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INVERSE PROBLEM OF THE CLASSICAL CALCULUS OF VARIATIONS

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In its widest scope, the mentioned problem is concerned with specifying a functional, if we have some information about its differential. The problem has been solved, and recently also resolved by several authors in the particular case when the mentioned differential is completely known. Among others, it was studied in the Seminar on the calculus of variations in Brno led by D. Krupka who used certain special exterior differential forms named *Lepagian forms,* and obtained very explicite results. For more information and also for the history and recent development see [1].

The purpose of the present note is to derive the above results by elementary and extremly simple methods, and to improve some details.

1. Setting of the problem. We deal with the functional

$$F(u) = \int_{\omega} f(x, u, \ldots, u_{\alpha}, \ldots)\,\mathrm{d}x, \tag{1}$$

where $x = (x_1, \ldots, x_n) \in \omega \subset \mathbf{R}^n$, $\mathrm{d}x = \mathrm{d}x_1, \ldots, \mathrm{d}x_n$, $u = (u^1(x), \ldots, u^m(x))$, $u_{\alpha} = (\partial^{|\alpha|}u^1(x)/\partial x^{\alpha}, \ldots, \partial^{|\alpha|}u^m(x)/\partial x^{\alpha})$. Here, as usual, $\alpha = (\alpha_1, \ldots, \alpha_n)$ denotes a multiindex consisting of non-negative integers, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\partial x^{\alpha} = \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$. We suppose all functions and all boundaries of integration domains to be sufficiently smooth and our considerations will be local.

Remember that the *variation* δF of the functional F is defined by using a function $U(x, \lambda)$, $x \in \omega$, $a \leqq \lambda \leqq b$, where $U(., c) = u$ for certain c, $a \leqq c \leqq b$, and then

$$\begin{aligned} \delta F(u, \delta u) &= \frac{\mathrm{d}}{\mathrm{d}\lambda} F(U(., \lambda))|_{\lambda=0} = \\ &= \int_{\omega} \sum_{i} \varepsilon^i(x, u, \ldots, u_{\alpha}, \ldots)\,\delta u^i(x)\,\mathrm{d}x + \\ &+ \int_{\partial\omega} \sum_{i, j, \alpha'} \varepsilon_{\alpha'}^{i, j}(x, u, \ldots, u_{\alpha}, \ldots)\,\delta u_{\alpha'}^i(x)\,\mathrm{d}x^j. \end{aligned} \tag{2}$$

Here

$$\varepsilon^i = \sum_\alpha (-1)^{|\alpha|} \frac{\mathrm{d}^{|\alpha|}}{\mathrm{d}x^\alpha} \left(\frac{\partial f}{\partial u^i_\alpha} \right) \tag{3}$$

are the *Euler—Lagrange operators* related with the functional F, $\varepsilon^{i,j}_\alpha$ are certain differential operators called *boundary operators* related with F, $\mathrm{d}x^j = \mathrm{d}x_1 \ldots \mathrm{d}x_{j-1}\, \mathrm{d}x_{j+1} \ldots \mathrm{d}x_n$, $\mathrm{d}x^\alpha = \mathrm{d}x_1^{\alpha_1} \ldots \mathrm{d}x_n^{\alpha_n}$, and $\delta u^i = \partial U^i/\delta\lambda\,|_{\lambda=c}$, $\delta u^i_\alpha = \delta U^i_\alpha/\partial\lambda\,|_{\lambda=c}$.

An explicite expression of the operators $\varepsilon^{i,j}_\alpha$ will not be needed in what follows. Remind only the fact that if the highest order derivatives involved in the function f are of order s then ε^i are the operators of order at most $2s$, the operators $\varepsilon^{i,j}_\alpha$ are of order at most $2s - 1$, and $\varepsilon^{i,j}_\alpha \equiv 0$ for $|\alpha| \geqq 2s$.

Now, the *inverse problem,* we shall deal with, consists in the question to find the conditions for a-priori given differential operators $\varepsilon^1, \ldots, \varepsilon^m$ to be exactly the Euler—Lagrange operators related with an appropriate functional F.

