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THE RICCATI DIFFERENTIAL EQUATION
WITH COMPLEX-VALUED COEFFICIENTS
AND APPLICATION TO THE EQUATION
\[ x'' + P(t) x' + Q(t) x = 0 \]

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Consider the Riccati differential equation
\[ z' = q(t) - p(t) z^2, \]
where \( q(t) \) and \( p(t) \) are certain continuous complex functions of the real variable \( t \in [t_0, \infty) \) and \( z \) is the complex variable.

The aim of the present paper is to study the asymptotic behavior of solutions of (1) supposing \( q(t) \) is "close enough" to the zero and \( p(t) \) to the complex constant different from the zero.

The basic idea is to consider (1) as a perturbation of
\[ w' = -aw^2, \]
where \( a \neq 0 \) is a complex number. The results are presented in a general form using the Ljapunov function method and comprehend some results of [1], [2] (Theorem 1, 2). The equation (1) is studied by M. Ráb in [3], [4] under the assumption \( q(t) \) is "close enough" to the non-zero complex constant.

The results will be applied to the differential equation
\[ x'' + P(t) x' + Q(t) x = 0 \]
under the corresponding assumptions on functions \( P(t) \), \( Q(t) \). This idea is used in [5] supposing \( \lim_{t \to \infty} [P^2(t) - 4Q(t)]^{1/2} = A, \text{Re } A^{1/2} > 0 \). Some results concerning these problems are generalized in [6], [7], [8], [9].

1. PRELIMINARIES

Let \( R \) or \( K \) denote the sets of all real or complex numbers, respectively. If \( z = u + iv, \ u, v \in R \), we denote \( \text{Re } z = u, \text{Im } z = v, \ z = u - iv, \ z = (zz)^{1/2} \).

In what follows we shall use "Ljapunov" functions \( W(z), \ W_j(z), \ V_j(z), \ j = 1, 2 \)
defined by
\begin{align*}
(3) \quad W(z) &= \text{Re} \left[ \frac{\bar{a}}{z} \right], \quad z \in K \setminus \{0\}, \\
(4) \quad W_1(z) &= \text{Re} \left[ \frac{(1 + i) \bar{a}}{z} \right], \quad W_2(z) = \text{Re} \left[ \frac{(1 - i) \bar{a}}{z} \right], \quad z \in K \setminus \{0\}, \\
(5) \quad V_j(z) &= |z|^j, \quad j = 1, 2, \quad z \in K,
\end{align*}
where \( a \in K \setminus \{0\} \) is fixed.

Let \( A \in K \setminus \{0\} \) and let \( \gamma \) be a real parameter, \( \gamma \neq 0 \). Then the equation
\[
\gamma = \text{Re} \left[ \frac{A}{z} \right]
\]
represents a pencil of circles not-involving the point \( z = 0 \), where the function \( \text{Re} \left[ \frac{A}{z} \right] \) is not defined. The circle \( K, \) corresponding to the value \( \gamma \) has the center \( \frac{A}{2\gamma} \) and the radius \( r = \frac{|A|}{2|\gamma|} \). The straight-line \( \text{Re} [Az] = 0 \) being the axis of the pencil corresponds to the value \( \gamma = 0 \).

Define for the real function \( U(z) \) the differentiation of \( U(z) \) with respect to (1) as follows:
\[
D_f U(t, z) = \frac{\partial U(z)}{\partial \text{Re} z} \text{Re} f(t, z) + \frac{\partial U(z)}{\partial \text{Im} z} \text{Im} f(t, z),
\]
where \( f(t, z) = q(t) - p(t) z^2 \).

Then it holds
\begin{align*}
(6) \quad D_f W(t, z) &\geq \text{Re} [\bar{a} p(t)] - \frac{|a| |q(t)|}{|z^2|}, \\
(7) \quad D_f W_j(t, z) &\geq \text{Re} [(1 \pm i) \bar{a} p(t)] - \frac{\sqrt{2} |a| |q(t)|}{|z^2|},
\end{align*}
where \( z \in K \setminus \{0\}, \quad t \in [t_0, \infty) \).

