since $f(S) \in \exp' X$ implies $S \in \exp' Y$, the set $f(S) = f(P_1) \cup f(P_2)$ belongs to σ and $\sigma \setminus \exp' X = \lambda (\cup) \exp' X$ for $\lambda \subset \exp(M \setminus t_2 X)$, we have $f(P_1) \in \lambda$. Let $Q \subset Y$, be a non-void subset. There is $f(Q) \subset f(Y) = X \cap f(M) \subset X$, i.e. $f(Q) \in \exp' X$, $f((X \cap (M \setminus t_1 Y)) \cup Q) = f(P_1) \cup f(Q) \in \lambda (\cup) \exp' X = \sigma \setminus \exp' X$. This means $(S \cap (M \setminus t_1 Y)) \cup Q \in \hat{f}^{-1}(\sigma \setminus \exp' X) = f^{-1}(\sigma) \setminus \hat{f}^{-1}(\exp' X) = \hat{f}^{-1}(\sigma) \setminus \exp' Y$ since $f^{-1}(X) \neq \emptyset$ implies $f^{-1}(\exp X) = \exp' f^{-1}(X)$. The family $\hat{f}^{-1}(\sigma)$ satisfies condition (ii). The inclusion (iii) can be easily verified. Indeed, if $S \in \exp' Y$ then $f(S) \subset f(Y) = X \cap f(M)$ thus $f(S) \in \exp' X \subset \sigma$, i.e. S belongs to $\hat{f}^{-1}(\sigma)$. Now, if we put $\xi = \{S \cap (M \setminus t_1 Y) : S \in f^{-1}(\sigma) \setminus \exp' Y\}$ we get with respect to (i), (ii) and (iii) the equality

$$f^{-1}(\sigma) = (\xi (\cup) \exp' Y) \cup \exp' Y.$$

Consequently $\hat{f}^{-1}(\sigma)$ belongs to $\mathscr{S}_M(t_1)$.

Theorem. *Let M be a non-void set. There exists an injective mapping $\mathscr{S} : \mathfrak{C}(M) \to \exp' \exp' M$ of the system of all Fréchet–Čech closure operations on M into the system of all stackbases in $\exp M$ such that for every continuous mapping $f : (M, t_1) \to (M, t_2)$, with $t_1, t_2 \in \mathfrak{C}(M)$, the induced self-map $\hat{f}$ of $\exp M$ is $(\mathscr{S}(t_1), \mathscr{S}(t_2))$-continuous.*

Proof. Consider $S_M : \mathfrak{C}(M) \to \exp' \exp' M$ defined above and apply **Lemma 1** and Lemma 2.

Remark. From the above theorem it follows that for every continuous mapping of a neighbourhood space (M, t_1) into a neighbourhood space (M, t_2) and every non-empty subset $X \subset M$ we have $(\mathcal{S}(t_2) \cap [\hat{f}(X)]) \prec \hat{f}(\mathcal{S}(t_1) \cap [X])$. As the proof of this statement there can be used the proof of implications (3) $\Rightarrow$ (4) $\Rightarrow$ (2) from Theorem 1.4.6 [6].

There is a quite natural construction of a stackbase in $\exp M$ determined by a Fréchet–Čech closure operation t on M and a subset A of M as follows: For $\sigma \in \exp' \exp M$ we put $T(\sigma) = \sigma \cup \{tX : X \in \sigma\}$, $T(\emptyset) = \emptyset$ and

$$\mathcal{T}_A(t) = \{\sigma \in \exp' \exp M : A \in \exp M \setminus T(\exp M \setminus \sigma)\}.$$

It is easy to verify that $(\exp M, T)$ is a neighbourhood space moreover with the completely additive closure operation (i.e. each point $X \in \exp M$ possesses the least T-neighbourhood). Hence by [4] 1.6, 7.1 and 7.6, the stackbase $\mathcal{T}_A(t)$ is in fact a filter (the T-neighbourhood filter of A) on $\exp M$. But as the following example shows $\mathcal{T}_A$ does not preserve the continuity of mappings.

Consider the topological space (ω, t^*), where ω is the set of all positive integers and $t^*\{n\} = \{n, n+1, n+2, \ldots\}$ for $n \in \omega$, $t^*X = \bigcup_{x \in X} t^*\{x\}$ for a nonempty subset $X \subset \omega$ and $t^*\emptyset = \emptyset$. Define a mapping $f : \omega \to \omega$ by $f(n) = n$ for n even and $f(n) = n - 1$ for n odd. Evidently f is a continuous self-map of (ω, t^*). But denoting by T^* the Fréchet–Čech closure operation assigned as above to t^* we have $\omega \in \{\{1, 2\}, \omega\} = T^*\{\{1, 2\}\}$, $\hat{f}(\omega) = \{2n - 1 : n \in \omega\} \notin \{\{1\}, \omega\} = T^*\{\{1\}\} = T^*\{\hat{f}(\{1, 2\})\}$. Further see [4] 1.6 and 1.12.

It is easy to see that the stackbase $\mathcal{S}(t)$ constructed above is not a stack in general (i.e. the condition $\sigma \in \mathcal{S}(t)$, $\sigma \subset \tau \Rightarrow \tau \in \mathcal{S}(t)$ is not satisfied). Thus to assign a Fréchet-Čech closure operation $T_{\mathcal{S}(t)}$ to the stackbase $\mathcal{S}(t)$ is possible in the following way: For $A \subset M$ we put $\mathcal{V}(A) = \{\tau \subset \exp M : \exists\, \sigma \in \mathcal{S}(t) \cap [A] \text{ with } \tau \subset \sigma\}$ and $T_{\mathcal{S}(t)}(\sigma) = \{A \subset M : \sigma \cap \tau \neq \emptyset \text{ for every } \tau \in \mathcal{V}(A)\}$, where $\sigma \subset \exp M$. Then by the definition of $\mathcal{S}(t)$ and [4] 1.6, 1.12, 7.1, 7.6, we have $(\exp M, T_{\mathcal{S}(t)})$ is a neighbourhood space and for an arbitrary continuous mapping $f : (M, t_1) \to (M, t_2)$ the mapping $\hat{f}$ of the space $(\exp M, T_{\mathcal{S}(t_1)})$ into $(\exp M, T_{\mathcal{S}(t_2)})$ is also continuous.

REFERENCES

[1] Čech, E.: *Topological Spaces*, Topological Papers of Eduard Čech, Academia Prague (1968), 436—472.

[2] Chvalina, J. and Sekanina, M.: *Realizations of Closure Spaces by Set Systems.* Proc. Third Sym. Gen. Topology and its Rel. to Modern Analysis and Algebra, Prague 1971. Academia Prague (1972), 85—87.

[3] Grimeisen, G.: *Convergence Structures*, Mimeographed notes of lectures given at the University of Colorado in the Spring Semester 1970, 42 pp.

[4] Hamburg, P.: *Konvergenz in verallgemeinerten topologischen Räumen*. Portugal. Math. 35, 3 (1976), 137—168.
[5] Mamuzič, Z. P.: *Introductior to General Topology*. Nordhoff, Groningen (1963).
[6] Preuss, G.: *Allgemeine Topologie*, Springer-Verlag, Berlin—Heidelberg—New York (1975).
[7] Thampuran, D. V.: *Neighbourhood Spaces a•d Co•vergence*, Portugal. Math. 33, 1 (1974), 43—49.

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