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Association schemes. I

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ASSOCIATION SCHEMES I

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INTRODUCTION

Delsarte [4] has shown how to use association schemes introduced by Bose and Nair [1] in the coding theory. Bose, Nair [1] and Bose, Shimamoto [2] were concerned with the properties of symmetric association schemes and Bose [3] with their using in graph theory and in geometry. Further in [4] the definition of association schemes is extended to the nonsymmetric case.

The present paper is concerned with properties of association schemes especially nonsymmetric. The first chapter deals with the definition and its comparison with those of Bose [1] and Delsarte [4] and with the algebraic properties of association schemes. They are used for investigating some cases of nonsymmetric association schemes in chapters 2—4. In chapter 5 a survey of association schemes obtained by means of a computer is given.

1. Some algebraic relations for parameters of association schemes

Notation. Let R_i, R_j be binary relations on X, $a, b \in X$. Put

$$R_i^{-1}: = \{(y, x) \mid (x, y) \in R_i\},$$
$$R_i(x): = \{y \in X \mid (x, y) \in R_i\},$$
$$p_{ij}(a, b): = \mid R_i(a) \cap R_j^{-1}(b) \mid.$$

1.1. Definition. Let X be a finite set, $\mid X \mid \geqq 2$. For arbitrary natural number n let $\boldsymbol{R} = \{R_0, R_1, \dots, R_n\}$ be a system of binary relations on X. A pair $(X, \boldsymbol{R})$ will be called an *association scheme with n classes* if and only if it satisfies the axioms A1 – A4:

A1. The system $\boldsymbol{R}$ forms a partition of the set X^2 and R_0 is the diagonal relation, i.e. $R_0 = \{(x, x) \mid x \in X\}$.

A2. For each $i \in \{0, 1, \dots, n\}$ it holds $R_i^{-1} \in \boldsymbol{R}$.

A3. For each $i, j, k \in \{0, 1, \dots, n\}$ it holds

$$(x, y) \in R_k \wedge (x', y') \in R_k \Rightarrow p_{ij}(x, y) = p_{ij}(x', y').$$

Then define $p_{ij}^k := p_{ij}(x, y)$ where $(x, y) \in R_k$.

A4. For each $i, j, k \in \{0, 1, \dots, n\}$ it holds

$$p_{ij}^k = p_{ji}^k.$$

The set X will be called a *carrier* of the association scheme $(X, \boldsymbol{R})$.

Remark. Association schemes have been introduced by Bose and Nair [1]. In their definition they used the axiom

A2'. For each $i \in \{0, 1, \dots, n\}$ it holds $R_i^{-1} = R_i$ (i.e. all relations are symmetric) instead of the axiom A2 and instead of the axioms A3, A4 they used the axiom

A3'. For each triple $i, j, k \in \{0, 1, \dots, n\}$ there exists a number $p_{ij}^k = p_{ji}^k$ such that for any pair $(x, y) \in R_k$ it holds

$$p_{ij}(x, y) = p_{ij}^k.$$

Delsarte [4] has generalized association schemes by replacing the axiom A2' by the axiom A2. But with the exception of some illustrative examples he was concerned only with the schemes satisfying A2'.

As it will be shown in 1.10., a system of relations satisfying A1, A2', A3 satisfies also A4. These association schemes will be called symmetric. From the axioms A1, A2 (generalization introduced by Delsarte) and A3 there does not follow the equality $p_{ij}^k = p_{ji}^k$ in general. Theorems 2.1., 3.1., 4.1. give some sufficient conditions for the dependence of A4 on A1, A2, A3.

Example 1.

$$X = \{a, b, c, d\}, \qquad \boldsymbol{R} = \{R_0, R_1, R_2, R_3\},$$

$$R_0 = \{(a, a), (b, b), (c, c), (d, d)\}, \qquad R_1 = \{(a, b), (b, c), (c, d), (d, a)\},$$

$$R_2 = \{(a, c), (b, d), (c, a), (d, b)\}, \qquad R_3 = \{(a, d), (b, a), (c, b), (d, c)\},$$

$(X, \boldsymbol{R})$ forms an association scheme with 3 classes. (It holds $R_1 = R_3^{-1}$, $R_2 = R_2^{-1}$; $p_{00}^0 = 1, p_{11}^2 = 1, p_{22}^0 = 1, p_{33}^2 = 1, p_{01}^1 = p_{10}^1 = 1, p_{12}^3 = p_{21}^3 = 1, p_{20}^2 = p_{02}^2 = 1$, $p_{23}^1 = p_{32}^1 = 1$, and the numbers p_{ij}^k for the other values of i, j, k are equal to zero).

