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Best approximation and strict convexity of metric spaces

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The notion of strict convexity in metric spaces was introduced in [1] and certain existence and uniqueness theorems on best approximation in such a space were proved in [1] and [2]. In this note we take a stronger version of the notion of strict convexity and characterize such metric spaces. As a result we get the unicity theorem of best approximation 'Every convex proximinal set in a strictly convex metric space is chebyshev' and its converse.

Before proceeding to our main results, we recall few definitions:

Let \((X, d)\) be a metric space and \(x, y, z \in X\). We say that the point \(z\) is between \(x\) and \(y\) (writing \(xzy\)) if

\[
d(x, z) + d(z, y) = d(x, y).
\]

For any two points \(x, y\) of \(X\), the set

\[
\{z \in X : d(x, z) + d(z, y) = d(x, y)\},
\]

i.e. the set of all those points which lie between \(x\) and \(y\), is called the segment \([x, y]\).

A metric space \((X, d)\) is said to be convex [4] if for every two points \(x\) and \(y \in X\), there exists \(z \in X\) such that \(x \neq y \neq z\) and \(xzy\) i.e. if for every \(x, y \in X\) and for every \(t, 0 \leq t \leq 1\) there exists at least one point \(z\) such that

\[
d(x, z) = (1 - t) d(x, y) \quad \text{and} \quad d(z, y) = td(x, y).
\]

The space is said to be strongly convex [4] if such a \(z\) exists and is unique for each pair \(x\) and \(y\) of \(X\).

Thus for strongly convex metric spaces each \(t, 0 \leq t \leq 1\), determines a unique point of the segment \([x, y]\).

A strongly convex metric space \((X, d)\) is said to be strictly convex if for every \(x, y\) of \(X\) and \(r > 0\),

\[
d(x, p) \leq r, d(y, p) \leq r \implies d(z, p) < r \quad \text{unless} \quad x = y, \quad \text{where} \quad p \quad \text{is arbitrary but fixed point of} \quad X \quad \text{and} \quad z \quad \text{is any point in the open segment} \quad ]x, y[.
\]
Therefore, in a strictly convex metric space if $x$ and $y$ are any two points on the boundary of a sphere then $[x, y]$ lies strictly inside the sphere.

A subset $K$ of a metric space $(X, d)$ is said to be convex [1] if for every $x, y \in K$, any point between $x$ and $y$ is also in $K$ i.e. for each $x, y$ in $K$, the segment $[x, y]$ lies in $K$.

Let $S$ be a subset of a metric space $(X, d)$ and $z$ be a point of $S$. Let

$$S_z = \{x \in X : d(x, z) = d(x, S)\},$$

i.e. $S_z$ is the set of all those points of $X$ having $z$ as a nearest point in $S$.

$S$ is said to be proximinal if for each point $x$ in $X$ there is a point of $S$ nearest to $x$ i.e. for each $x$ in $X$ there exists at least one point $z \in S$ such that $x \in S_z$. If there is a unique such point $z$ for each $x$ in $X$ then $S$ is said to be uniquely proximinal or Chebyshev.

In [1] and [2] the conditions under which $S$ is uniquely proximinal have been studied. We have the following unicity theorem of best approximation, the proof of which is contained in Theorem 2 of [1].

**Theorem 1.** In a strictly convex metric space whenever a convex set is proximinal, it is uniquely proximinal.

In order to show that the converse of the above theorem also holds, we establish a lemma.

**Lemma.** For any two points $x, y$ in a strongly convex metric space $(X, d)$ the function

$$\Phi = \Phi_{x,y} : [x, y] \to [0, d(x, y)] \subseteq \mathbb{R},$$

taking $z \in [x, y]$ to the real number $d(x, z)$ is an isometry.

