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A NOTE ON FUNCTIONALLY COMPLETE ALGEBRAS

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The aim of this note is to prove that an algebra in a variety with a "majority polynomial" has the Interpolation Property, [2], for all k-ary functions ($k \geqq 1$ arbitrary) if and only if it has this property for all unary functions. The obtained result is a consequence of Pixley's Theorem, see [1] and [3].

Let $\mathfrak{A} = (A, F)$ be an algebra, S a finite subset of A^k and $f : S \to A$. A mapping g: $A^k \to A$ is an *interpolating mapping* if $g|_S = f$. $\mathfrak{A}$ is said to be *functionally complete* if for every integer k all functions f: $A^k \to A$ are algebraic. We say that a variety $\mathcal{V}$ has a *majority polynomial* if there exists a ternary polynomial m over $\mathcal{V}$ obeying the identities

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x.$$

A subalgebra S of the direct product $\mathfrak{A} \times \mathfrak{A}$ is called a diagonal *subalgebra* if it contains the *diagonal* $\Delta = \{(x, x);\ x \in A\}$. $\mathfrak{A} \times \mathfrak{A}$ *has no proper diagonal subalgebra* if Δ and $\mathfrak{A} \times \mathfrak{A}$ are the only diagonal subalgebras of $\mathfrak{A} \times \mathfrak{A}$. The set of all diagonal subalgebras of $\mathfrak{A} \times \mathfrak{A}$ forms a complete lattice with respect to the set inclusion. Denote by $R(a, b)$ the least diagonal subalgebra of $\mathfrak{A} \times \mathfrak{A}$ containing the pair (a, b).

Theorem. *Let $\mathcal{V}$ be a variety with a majority polynomial and $\mathfrak{A} = (A, F) \in \mathcal{V}$. The following conditions are equivalent;*

(1) For each $a_1, a_2, b_1, b_2 \in A$ *with $a_1 \neq a_2$ there exists a unary algebraic function φ over $\mathfrak{A}$ such that $b_1 = \varphi(a_1)$, $b_2 = \varphi(a_2)$.*

(2) *For any integer $k \geqq 1$ and every finite partial function f: $A^k \to A$, f has an interpolating algebraic function.*

(3) *$\mathfrak{A} \times \mathfrak{A}$ has no proper diagonal subalgebra.*

Corollary. *Let $\mathcal{V}$ be a variety with a majority polynomial* and *$\mathfrak{A} \in \mathcal{V}$ be a* finite *algebra. The following conditions are equivalent;*

(1) *$\mathfrak{A}$ is functionally complete.*

(2) *For each a_1, a_2, b_1, b_2 of $\mathfrak{A}$ with $a_1 \neq a_2$ there exists a unary algebraic function φ over $\mathfrak{A}$ such that $b_1 = \varphi(a_1)$, $b_2 = \varphi(a_2)$.*

(3) *$\mathfrak{A} \times \mathfrak{A}$ has no proper diagonal subalgebra.*

The Corollary is an immediate consequence of the Theorem; it suffices to regard every function on $\mathfrak{A}$ as a finite partial function.

Lemma. *Let a_1, a_2, b_1, b_2 be elements of an algebra $\mathfrak{A}$. $(b_1, b_2) \in R(a_1, a_2)$ if and only if there exists a unary algebraic function φ over $\mathfrak{A}$ with $b_1 = \varphi(a_1)$, $b_2 = \varphi(a_2)$.*

Proof. Let R be the set of all pairs (b_1, b_2) such that $b_1 = \varphi(a_1)$, $b_2 = \varphi(a_2)$ for some unary algebraic function φ over $\mathfrak{A}$. Evidently, $(a_1, a_2) \in R$, R contains the diagonal Δ and R is a subalgebra of $\mathfrak{A} \times \mathfrak{A}$. Thus R is a diagonal subalgebra of $\mathfrak{A} \times \mathfrak{A}$ and $R(a_1, a_2) \subseteq R$. The converse inclusion is evident.

Proof of the Theorem. (2) $\Rightarrow$ (3): Suppose R is a diagonal subalgebra of $\mathfrak{A} \times \mathfrak{A}$ different from Δ and $\mathfrak{A} \times \mathfrak{A}$. Then there exist pairs $(a_1, a_2) \in R$ with $a_1 \neq a_2$ and $(b_1, b_2) \in \mathfrak{A} \times \mathfrak{A} - R$. Since $a_1 \neq a_2$, $a_1 \to b_1$, $a_2 \to b_2$ is a finite partial function of A into A and, by (6) of Theorem 0 in [3], R is closed under φ, which is a contradiction.

(3) $\Rightarrow$ (2) is a direct consequence of (6) of Theorem 0 in [3].

(3) $\Rightarrow$ (1): Let $a_1 \neq a_2$ and $a_1, a_2, b_1, b_2 \in A$. Then, by (3), $(b_1, b_2) \in \mathfrak{A} \times \mathfrak{A} = R(a_1, a_2)$ and (1) is a conclusion of the Lemma.

(1) $\Rightarrow$ (3): Suppose R is a diagonal subalgebra of $\mathfrak{A} \times \mathfrak{A}$ different from $\triangle$ and $\mathfrak{A} \times \mathfrak{A}$. Then there exist pairs $(b_1, b_2) \in \mathfrak{A} \times \mathfrak{A} - R$ and $(a_1, a_2) \in R$ with $a_1 \neq a_2$. By (1), there exists an algebraic function φ over $\mathfrak{A}$ such that $b_1 = \varphi(a_1)$, $b_2 = \varphi(a_2)$. By (6) of Theorem 0 in [3], R is closed under φ which is a contradiction with the choosing of (b_1, b_2). Q.E.D.

Remark. If a $(d + 1)$-ary "near unanimity" polynomial (see [3]) is considered instead of the majority polynomial and $\mathfrak{A}^d$ instead of $\mathfrak{A} \times \mathfrak{A}$, the Theorem is not valid for $d \geq 3$. Namely, for $d \geq 3$, $\mathfrak{A}^d$ has a diagonal subalgebra $R = \mathfrak{A} \times \triangle_{d-1}$, where $\triangle_{d-1} = \{(x, \ldots, x) \in A^{d-1}\}$, which makes it impossible to apply the above way of proving (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (3).

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