Notation. The expressing of an association scheme by enumeration of separate relations is much complicated (ex. 1). For an arbitrary pair $(x, y) \in X^2$ there exists one and only one $i \in \{0, 1, \dots, n\}$ such that $(x, y) \in R_i$ (A1) and so for expressing association scheme $(X, \boldsymbol{R})$ we can use the matrix (r_{xy}), $x, y \in X$ where $r_{xy} = i$ iff $(x, y) \in R_i$.

The association scheme from ex. 1 will be written as follows:

	a	b	c	d
a	0	1	2	3
b	3	0	1	2
c	2	3	0	1
d	1	2	3	0

Sometimes for illustrating an association scheme it is better to use an oriented graph whose edges are denoted by numbers $1, \ldots, n$. x and y are joined by an edge denoted by number i if and only if $(x, y) \in R_i$. The association scheme from Ex. 1. is illustrated on Fig. 1. For better illustration there are omitted edges corresponding to the relation R_0 and symmetric relations are expressed by nonoriented edges replacing the pair of opposite oriented edges denoted by the same number.

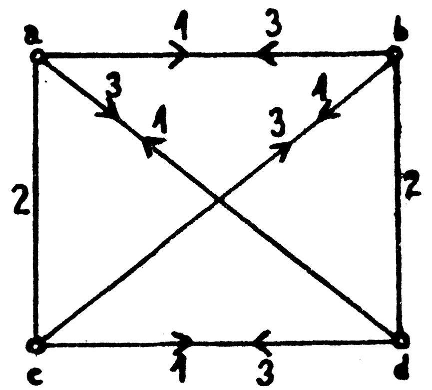

Fig. 1

Notation. Let $(X, \boldsymbol{R})$ satisfy A1, A2. Put $j = i^*$ iff $R_j = R_i^{-1}$.

1.2. Definition. Let $(X, \boldsymbol{R})$ satisfy A1, A2, A3. The number $v_i = p^0_{ii^*}$ will be called the *valency* of the relation R_i.

1.3. Definition. Let $R_i \in X^2$. R_i be called a *regular relation* if and only if there exists the number v_i, i.e. the numbers $p_{ii^*}(x, x)$ do not depend on the choise of $x \in X$.

1.4. Theorem. *Let $(X, \boldsymbol{R})$ be an association scheme with n classes. Then all relations R_i, $i \in \{0, 1, \ldots, n\}$ are regular.*

Proof follows immediately from A3.

Remark. When expressing an association scheme by matrix, the regularity of a relation R_i is expressed in such a way that in every row and every column of the matrix, the number i has the same number of occurrences.

Remark. The opposite assertion for 1.4. is not valid. For example the system

	a	b	c	d	e
a	0	1	3	3	2
b	2	0	1	3	3
c	3	2	0	1	3
d	3	3	2	0	1
e	1	3	3	2	0

is composed of regular relations and satisfies A1, A2 but does not form association scheme (e.g. there does not exist the number p^3_{11}, there hold $p_{11}(a, c) = 1, p_{11}(a, d) = = 0$).

1.5. Theorem. *Let* $(X, \boldsymbol{R})$ *satisfy* A1, A2. *Then for any* $j, k \in \{0, 1, \ldots, n\}$ *there exist the numbers* p^k_{0j}, p^k_{j0} *and it holds*

$$p^k_{0j} = \delta_{jk} = p^k_{j0}.$$

Proof: For every $x \in X$ there holds $R_0(x) = \{x\}$. Thus for any $y \in X$ there holds $p_{0j}(x, y) \leqq 1$. Simultaneously there holds $p_{0j}(x, y) > 0$ iff $x \in R_j^{-1}(y)$, i.e. $(x, y) \in R_j$. Consequently $p^j_{0j} = 1$ and $p^k_{0j} = 0$ for $k \neq j$. (The axiom A1 implies that if $(x, y) \in R_j$ then $(x, y) \notin R_k$ for $k \neq j$.)

Analogously for p^k_{j0}.

