Arbitrarily traceable Eulerian graph has the Hamiltonian square
We shall deal with finite undirected graphs $G = (V, E)$ without loops and multiple edges. $V_k(G)$ is the set of vertices of degree $k$ in $G$. If $G$ is connected, then $d_G$ denotes the usual metric on $V$ and $n$ being a positive integer $G^n = (V, E^n)$ is the $n$-power (called the square for $n = 2$) of $G$, i.e. $(x, y) \in E^n$ iff $1 \leq d(x, y) \leq n$. It is known ([8], [4]) that the 3-power is always Hamiltonian. Several papers [5], [1], [2], [3] et al. were devoted to the question on hamiltonicity of the square of a connected graph. In [2] an important role of Eulerian graphs has been discovered.

In this note, we are interested in so called Eulerian graphs arbitrarily traceable from a vertex, which were introduced and studied by O. Ore in [6] (we admit no loops and no multiple edges, but this is not essential, in fact). These are such Eulerian graphs, in which an Eulerian circle can be drawn starting from a suitable vertex and obeying one rule only: to draw every edge only once. In [6] (see also [7], chapter IV, 4.5.6) it is proved that every such arbitrarily traceable graph is given by the following construction: we take a forest $G_1 = (V_1, E_1)$ without one vertex components and $v \in V_1$. We add $v$ to $G_1$ and we connect $v$ by an edge to each vertex from $V_1$ having an uneven degree in $G_1$. So we get the Eulerian graph $G$ arbitrarily traceable from the vertex $v$. We shall prove that such a graph has the Hamiltonian square. This assertion will be a corollary of an auxiliary statement on forests, generalizing the notion of the square for trees, which can be of some interest by itself.

Proposition. Let $G = (V, E)$ be a forest. There exists such an ordering of $V$ in a sequence $v_1, \ldots, v_p$ ($p = |V|$), for which

1. $v_1, v_p \in V_1(G)$,
2. for all $i \ d_G(v_i, v_{i+1}) \leq 2$ or $v_i, v_{i+1} \in V_1(G)$.

Proof (by induction on $p$). If $p = 1$, clear. Let $p > 1$. If $G$ is no tree, we can use induction for connected components of $G$. So let $G$ be a tree.

a) Let there be two vertices $v, w \in V$, which are neighbors and both of them of degrees at least 3. Delete the edge $(v, w)$ from $G$. After that $G$ decomposes in $G_1$
and $G_2$ for which $V_1(G) = V_1(G_1) \cup V_1(G_2)$. We can apply the induction assumption to $G_1$ and $G_2$.

b) If the assumption in a) is not valid in $G$ (i.e. $u$, $w$ do not exist) we take some maximal (further non-prolongable) way in $G$ with vertices $x_1, \ldots, x_k$. So $d_G(x_i, x_{i+1}) = 1$ and $x_1, x_k \in V_1(G)$. From the sequence $x_1, \ldots, x_k$ select the vertices belonging to $V_1(G) \cup V_2(G) \cup V_3(G)$. We get the sequence $\mu : x_1, x_2, \ldots, x_j, x_{j+1}, \ldots, x_{j+1}, \ldots, x_{j+k}, \ldots, x_{j+k+1}, \ldots, x_k$. Our assumption b) implies $j_1 + 1 < j_2 < \ldots$. Let $x$ be a vertex from the sequence $\mu$ belonging to $V_3(G)$. Let $G_x$ be the part of the branch in $G$ starting in $x$ contained in $[G - \{x_1, \ldots, x_k\}] \cup \{x\}$ up to the first vertex (if it exists) of degree at least 3 in $G$ (this vertex not including). I.e. $G_x$ is a way $x, x^1, x^2, \ldots, x^r$ in $G$ which is „longest possible” such that $x^1, x^2, \ldots, x^{r-1}$ are of degree 2 in $G$, the degree of $x^r$ is 2 or 1. Take now the sequence of vertices $x_1, \ldots, x_{j-1}$ and let $x', x'', \ldots, x^{(m)}$ be from $V_3(G)$ among them. Put $H_1$ to be the subgraph in $G$ with the vertices $x_1, \ldots, x_{j-1}$ and those of $G_{x'}, G_{x''}, \ldots, G_{x^{(m)}}$.

$H_1$ is a graph of the type (we put $x = x'$) Similarly $H_2, \ldots, H_{n+1}$ are defined. By [5] (and it is easily seen here) there exists an ordering of the set of vertices in $H_1$ in such a sequence that the first vertex is $x_1$, the last vertex $x_{j-1}$ and the neighbors in this sequence have in $H_1$ (i.e. in $G$) the distance at most 2. Let us denote this sequence by $\pi_1$. We get a sequence of such sequences $\pi_1, \pi_2, \ldots, \pi_{n+1}$. The end of $\pi_{n+1}$ is the vertex $x_k$ and the end of $\pi_i$ has the distance 2 from the starting vertex of $\pi_{i+1}$. Therefore the sequence $\pi = (\pi_1, \pi_2, \ldots, \pi_{n+1})$ obtained by juxtaposition of the sequences $\pi_1, \pi_2, \ldots, \pi_{n+1}$ has the property 1. and 2. from the Proposition. We now delete the vertices participating in the sequence $\pi$ and the edges incident to them from $G$. We get the forest $W$ such that $V_1(W) \subset V_1(G)$. By the induction assumption we can order the set of vertices of $W$ in a sequence $\sigma$ fulfilling 1. and 2. from the Proposition. Then the sequence $\sigma = (\pi, \sigma)$ is a required sequence.

Corollary 1. Let $G = (V, E)$ be a graph with a vertex $v$ having the following property:
$H = G - \{v\}$ is a forest and $u \in V_1(H) \Rightarrow$ the edge $(u, v)$
is in $E$. Then $G^2$ is Hamiltonian.

**Corollary 2.** Let $G$ be an Eulerian graph arbitrarily traceable from a vertex. Then $G^2$ is Hamiltonian.

Proof is an immediate consequence of Corollary 1. and Ore's construction of such graphs.

**REFERENCES**


*M. Sekánina, A. Sekaninová*
662 95 Brno, Janáčkovo nám. 2a
Czechoslovakia