2. Necessary conditions. Let ε^i be the Euler—Lagrange operators related with the functional (1). Then (2) holds, and by integration we get

$$\begin{aligned} F(U(.,b)) - F(U(.,a)) &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}\lambda} F(U(.,\lambda))\,\mathrm{d}\lambda = \\ &= \int_b^a \int_\omega \sum_i \varepsilon^i(x, U, \ldots, U_\alpha, \ldots) \frac{\partial U}{\partial\lambda}\,\mathrm{d}x\,\mathrm{d}\lambda + \\ &+ \int_a^b \int_{\partial\omega} \sum_{i,j,\alpha'} \varepsilon^{i,j}_{\alpha'}(x, U, \ldots, U_\alpha, \ldots) \frac{\partial U_{\alpha'}}{\partial\lambda}\,\mathrm{d}x^j \mathrm{d}\lambda. \end{aligned} \tag{4}$$

Denote $x_{n+1} = \lambda$ and let I be the interval $a \leqq x_{n+1} \leqq b$. The cartesian product $\Omega = \omega \times I \subset \mathbf{R}^{n+1}$ is a nice integration domain, and we may introduce the functional

$$R(U) = \int_\Omega \sum_i \varepsilon^i(x, U, \ldots, U_\alpha, \ldots) \frac{\partial U}{\partial x_{n+1}}\,\mathrm{d}x\,\mathrm{d}x_{n+1}. \tag{5}$$

But the relation (4) is of the form

$$R(U) = \int_{\partial\Omega} (\text{certain differential form}),$$

so we see can that the value $R(U)$ does not depend on the behaviour of the function U in inner points of the domain Ω. Consequently, all the Euler—Lagrange operators related with the functional R vanish identically:

$$\sum_\beta (-1)^{|\beta|} \frac{\mathrm{d}^{|\beta|}}{\mathrm{d}x^\beta}\, \partial\left(\sum_i \varepsilon^i \frac{\partial\ U^i}{\partial x_{n+1}} \right) \Big/ \partial U^j_\beta \equiv 0, \tag{6}$$

where $\beta = (\beta_1, \ldots, \beta_{n+1})$ is an $n + 1$-tuple of non-negative integers, $\mathrm{d}x^\beta = \mathrm{d}x_1^{\beta_1} \ldots \mathrm{d}x_{n+1}^{\beta_{n+1}}$. The variable plays an exceptional role here, and using the abbreviated notation $U^i_k = \partial U^i/\partial x_k$, $U^i_{k,l} = \partial^2 U^i/\partial x_k\,\partial x_l$, ... we may write (6) more distinctly as follows:

$$\sum_i\left(\frac{\partial\varepsilon^i}{\partial U^j}U^i_{n+1}-\left(\frac{\mathrm{d}}{\mathrm{d}x_{n+1}}\varepsilon^j+\sum_{k=1}^{n}\frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial\varepsilon^i}{\partial U^j_k}U^i_{n+1}\right)\right)+\right.$$
$$\left.+\sum_{k,l=1}\frac{\mathrm{d}}{\mathrm{d}x_k}\frac{\mathrm{d}}{\mathrm{d}x_l}\left(\frac{\partial\varepsilon^i}{\partial U^j_{k,l}}U^i_{n+1}\right)-\ldots\right)\equiv 0. \tag{7}$$

The coefficients of U^i_{n+1}, $U^i_{n+1,k}$, ... vanish separately:

$$\frac{\partial\varepsilon^i}{\partial U^j}-\left(\frac{\partial\varepsilon^j}{\partial U^i}+\sum_k\frac{\mathrm{d}}{\mathrm{d}x_k}\frac{\partial\varepsilon^i}{\partial U^j_k}\right)+\sum_{k,l}\frac{\mathrm{d}}{\mathrm{d}x_k}\frac{\mathrm{d}}{\mathrm{d}x_l}\frac{\partial\varepsilon^i}{\partial U^j_{k,l}}-\ldots\equiv 0, \tag{8}$$
$$-\left(\frac{\partial\varepsilon^j}{\partial U^i}+\frac{\partial\varepsilon^i}{\partial U^j_k}\right)+2\sum_l\frac{\mathrm{d}}{\mathrm{d}x_l}\frac{\partial\varepsilon^i}{\partial U^j_{k,l}}-\ldots\equiv 0,\ \ldots .$$

We hope that the general rule is quite clear and that there is no need to write down all relations (8), however see [1].