1.6. Theorem. *Let* $(X, \boldsymbol{R})$ *satisfy* A1, A2, A3. *Then for any* $i \in \{0, 1, \ldots, n\}$ *it holds*

$$v_i = v_{i*}.$$

Proof: $v_i = p^0_{ii*} = p_{ii*}(x, x) = |R_i(x) \cap R_i^{-1}(x)| = p_{i*i}(x, x) = p^0_{i*i} = v_{i*}$ $(x \in X)$.

1.7. Theorem. *Let* $(X, \boldsymbol{R})$ *satisfy* A1, A2, A3. *Then for* $i, j \in \{0, 1, \ldots, n\}$ *such that* $R_i^{-1} \neq R_j$ *it holds*

$$p^0_{ij} = 0 = p^0_{ji}.$$

Proof: Suppose $p^0_{ij} > 0$. Then for $x \in X$ there is $p_{ij}(x, x) > 0$ and thus there exists $y \in X$ such that $y \in R_i(x) \cap R_j^{-1}(x)$ consequently $(x, y) \in R_i$ and simultaneously $(x, y) \in R_j^{-1}$. From this and from A1 there follows $R_i = R_j^{-1}$, contradicting the supposition. Thus $p^0_{ij} = 0$. Analogously for p^0_{ji}.

1.8. Remark. The theorems 1.6., 1.7. can be summarized as follows:

Let $(X, \boldsymbol{R})$ *satisfy* A1, A2, A3. *Then for any* $i, j \in \{0, 1, \ldots, n\}$ *it holds*

$$p^0_{ij} = v_i \,.\, \delta_{i*j} = p^0_{ji}.$$

1.9. Theorem. *Let* (X, R) *satisfy* A1, A2, A3. *Then for any* $i, j, k \in \{0, 1, \cdots, n\}$ *it holds*

$$p_{ij}^k = p_{j^*i^*}^{k^*}.$$

Proof: Let $(x, y) \in R_k$. Then it holds $(y, x) \in R_{k^*}$ and

$$p_{ij}^k = p_{ij}(x, y) = | R_i(x) \cap R_j^{-1}(y) | = | R_{i^*}^{-1}(x) \cap R_{j^*}(y) | =$$
$$= p_{j^*i^*}(y, x) = p_{j^*i^*}^{k^*}.$$

1.10. Corollary. *Let* (X, R) *satisfy* A1, A2′, A3. *Then* (X, R) *satisfies* A4.

Proof: A2′ implies that $i^* = i$ for all $i \in \{0, 1, \ldots, n\}$. Hence for any $i, j, k \in \{1, \ldots, n\}$ it holds $p_{ij}^k = p_{j^*i^*}^{k^*} = p_{ji}^k$.

1.11. Theorem. *Let* (X, R) *satisfy* A1, A2, A3. *Then for any* $k \in \{0, 1, \cdots, n\}$ *it holds*

$$\sum_{i=0}^{n} \sum_{j=0}^{n} p_{ij}^k = | X |.$$

Proof: Let $(x, y) \in R_k$. The systems $\{R_i(x) \mid i \in \{0, 1, \ldots, n\}\}$, $\{R_j^{-1}(y) \mid j \in \{0, 1, \ldots, n\}\}$ form partitions of the set X. (A1 implies that for every $z \in X$ there exists $i \in \{0, 1, \ldots, n\}$ such that $z \in R_i(x)$ and simultaneously $R_i(x) \cap R_h(x) = \emptyset$ for $i \neq h$, analogously for R_j^{-1}.) For fixed i it holds

$$\sum_{j=0}^{n} p_{ij}^k = \sum_{j=0}^{n} | R_i(x) \cap R_j^{-1}(y) | = | R_i(x) |.$$

Hence

$$\sum_{i=1}^{n} \sum_{j=0}^{n} p_{ij}^k = \sum_{i=0}^{n} | R_i(x) | = | X |.$$

1.12. Corollary. *Let* (X, R) *satisfy* A1, A2, A3. *Then it holds*

$$\sum_{i=0}^{n} v_i = | X |.$$

Proof: In 1.11. put $k = 0$. By 1.7. all numbers p_{ij}^0, where $i \neq j^*$, are equal to zero thus $\sum_{j=0}^{n} p_{ij}^0 = p_{ii^*}^0$ and hence

$$\sum_{i=0}^{n} v_i = \sum_{i=0}^{n} p_{ii^*}^0 = \sum_{i=0}^{n} \sum_{j=0}^{n} p_{ij}^0 = | X |.$$