**Proof.** We can assume $x \neq y$. Let $z \in [x, y]$ and $z' \in [z, y]$. Then

$$d(x, y) = d(x, z) + d(z, y) =$$

$$= d(x, z) + d(z, z') + d(z', y) \geq d(x, z') + d(z', y) \geq d(x, y).$$

Hence

(1) $$d(x, y) = d(x, z') + d(z', y)$$

and

$$\Phi(z') = d(x, z') = d(x, z) + d(z, z') = \Phi(z) + d(z, z')$$

implying

(2) $$|\Phi(z') - \Phi(z)| = d(z, z').$$

The equality (1) shows that $z' \in [x, y]$ and implies that $[x, y]$ is convex, and the equality (2) shows that $\Phi$ is an isometry.

**Corollary.** For any two points $x, y$ in a strongly convex metric space $(X, d)$ the segment $[x, y]$ is a compact set.
The following theorem shows that the converse of the unicity Theorem (Theorem 1) is also true.

**Theorem 2.** Let \((X, d)\) be a strongly convex metric space. Then the following statements are equivalent:

(i) \(X\) is strictly convex.

(ii) For each convex set \(S\) and distinct points \(x\) and \(y\) of \(S\), \(S_x \cap S_y = \emptyset\).

(iii) Whenever a convex set is proximinal, it is uniquely proximinal.

**Proof.** (i) \(\Rightarrow\) (ii).

Let, if possible, \(S_x \cap S_y \neq \emptyset\) and let \(z \in S_x \cap S_y\). This implies
\[
d(z, x) = d(z, y) = d(z, S).
\]

Now \(x, y \in X\) and \(X\) is a convex space, therefore there exists \(q \in X\) such that \(xqy\).

\(q \in [x, y]\) and \(S\) is a convex set, therefore \(q \in S\).

Strict convexity of the space implies \(d(z, q) < d(z, S)\), which is a contradiction.

(ii) \(\Rightarrow\) (iii).

Let a convex set \(S\) be proximinal. Let \(p \in X\). Since \(S\) is proximinal, there exists \(x \in S\) such that \(p \in S_x\).

Let if possible, \(y \neq x\) be also nearest to \(p\), then \(p \in S_y\). Thus \(p \in S_x \cap S_y, x \neq y\), which is a contradiction.

(iii) \(\Rightarrow\) (i).

Let \(x \neq y\), \(p\) be points of \((X, d)\) with \(d(x, p) = d(y, p) = r\) (say). Define
\[
f : I := [0, d(x, y)] \to \mathbb{R},
\]
as
\[
f(t) = d(p, \Phi_{x, y}^{-1}(t)).
\]

Then \(f\) is continuous. Moreover, since \([x, y]\) is a compact, convex subset, the hypothesis (iii) implies that there exists no subinterval \([t_1, t_2] \subseteq I, t_1 < t_2\), such that
\[
f(t_1) = f(t_2) = \min \{f(t) : t_1 \leq t \leq t_2\}.
\]

We affirm that all interior points \(t \in ]0, d(x, y)[\) satisfy
\[
f(t) < \max f = f(0) = f(d(x, y)).
\]

Let, if possible, \(f(t_0) \geq \max f\) for some interior point. Set
\[
m' = \min \{f(t) : t \leq t_0\},
\]
\[
m'' = \min \{f(t) : t \geq t_0\}.
\]

Suppose \(m' \geq m''\). Define
\[
t_0' = \inf \{t : t \leq t_0, \min \{f(t_1) : t \leq t_1 \leq t_0\} \geq m'\},
\]
\[
t_0'' = \sup \{t : t \geq t_0, \min \{f(t_2) : t_0 \leq t_2 \leq t\} \geq m'\}.
\]
Since $f$ is continuous it follows that
\[ f(t_0') = f(t_0'') = \min \{ f(t) : t_0' \leq t \leq t_0'' \}. \]
If $t_0' < t_0''$ then $[t_0', t_0'']$ is the subinterval leading to a contradiction, if $t_0' = t_0''$ then $I = [0, d(x, y)]$ is the subinterval leading to a contradiction. If $m' \leq m''$, a contradiction can be reached in an analogous fashion.

Since $\phi$ is an isometry, (3) implies $d(z, p) < r$ for any point $z$ in the open segment $]x, y[$. Hence the space is strictly convex.

REFERENCES


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