Further for \( j = 1 \) or \( j = 2 \) it holds
\[
(8) \quad j |z|^{j-1}(- |q(t)| - |z| \text{Re} [p(t) z]) \leq D_f V_j(t, z) \leq \quad \\
\leq j |z|^{j-1}(|q(t)| - |z| \text{Re} [p(t) z])
\]
where \( z \in K \setminus \{0\} \) or \( z \in K \), respectively.

**Remark 1.** Trajectories \( w(t) \) of (3) satisfying the initial condition \( w(t_0) = w_0 \neq 0 \) have the following properties:

(i) If \( \text{Im} [aw_0] \neq 0 \), then \( \text{Re} \left[ \frac{i\bar{a}}{w(t)} \right] = \gamma \), where \( \gamma \in R \setminus \{0\} \) is determined by the initial condition, for all \( t \geq t_0 \) and \( w(t) \to 0 \) as \( t \to \infty \);
(ii) If $\text{Im}[aw_0] = 0$, $\text{Re}[aw_0] > 0$, then $\text{Im}[aw(t)] = 0$ for all $t \geq t_0$ and $w(t) \to 0$ as $t \to \infty$; 
(iii) if $\text{Im}[aw_0] = 0$, $\text{Re}[aw_0] < 0$, then $\text{Im}[aw(t)] = 0$ for $t \in [t_0, \omega)$, where $\omega < \infty$, and $\lim_{t \to \omega} |z(t)| = \infty$.

The following lemmas are necessary for our later considerations.

**Lemma 1.** Let $t_* < t^*$ and let $z(t)$ be a solution of (1). Assume $a \in K \setminus \{0\}$. Suppose (i) for $t \in [t_*, t^*]$ it holds

\begin{equation}
\text{Re}[az(t)] > 0
\end{equation}

and

\begin{equation}
|z(t)| \geq |z(t_*)|
\end{equation}

(ii) for $t \in [t_*, t^*]$ and $z \in M = \{z : \text{Re}[az] > 0, |z| \geq |z(t_*)|\}$ it holds

\begin{equation}
D_jW_j(t, z) \geq 0, \quad j = 1, 2,
\end{equation}

where $W_j(z)$ is defined by (4).

Then, it holds

\[|z(t)| < 2|z(t_*)| \quad \text{for} \quad t \in [t_*, t^*].\]

**Proof.** It follows from the assumptions (9), (10), (11) that there exist $\gamma(t)$, $\gamma(t) > 0$ and $j \in \{1, 2\}$ such that $W_j(z(t)) = \gamma(t)$ for $t \in [t_*, t^*]$. By definition $W_j(z)$ we obtain

\[\frac{|z(t)|}{2} \leq r(t) \leq \frac{|z(t)|}{\sqrt{2}},\]

where $r(t)$ is the radius of the circle. This together with (10), (11) implies the statement of Lemma 1.

**Lemma 2.** Let the hypothesis of Lemma 1 be satisfied with the exception that $\text{Re}[az(t)] > 0$ and $|z(t)| \geq |z(t_*)|$ are replaced by $\text{Re}[az(t)] < 0$ and $|z(t)| \geq |z(t_*)|$, respectively. Then, it holds

\[|z(t)| < 2|z(t_*)| \quad \text{for} \quad t \in [t_*, t^*].\]

**Proof.** The proof is analogous to that of the previous lemma.

2. **MAIN RESULTS**

**Theorem 1.** Suppose

\begin{equation}
limit_{t \to \infty} q(t) = 0,\end{equation}

\begin{equation}
limit_{t \to \infty} p(t) = a,\end{equation}

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Re \[aq(t)\] \geq 0, \quad q(t) \neq 0

and

Re [ap(t)] > 0

for \( t \geq t_0 \), where \( a \in K \setminus \{0\} \).

Then every solution \( z(t) \) of (1) satisfying at \( t_1 \geq t_0 \) the condition

\[ \text{Re} [az(t_1)] \geq 0 \]

exists for all \( t \geq t_1 \) and it holds

\[ \lim_{t \to \infty} z(t) = 0. \]

Proof. Let \( z = z(t) \) be any solution of (1) satisfying (16).