1.13. Theorem. *Let* (X, R) *satisfy* A1, A2, A3. *Then for any* $i, j \in \{0, 1, \ldots, n\}$ *it holds*

$$\sum_{k=0}^{n} v_k p_{ij}^k = v_i v_j.$$

Proof: Choose a fixed $x \in X$. It holds $| \{y \mid (x,y) \in R_i\} | = | R_i(x) | = v_i$. For every $y \in X$ it holds $| \{z \mid (y, z) \in R_j\} | = | R_j(y) | = v_j$. Thus

$$|\{(y, z) \mid (x, y) \in R_i, (y, z) \in R_j\}| = v_i v_j. \tag{+}$$

For fixed $z \in X$ such that $(x, z) \in R_k$ it holds

$$|\{y \mid (x, y) \in R_i, (y, z) \in R_j\}| = p_{ij}^k$$

simultaneously $|\{z \mid (x, z) \in R_k\}| = v_k$, thus

$$|\{(y, z) \mid (x, z) \in R_k, (x, y) \in R_i, (y, z) \in R_j\}| = v_k p_{ij}^k.$$

A1 implies that for every $z \in X$ there exists one and only one $k \in \{0, 1, \ldots, n\}$ such that $(x, z) \in R_k$. Hence

$$|\{(y, z) \mid (x, y) \in R_i, (y, z) \in R_j\}| =$$
$$= \sum_{k=0}^{n} |\{(y, z) \mid (x, z) \in R_k, (x, y) \in R_i, (y, z) \in R_j\}| = \sum_{k=0}^{n} v_k p_{ij}^k. \tag{++}$$

Comparing the relations (+), (+ +), we get the assertion.

1.14. Theorem. *Let* (X, R) *satisfy* A1, A2, A3. *Then for every* $i \in \{1, \ldots, n\}$ *it holds*

$$p_{ii}^i < v_i.$$

Proof: Let $(x, y) \in R_i$. Then $p_{ii}^i = |R_i(x) \cap R_i^{-1}(y)|$, simultaneously $|R_i(x)| = v_i$, $|R_i^{-1}(y)| = v_i$. Thus $p_{ii}^i \leqq v_i$. Suppose that $p_{ii}^i = v_i$. Then $R_i(x) = R_i^{-1}(y)$. Because $(x, y) \in R_i$, it holds $y \in R_i(x) = R_i^{-1}(y)$ and thus $(y, y) \in R_i$ contradicting the supposition $i \in \{1, \ldots, n\}$. Hence $p_{ii}^i < v_i$.

Remark. In the situation described in the preceding theorem, for $i = 0$ it holds $p_{00}^0 = v_0 = 1$ (according to 1.5.).

1.15. Theorem. *Let* (X, R) *satisfy* A1, A2, A3. *Then for any* $i, k \in \{0, 1, \ldots, n\}$ *it holds*

$$\sum_{j=0}^{n} p_{ij}^k = v_i.$$

Proof: Let $(x, z) \in R_k$, then

$$\sum_{j=0}^{n} p_{ij}^k = \sum_{j=0}^{n} |R_i(x) \cap R_j^{-1}(z)| = |R_i(x) \cap (\bigcup_{j=0}^{n} R_j^{-1}(z))|,$$

since **A1** implies $R_j^{-1}(z) \cap R_h^{-1}(z) = \emptyset$ for $j \neq h$ and

$$\bigcup_{j=0}^{n} R_j^{-1}(z) = X,$$

hence

$$\sum_{j=0}^{n} p_{ij}^k = |R_i(x) \cap X| = |R_i(x)| = v_i.$$

1.16. Theorem. *Let* (X, R) *satisfy* A1, A2, A3. *Then for any* $i, j, k \in \{0, 1, \ldots, n\}$ *it holds*

$$v_k p_{ij}^k = v_i p_{kj*}^i = v_j p_{i*k}^j.$$

Proof: Choose a fixed $x \in X$. It holds $|\{y \mid (x, y) \in R_k\}| = v_k$ and simultaneously for given $y \in X$ such that $(x, y) \in R_k$ it holds $|\{z \mid (x, z) \in R_i, (z, y) \in R_j\}| = p_{ij}^k$. Thus

$$|\{(y, z) \mid (x, y) \in R_k, (x, z) \in R_i, (z, y) \in R_j\}| = v_k p_{ij}^k. \qquad (+)$$

