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AN EXTENSION OF A THEOREM OF ABIAN

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Abstract. In [1], Abian has proved a theorem to the effect that in every deleted neighborhood A of an isolated essential singularity of an analytic function f there exists a sequence of complex numbers c_n which are zeros of finite partial sums (which converge to f in A) of the corresponding Laurent series of f such that $\lim_n f(c_n) = 0$ provided 0 is not a Picard exceptional value of f. Below we extend Abian's theorem by dropping his provision, but by requiring that the isolated essential singularity of f be of finite nonintegral order. A further extension by Edrei and some open questions are mentioned in Remark 2.

Let a be an isolated essential singularity of an analytic function f and let $\Sigma_{-\infty}^{\infty} a_m(z-a)^m$ be the Laurent series (around a) of f. We recall that a is called *an isolated essential singularity of finite nonintegral order* of f if the entire function $\Sigma_{-\infty}^{-1} a_m(z-a)^{-m}$ is of finite nonintegral order [2, p. 142]. Also, in what follows, for all nonnegative integers k and p we call the function $\Sigma_{-k}^{p} a_m(z-a)^m$ a *finite partial sum* of the corresponding Laurent series of f.

Based on the above notions we prove:

Theorem 1. *Let $\Sigma_{-\infty}^{\infty} a_m z^m$ be the Laurent series of a function f which is analytic in the annulus A given by $0 < |z| < r$ and let 0 be an (isolated) essential singularity of f of finite nonintegral order.*

Then there exist a sequence of complex numbers c_n and a sequence of finite partial sums T_n of the Laurent series of f such that:

(1) $c_n \in A$ *for every* $n \in \omega$, *and* $\lim_n c_n = 0$

(2) $T_n(c_n) = 0$ *for every* $n \in \omega$ *and* $\lim_n T_n = f$ *in* A

(3) $\lim_n f(c_n) = 0$

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Proof. In what follows all annuli have their centers at 0 and are disks punctured at 0.

Also, for every positive integer n,

(4) $\quad A_n$ *denotes an annulus whose radius is* $1/n$ *of that of* A.

Now, let n be a fixed positive integer.

Since $\Sigma_{m>0}^{\infty} a_m z^m$ is an analytic function in annulus A and since it assumes 0 at $z = 0$, it is clear that given a nonnegative integer K_n, there exists an annulus B_n such that

(5) $$B_n \subseteq A_n \quad \text{and} \quad |\Sigma_{m>K_n}^{\infty} a_m z^m| < n^{-1} \quad \text{for} \quad z \in B_n.$$

Since the order of the isolated essential singularity 0 of f is finite and nonintegral, it follows from our earlier remark that $E(w)$ given by

$$E(w) = \Sigma_{-\infty}^{-1} a_m w^{-m}$$

is an entire function of finite nonintegral order. So also is $E_1(w)$ given by

$$E_1(w) = (\Sigma_{-\infty}^{-1} a_m w^{-m})\, w^{K_n} + \Sigma_0^{K_n} a_m w^{K_n - m},$$

(where K_n is the nonzero integer appearing in (5)) which is obtained from $E(w)$ by multiplying it by w^{K_n} and by adding a polynomial to the result.

But then by [2, p. 155], outside every circle, $E_1(w)$ assumes the value 0. Thus, inside every annulus, $E_2(z)$ given by

$$E_2(z) = (\Sigma_{-\infty}^{-1} a_m z^m)\, z^{-K_n} + \Sigma_0^{K_n} a_m z^{-(K_n - m)}$$

assumes the value 0. The same holds for

$$(E_2(z))\, z^{K_n} = \Sigma_{-\infty}^{K_n} a_m z^m$$

Hence, there exists b_n such that

(6) $$b_n \in B_n \quad \text{and} \quad b_n \quad \text{is a zero of} \quad \Sigma_{-\infty}^{K_n} a_m z^m$$

From (6) and the fact that $\Sigma_{-\infty}^{K_n} a_m z^m$ is a nonconstant analytic function in B_n, it follows that there exists a closed disk D_n such that