Note only briefly that the relations (8) which we get for the higher order derivatives are the most simple ones. For example, let the operators ε^i involve the derivatives u_α of order $|\alpha| = r$ but not of order $r+1$. Then the coefficients of $\partial U^i_\alpha/\partial x_{n+1}$ in (7) are

$$-\frac{\partial\varepsilon^j}{\partial U^i_\alpha}+(-1)^r\frac{\partial\varepsilon^i}{\partial U^j_\alpha}\equiv 0,\ |\alpha| = r.$$

At first, suppose r to be even. Then $(-1)_r = 1$, and the last identity means-that the differential form $\sum_{i,j,\alpha}\varepsilon^i\,\mathrm{d}U^j_\alpha$ is exact as a function of the highest order derivatives U^i_α, $|\alpha = r$. At second, let r be odd. Then $(-1)_r = -1$, and in the particular case, if ε^i are quasilinear operators, the last identities mean that the matrix of the coeficients of U^j_α in ε^i is skew-symmetric. Especially, this is not possible if $m = 1$, so the Euler–Lagrange operator is necessary of even order.

3. Theorem. *Let $\varepsilon^i(x, u, \ldots, u_\alpha, \ldots)$, $i = 1, \ldots, m$ be certain differential operators. Then these operators are the Euler–Lagrange operators related with an appropriate functional F is and only if all identities (8) are true.*

Proof: Necessity of the identities (8) was already proved. Reversely, let ε^i be operators satisfying (8), we have to find the functional (1) such that (2) holds with our operators ε^i and certain unspecified operators $\varepsilon^{i,j}_\alpha$. However, the boundary terms are without any influence on the Euler–Lagrange operators, thus the relation (4) suggests the formula

$$F(u) = \int_\omega\int_a^b\sum_i\varepsilon^i(x, U, \ldots, U_\alpha, \ldots)\frac{\partial U^i}{\partial\lambda}\,\mathrm{d}\lambda\,\mathrm{d}x,\ u = U(.,b), \tag{9}$$

but some caution is necessary.

We start with the simple observation that the relations (8) are equivalent to the fact that the value $R(U)$ defined by (5) depends only on the behaviour of the function U in inner points of the domain Ω. And looking for the shape of the function under the sign of the integral, we can see that $R(U)$ depends actually only on values of the functions $U(.,\lambda)$ near the boundary $\partial\omega$ and also on the functions $U(.,a)$, $U(.,b)$. So if we fix once for all the function $\bar{u} = U(.,a)$, and if we suppose $U(x,\lambda) = u(x) = \bar{u}(x)$ for all x near the boundary $\partial\omega$, then the number $F(u)$ is uniquely determined by (9). Namely, the value $F(u)$ actually depends only on the function $u = U(.,b)$. At last, observe that

$$\frac{\mathrm{d}}{\mathrm{d}b} F(U(.,b)) = \int_\omega \sum_i \varepsilon^i(x, U, \ldots, U_\alpha, \ldots) \frac{\partial U^i}{\partial \lambda} \mathrm{d}x \,|_{\lambda=b}, \tag{10}$$

an encouraging formula, in the main identical with (2).

However, so far we operate only on the restricted class of functions u with the property $u(x) = \bar{u}(x)$ for x near $\partial\omega$. But now, let us define the value $F(u)$, for an arbitrary u by the formula (1), where

$$f(x, u, \ldots, u_\alpha, \ldots) =$$
$$= \int_0^1 \sum_i \varepsilon^i(x, \bar{U}, \ldots, \bar{U}_\alpha, \ldots) \frac{\partial \bar{U}^i}{\partial \lambda} \mathrm{d}x, \quad \bar{U}(.,\lambda) = \bar{u} + \lambda(u - \bar{u}).$$

Since the function $\bar{U}$ is uniquely determined by u, the last definition of $F(u)$ is authomatically correct. In addition, it agrees with the old one in the previous case, if $u(x) = \bar{u}(x)$ near the boundary.

To complete the proof of the Theorem, it remains to verify (2) for the new, enlarged definition of the functional F. We shall do it using the result (10).