Further it holds $|\{z \mid (x, z) \in R_i\}| = v_i$ and simultaneously for a given $z \in X$ such that $(x, z) \in R_i$ it holds $|\{y \mid (x, y) \in R_k, (z, y) \in R_j\}| = p_{kj*}^i$. Thus

$$|\{(y, z) \mid (x, y) \in R_k, (x, z) \in R_i, (z, y) \in R_j\}| = v_i p_{kj*}^i. \qquad (++)$$

Comparing the relations $(+)$, $(++)$, we get $v_k p_{ij}^k = v_i p_{kj*}^i$ $\qquad (+++)$
what is the first part of the assertion.

By 1.9. it holds $v_i p_{kj*}^i = v_i p_{jk*}^{i*} = v_{i*} p_{jk*}^{i*}$ and using $(+++)$ we get $v_{i*} p_{jk*}^{i*} = v_j p_{i*k}^j$, hence $v_i p_{kj*}^i = v_j p_{i*k}^j$, being the second part of the assertion.

Remark. The theorems 1.15., 1.16. represent the generalization of the theorems proved by Bose and Nair [1] for symmetric association schemes.

The theorem 1.11. can be derived as a consequence of 1.12. and 1.15. (1.12. can be proved without using of 1.11.—Bose, Nair [1]).

1.17. Theorem. *Let X be a finite set and let there exist an areflexive symmetric relation R on X such that for any $x \in X$ there exists one and only one $y \in X$ such that $(x, y) \in R$. Then $|X|$ is an even number.*

Proof; Let $R' = R \cup \{(x, x) \mid x \in X\}$. Then R' is an equivalence (reflexivity is obvious, symmetry follows from the defining equality of R' and of the symmetry of R, transitivity: if $(x, y) \in R'$, $(y, z) \in R'$ then according to the assumption of the theorem there holds $x = y$ or $y = z$ and thus $(x, z) \in R'$). If we choose an arbitrary $x \in X$, then according to the assumptions there exists one and only one element $y \in X$, $y \neq x$ such that $(x, y) \in R'$. Thus any class of the partition of the set X corresponding to R' has two elements. The set X is the union of all the classes of the partition and therefore it has an even number of elements.

1.18. Corollary. Let $(X, \boldsymbol{R})$ satisfy A1, A2, A3 and let there exist $R_i \in \boldsymbol{R}$ such that $R_i \neq R_0$, $R_i^{-1} = R_i$, $v_i = 1$. Then $|X|$ is an even number.

Proof: R_i satisfies the assumptions of Theorem 1.17. (The assumption $R_i \neq R_0$ and the axiom A1 imply the areflexivity.)

2. Association schemes with two classes

2.1. Theorem. *Let $\boldsymbol{R} = \{R_0, R_1, R_2\}$ hold and $(X, \boldsymbol{R})$ satisfy* A1, A2, A3. *Then $(X, \boldsymbol{R})$ satisfies* A4.

Proof: In the symmetric case ($R_1 = R_1^{-1}$, $R_2 = R_2^{-1}$) the assertion holds (1.10.). Let $R_2 = R_1^{-1}$. According to 1.6. it holds:

(1) $$p^0_{12} = v_1 = v_2 = p^0_{21}.$$

According to 1.5. it holds:

(2) $$p^0_{01} = p^0_{10} = p^2_{01} = p^2_{10} = 0,$$

(3) $$p^1_{01} = p^1_{10} = 1,$$

(4) $$p^0_{02} = p^0_{20} = p^1_{02} = p^1_{20} = 0,$$

(5) $$p^2_{02} = p^2_{20} = 1.$$

According to 1.9. it is

(6) $$p^1_{12} = p^2_{12},$$

(7) $$p^1_{21} = p^2_{21}.$$

According to 1.13. it holds for $i = 1, j = 2$

(8) $$v_0p^0_{12} + v_1p^1_{12} + v_2p^2_{12} = v_1v_2$$

and for $i = 2, j = 1$

(9) $$v_0p^0_{21} + v_1p^1_{21} + v_2p^1_{21} = v_2v_1.$$

Using relations (1), (6), we get from (8)

(10) $$1 + 2p^1_{12} = v_1$$

and using (1), (7) we get from (9)

(11) $$1 + 2p^1_{21} = v_1.$$

Comparing (10), (11) we obtain

(12) $$p^1_{12} = p^1_{21}.$$

hence using (6), (7)

(13) $$p^2_{12} = p^2_{21}.$$

The relations (1), (2), (3), (4), (5), (12), (13) give the assertion.