(7) $$b_n \in D_n \subseteq B_n \quad \text{and} \quad \Sigma_{-\infty}^{K_n} a_m z^m \neq 0 \quad \text{on the boundary of } D_n$$

and

(8) $$|\Sigma_{-\infty}^{K_n} a_m z^m| < n^{-1} \quad \text{for} \quad z \in D_n.$$

But then, in view of the uniform convergence of the sequence $(\Sigma_{-t}^{K_n} a_m z^m)_{t\in\omega}$ to $\Sigma_{-\infty}^{K_n} a_m z^m$ on D_n and (6), (7), from Hurwitz's theorem [2, p. 162] it follows that there exists a nonnegative integer $M_n > K_n$ and a complex number c_n such that

$$c_n \in D_n \quad \text{and} \quad c_n \text{ is a zero of } \Sigma_{-M_n}^{K_n} a_m z^m.$$

Thus,

(9) $$c_n \in D_n \quad \textit{and} \quad \Sigma_{-M_n}^{K_n} a_m c_n^m = 0.$$

Denoting the finite partial sum $\Sigma_{-M_n}^{K_n} a_m z^m$ of Laurent series $\Sigma_{-\infty}^{\infty} a_m z^m$ of f by $T_n(z)$, from (9), (7), (5) we have

(10) $$c_n \in A_n \quad \textit{and} \quad T_n(c_n) = 0$$

On the other hand, from (5), (8) we obtain

(11) $$|\Sigma_{-\infty}^{\infty} a_m c_n^m| = |f(c_n)| < 2n^{-1}.$$

Now, we let n run through 1, 2, 3, ... and we choose K_n's to form an increasing sequence of nonnegative integers. But then (10), (4) imply the existence of a sequence of complex numbers c_n in A and a sequence of finite partial sums T_n of the Laurent series of f such that $\lim_n c_n = 0$ and $T_n(c_n) = 0$ for every $n \in \omega$. Also, $\lim_n T_n = f$ in A since $M_n > K_n$. Moreover, (11), (4) imply that $\lim_n f(c_n) = 0$. Thus, (1), (2), (3) are established.

Remark 1. Let b be any complex number. Let us replace $f(z)$ in Theorem 1 by $f(z) - b$ and let us change the origin of the z-plane to a. Also, let K_n be any preassigned sequence of increasing positive integers. Then applying almost verbatim the proof of Theorem 1, we can establish the following:

Theorem 2. *Let $\Sigma_{-\infty}^{\infty} a_m(z - a)^m$ be the Laurent series of a function f which is analytic in the annulus A given by $0 < |z - a| < r$ and let a be an (isolated) essential singularity of f of finite nonintegral order. Moreover let b be a complex number and let there be preassigned an increasing sequence of nonnegative integers K_n. Then there exist a sequence of complex numbers c_n and a sequence of nonnegative integers M_n such that:*

(i) $c_n \in A_n$ *for every* $n \in \omega$ *and* $\lim_n c_n = a$,

(ii) $\Sigma_{-M_n}^{K_n} a_m(c_n - a)^m = b$ *for every* $n \in \omega$ *and* $\lim_n \Sigma_{-M_n}^{K_n} a_m(z - a)^m = f$ *in* A,

(iii) $\lim_n f(c_n) = b$.

Remark 2. There are reasons to believe that any substantial extension of Theorem 2 will entail considerable theoretical and technical difficulties. This is evidenced by a written communication [3] of Professor Albert Edrei in which he extends Theorem 2 to the case of functions with isolated essential singularities of finite order (instead of finite nonintegral order). Despite Edrei's significant result, our straightforward proof of Theorem 1 has its mathematical merits.

A remaining open question is whether or not the statement of Theorem 2 is valid without making any restriction on the order of the essential singularity.

Let d be a complex number (not necessarily equal to b). Another open question is whether or not the statement of Theorem 2 is valid without making any restriction on the order of the essential singularity, however, by requiring that $\lim_{n} f(c_n) = d$.

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