Take an arbitrary function $u(.,\lambda)$ dependent on a parameter λ in such a manner that $u(.,\lambda)$ varies only inside ω. That is, $\partial u(.,\lambda)/\partial\lambda = 0$ near $\partial\omega$, or,

$$\frac{\partial u(x,\lambda)}{\partial \lambda} \equiv 0 \quad \text{if} \quad x \notin \omega, \omega' \subset\subset \omega. \tag{11}$$

Now, modify all functions $u(.,\lambda)$ in a unique manner near the boundary $\partial\omega$ to get some functions $U(.,\lambda)$ for which $U(.,\lambda) \equiv \bar{u}(x)$ near $\partial\omega$. (For example, we may set $U(.,\lambda) = \bar{u} + \chi\,.\,(u(.,\lambda) - \bar{u})$, where χ is an auxiliary function with the property $\chi(x) \equiv 0$ for x near the boundary $\partial\omega$, $\chi(x) \equiv 1$ for $x \in \omega'$.) Then

$$F(u(.,\lambda)) - F(U(.,\lambda)) = \int_\omega (f(x, u(x,\lambda), \ldots) - f(x, U(x,\lambda), \ldots))\, \mathrm{d}x =$$
$$= \int_{\omega-\omega'} (f(x, u(x,\lambda), \ldots) - f(x, U(x,\lambda), \ldots))\, \mathrm{d}x = \text{constant}.$$

Then we have, using (10), (11),

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} F(u(.,\lambda)) = \frac{\mathrm{d}}{\mathrm{d}\lambda} F(U(.,\lambda)) =$$

$$= \int_{\omega'} \sum \varepsilon^i(x, U(x, \lambda), \dots) \frac{\partial U}{\partial \lambda} \mathrm{d}x = \int_{\omega'} \sum \varepsilon^i(x, u(x, \lambda), \dots) \frac{\partial u}{\partial \lambda} \mathrm{d}x.$$

Comparing with (2), we see that ε^i are the Euler – Lagrange operators.

4. Another approach. Theorem 3 may be proved still more easily by using two parameter families of functions and the *plane divergence theorem.* We shall only briefly outline this approach. (See also [2], where this simple idea is covert in an unnecessary apparatus of non-linear functional analysis.)

Let us set $u = u(., \lambda, \mu)$ into the functional (1), and compare the second mixed derivatives, $\partial^2 F/\partial\lambda\, \partial\mu = \partial^2 F/\partial\mu\, \partial\lambda$. Using (2), we easily get

$$\text{(12)} \qquad \frac{\partial^2 F(u(., \lambda, \mu))}{\partial\lambda\, \partial\mu} = \int_\omega \sum_i \left(\sum_{j,\alpha} \frac{\partial \varepsilon^i}{\partial u^j_\alpha} \frac{\partial u^j_\alpha}{\partial \mu} \frac{\partial u^i}{\partial \lambda} + \varepsilon^i \frac{\partial^2 u^i}{\partial\mu\, \partial\lambda} \right) \mathrm{d}x + $$
$$+ \text{ boundary term,}$$

and, by integration per partes,

$$\text{(13)} \quad \frac{\partial^2 F(u(., \lambda, \mu))}{\partial\lambda\, \partial\mu} = \int_\omega \sum_i \left(\sum_{j,\alpha} (-1)^{|\alpha|} \frac{\mathrm{d}^{|\alpha|}}{\mathrm{d}x^\alpha} \left(\frac{\partial \varepsilon^i}{\partial u^j_\alpha} \frac{\partial u^i}{\partial \lambda} \right) \frac{\partial u^j}{\partial \mu} + \varepsilon^i \frac{\partial^2 u^i}{\partial\mu\, \partial\lambda} \right) \mathrm{d}x +$$
$$+ \text{ boundary term.}$$