First, we are going to establish domains where there occurs \( z(t) \). It follows from (13), (15) that there exist \( A > 0, B > 0 \) such that

\[ \text{Re} \left[ \frac{p(t)}{a} \right] \geq A, \quad \left| \text{Im} \left[ \frac{p(t)}{a} \right] \right| \leq B \quad \text{for } t \geq t_0. \]

Then, with respect to (14), it holds for \( t \geq t_0 \)

\[ \text{Re} [aq(t)] - \text{Re} [ap(t) z^2] \geq -A \text{Re} [a^2 z^2] - B \left| \text{Im} [a^2 z^2] \right|. \]

Define \( \Omega = \{z : -A \text{Re} [a^2 z^2] - B \left| \text{Im} [a^2 z^2] \right| > 0\} \). It is easy to see that \( \Omega \neq \emptyset \), and if \( w \in \Omega \), then \( -\text{Re} [a^2 w^2] > 0 \). Hence

\[ \text{Re} [aq(t)] - \text{Re} [ap(t) z^2] > 0 \]

for \( z \in \Omega, t \geq t_0 \), in the case \( z = 0 \) is valid (18) or

\[ \text{Re} [aq(t)] - \text{Re} [ap(t) z^2] \geq 0, \quad \text{Im} [aq(t)] - \text{Im} [ap(t) z^2] \neq 0 \]

for \( t \geq t_0 \).

That implies (i) \( \text{Re} [az'(t)] > 0 \) for \( t \geq t_1 \) such that \( z(t) \in \Omega \); (ii) \( \text{Re} [az'(t)] > 0 \) or \( \text{Re} [az'(t)] \geq 0, \text{Im} [az'(t)] \neq 0 \) for \( t \geq t_1 \) such that \( z(t) = 0 \).

This together with (16) implies

\[ \text{Re} [az(t)] \geq 0, \quad \text{Re} [az(t)] = 0 \iff \text{Im} [az(t)] = 0 \]

for all \( t \geq t_1 \) for which there exists \( z(t) \).

Choose "Ljapunov" functions \( W_j(z) \) defined by (4). Then there exists \( \gamma(t) > 0, j \in \{1, 2\} \) such that \( \gamma(t) = W_j(z(t)) \) for \( z(t) \neq 0, t \geq t_1 \). In view of (13), (15) we infer from (7) and (19) that \( z(t) \) is bounded for all \( t \geq t_1 \) for which there exists \( z(t) \).

From the fact that each limit point of the set \( M = \{(t, z(t)), t \geq t_1 \} \) is on the boundary of the domain on which the right-hand side of (1) is continuous, it follows that \( z(t) \) exists for all \( t \geq t_1 \).
Now, it remains to prove (17). Let \( \varepsilon > 0 \) be arbitrary. From (12), (13) there follows the existence of \( T = T(\varepsilon) \) such that for all \( t \geq T \) it holds

\[
\text{Re} \left[ (1 \pm i) \bar{a}p(t) \right] \geq \frac{2}{3} |a|^2,
\]

\[
|q(t)| \leq \frac{|a| \varepsilon^2}{12}.
\]

With respect to (7) we receive \( D_j W_j(t, z) > 0 \) for \( t \geq T, \ |z| \geq \frac{\varepsilon}{2} \).

Put \( J = \left\{ t \geq T : |z(t)| \geq \frac{\varepsilon}{2} \right\} \). Suppose \( J \neq \emptyset \). Then there exists \( \tau = \tau(\varepsilon) \) such that \( |z(\tau)| < \frac{\varepsilon}{2} \). We claim \( |z(t)| < \varepsilon \) for all \( t \geq \tau \). If this were not true, there would exist a \( t^* > \tau \) such that \( |z(t^*)| \geq \varepsilon \), and define \( t_2 = \sup \left\{ t \in [\tau, t^*] : |z(t)| < \frac{\varepsilon}{2} \right\} \). Clearly \( t^* > t_2 > \tau \). Then,

\[
|z(t_2)| = \frac{\varepsilon}{2}, \quad |z(t)| \geq \frac{\varepsilon}{2} \quad \text{for} \quad t \in [t_2, t^*].
\]

Since \( [t_2, t^*] \subset J \), we have \( D_j W_j(t, z) > 0, j = 1, 2 \), for \( t \in [t_2, t^*] \) and \( z \in M = \{ z : |z| \geq |z(t_2)| \} \). Using Lemma 1 we obtain

\[
|z(t)| < 2 \frac{\varepsilon}{2} = \varepsilon \quad \text{for} \quad t \in [t_2, t^*],
\]

which contradicts \( |z(t^*)| \geq \varepsilon \). The proof is complete.