2.2. Theorem. *Let* $\boldsymbol{R} = \{R_0, R_1, R_2\}$, $R_1^{-1} = R_2$ *hold and* $(X, \boldsymbol{R})$ *be an association scheme. Then*

14) $$p^1_{11} = \frac{|X| - 3}{4},$$

(15) $$p^1_{12} = p^1_{21} = \frac{|X| - 3}{4},$$

(16) $$p^1_{22} = \frac{|X| + 1}{4},$$

(17) $$p_{11}^{2} = \frac{|X| + 1}{4},$$

(18) $$p_{12}^{2} = p_{21}^{2} = \frac{|X| - 3}{4},$$

(19) $$p_{22}^{2} = \frac{|X| - 3}{4},$$

(20) $$v_1 = v_2 = \frac{|X| - 1}{2}.$$

Proof: Theorem 1.9. implies

(21) $$p_{11}^{1} = p_{22}^{2},$$

(22) $$p_{22}^{1} = p_{11}^{2}.$$

According to 1.12. it holds

(23) $$v_0 + v_1 + v_2 = |X|$$

and according to 1.5.

(24) $$v_0 = p_{00}^{0} = 1.$$

Substituting v_0 and v_2 to (23) by (1), (24) we get

$$1 + 2v_1 = |X|$$

and from this there follows the relation (20). According to 1.13. it holds for $i = 1$, $j = 1$

(25) $$v_0 p_{11}^{0} + v_1 p_{11}^{1} + v_2 p_{11}^{2} = v_1 v_1.$$

The relation (10) implies that

(26) $$p_{12}^{1} = \frac{v_1 - 1}{2}$$

and thus using (20) we obtain the relation (15) and using (6) the relation (18). By 1.15. we have for $i = 1$, $k = 1$

(27) $$p_{10}^{1} + p_{11}^{1} + p_{12}^{1} = v_1.$$

According to 1.7.

(28) $$p_{11}^{0} = 0.$$

From the relation (25), using (1), (22), (28), we get

(29) $$p_{11}^{1} + p_{22}^{1} = v_1.$$

From (27) by substituting according to (3) and (26), we get

$$1 + p_{11}^1 + \frac{v_1 - 1}{2} = v_1,$$

hence

(30)
$$p_{11}^1 = \frac{v_1 - 1}{2}.$$

Relations (20), (30) imply (14) and with use of (21) also (19). Substituting (30) into (29), we get

(31)
$$\frac{v_1 - 1}{2} + p_{22}^1 = v_1,$$

hence

$$p_{22}^1 = \frac{v_1 + 1}{2}$$

and using (20) we get (16), (16) and (22) imply (17).

2.3. Theorem. *Let (X, R) be a nonsymmetric association scheme with two classes (i.e. $R_1^{-1} = R_2$). Then $|X| \equiv 3 \pmod 4$.*

Proof: Suppose that $|X| \equiv 3 \pmod 4$ is not valid. Then $4 \nmid (|X| - 3)$ and (14) implies that p_{11}^1 is not an integer contradicting the definition of the number p_{ij}^k.

Example 2. An association scheme (nonsymmetric) with two classes on a carrier with cardinality 7:

0	1	2	1	2	1	2
2	0	2	2	1	1	1
1	1	0	2	1	2	2
2	1	1	0	2	2	1
1	2	2	1	0	2	1
2	2	1	1	1	0	2
1	2	1	2	2	1	0

$$p_{11}^1 = 1, \quad p_{12}^1 = p_{21}^1 = 1, \quad p_{22}^1 = 2,$$
$$p_{11}^2 = 2, \quad p_{12}^2 = p_{21}^2 = 1, \quad p_{22}^2 = 1.$$

Remark: Theorem 2.3. can be proved also without using 1.15., the same is true of the relations (15), (18). For $|X| = 7$ the relations (14), (16), (17), (19) can be proved without using 1.15., it suffices to use 1.14.

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