Then, using the duplicate of the formulae (12) with λ, μ interchanged, we obtain the property of *formal self-adjointness* of the Euler – Lagrange operators,

$$\text{(14)} \qquad \int_\omega \sum_{i,j,\alpha} \frac{\partial \varepsilon^i}{\partial u^j_\alpha} v^j_\alpha w^i \,\mathrm{d}x = \int_\omega \sum_{i,j,\alpha} \frac{\partial \varepsilon^i}{\partial u^j_\alpha} w^j_\alpha v^i \,\mathrm{d}x + \text{boundary term,}$$

where we denote $v = \partial u/\partial\lambda$, $w = \partial u/\partial\mu$, for clarity. Also, using the formulae (12) and the duplicate of (13), we have

$$\text{(15)} \quad \int_\omega \sum_{i,j,\alpha} \frac{\partial \varepsilon^i}{\partial u^j_\alpha} \frac{\partial u^j_\alpha}{\partial \mu} \frac{\partial u^i}{\partial \lambda} \mathrm{d}x = \int_\omega \sum_{i,j,\alpha} (-1)^{|\alpha|} \frac{\mathrm{d}^{|\alpha|}}{\mathrm{d}x^\alpha} \left(\frac{\partial \varepsilon^i}{\partial u^i_\alpha} \frac{\partial u^i}{\partial \mu} \right) \frac{\partial u^j}{\partial \lambda} \mathrm{d}x +$$
$$+ \text{ boundary term,}$$

and by comparing the coefficients of $\partial u^i/\partial\lambda$ on both sides, we could obtain the identities (7).

So we have derived the necessity of the conditions (4), (7). Following these lines, the sufficiency proof is also possible, but we shall not deal with it.

5. The case $\varepsilon^i \equiv 0$. The functional F is not uniquely determined by the related operators ε^i. Namely, suppose $\varepsilon^i = 0$. Then (2) is simplified to the equation

$$\text{(16)} \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda} F(U(., \lambda)) = \int_{\partial\omega} \sum_{i,j,\alpha} \varepsilon^{i,j}_\alpha(x, U(x, \lambda), \dots) \frac{\partial U^i_\alpha(x, \lambda)}{\partial \lambda} \mathrm{d}x^j.$$

$$F(U(.,b)) - F(U(.,a)) = \int_{\partial\omega} \sum g^j \,\mathrm{d}x^j, \tag{17}$$

$$g^j = \int_a^b \sum_{i,\alpha} \varepsilon_\alpha^{i,j}(x, U(x,\lambda), \dots) \frac{\partial U_\alpha^i(x,\lambda)}{\partial \lambda} \,\mathrm{d}\lambda. \tag{18}$$

Let us fix the function $\bar{u} = U(.,a)$, and set $a = 0$, $b = 1$, $U(.,\lambda) = \bar{u} + \lambda(u - \bar{u})$. Then the functions g^j are well defined even by the target function $u = U(.,1)$. Namely, $g^j = g^j(x, u(x), \dots, u_\alpha(x), \dots)$. So we have a formula

$$\begin{aligned} F(u) &= F(U(.,1)) = F(\bar{u}) + \int_{\partial\omega} \sum_j g^j \,\mathrm{d}x^j = \\ &= \int_\omega (f(x, \bar{u}(x), \dots) \,\mathrm{d}x + \int_{\partial\omega} \sum_j g^j \,\mathrm{d}x^j. \end{aligned} \tag{19}$$

Locally, there exists an $n - 1$-form φ in the variable x, for which $\mathrm{d}\varphi = f(x, \bar{u}(x), \dots) \,\mathrm{d}x$. We have proved

6. Theorem. *Let the Euler–Lagrange operators related with F all identically vanish, $\varepsilon^i = 0$. Then, locally, there exists an $n - 1$-form φ in the variable x and such that*

$$F(u) = \int_{\partial\omega} (\varphi + \sum_j g^j(x, u(x), \dots, u_\alpha(x), \dots) \,\mathrm{d}x^j),$$

for certain functions g^j. If the function f in (1) *does not depend on derivatives u_α of order higher than s, the functions g^j may be chosen in such a way that they do not involve the derivatives u_α of order $|\alpha| > 2s$.*

The last part of the theorem follows from (18) and from the remark on the order of boundary operators $\varepsilon_{i,j}^\alpha$ stated at the end of the Paragraph 1.

REFERENCES

[1] Krupka, D.: *On the local structure of the Euler—Lagrange mapping*, Proc. of the ČSSR-GDR-Polish Conference on Differential Geometry and its Applications, September 1980, Nové Město na Moravě, edited by O. Kowalski, to appear.

[2] Tonti, E.: *Variational formulation of non-linear differential eauations*. I, II, Bull. Classe Sciences Acad. R. de Belgique 55 (1969), 137—165, 262—268.

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