**Theorem 2.** Let the assumptions of Theorem 1 be satisfied with the exception (12) is replaced by

\[
(20) \quad \int_{t_0}^{\infty} |q(t)| \, dt < \infty
\]

and suppose in addition

\[
(21) \quad \text{Im} \left[ \bar{a}p(t) \right] \equiv 0 \quad \text{for} \quad t \geq t_0.
\]

Then, the conclusion of Theorem 1 is valid.

**Proof.** Let \( z = z(t) \) be any solution of (1) satisfying (16). To prove the boundedness and existence of \( z(t) \) choose \( V_j(z) \). In the proof of Theorem 1 we obtained (19) from (13), (14), (15) and (16). In addition it follows from (21)

\[
\text{Re} \left[ p(t) z(t) \right] = \text{Re} \left[ \frac{p(t)}{a} \right] \text{Re} \left[ az(t) \right],
\]

thus with respect to (15) and (19) it holds

\[
(22) \quad \text{Re} \left[ p(t) z(t) \right] \geq 0, \quad \text{Re} \left[ p(t) z(t) \right] = 0 \iff z(t) = 0
\]

for all \( t > t_1 \) for which there exists \( z(t) \).
Integrating the second inequality of (8), where $z = z(t)$, from $t_2 \geq t_1$ to $t$ we get according to (20)

$$V_1(z(t)) \leq V_1(z(t_2)) + \text{const}$$

for $t \geq t_2$ such that $z(t) \neq 0$. From the same reason as in the previous proof it follows that $z(t)$ is defined for all $t \geq t_1$.

First we are going to show $\liminf \mid z(t) \mid = 0$. Suppose for the sake of argument, that there exists an $\varepsilon > 0$ such that $\mid z(t) \mid \geq \varepsilon$ for $t \geq t_2 \geq t_1$. According the assumption (13) there exists $t_3 \geq t_2$ such that $\text{Re}\left[\overline{a}p(t)\right] \geq \frac{2}{3} \mid a \mid^2$. Choosing the function $W(z)$ and integrating (6), where $z = z(t) \neq 0$, from $t_3 \geq t_2$ to $t$ we obtain

$$W(z(t)) \geq W(z(t_3)) + \frac{2}{3} \mid a \mid^2(t - t_3) - \frac{\mid a \mid^2}{\varepsilon^2} \int_{t_3}^{t} \mid q(s) \mid ds,$$

$W(z(t)) \to \infty$ for $t \to \infty$, a contradiction.

Now, let us prove (17). Choose the function $V_2(z)$. There exists a sequence $\{s_n\}$, $s_n \to \infty$ such that for arbitrary $\varepsilon > 0$ there exists $n_1 \in N$ such that $V_2(z(s_n)) < \frac{\varepsilon}{2}$ for $n \geq n_1$. There exists a $L > 0$ such that $\mid z(t) \mid \leq L$ for $t \geq t_1$ and $n_2 \in N$ such that for $n \geq n_2$ it holds

$$\int_{s_n}^{\infty} \mid q(s) \mid ds < \frac{\varepsilon}{4L}.$$

Let $n_3 = \max(n_1, n_2)$. Using (8) we get

$$V_2(z(t)) \leq V_2(z(s_n)) + 2 \int_{s_n}^{t} \mid q(s) \mid \mid z(s) \mid ds - 2 \int_{s_n}^{t} \mid z(s) \mid^2 \text{Re}\left[p(s)z(s)\right] ds,$$

for $t \geq s_n$, $n \geq n_3$ and with respect to (22)

$$V_2(z(t)) < \varepsilon \quad \text{for} \quad t \geq s_n.$$

The proof is complete.

**Theorem 3.** Let the assumptions of Theorem 1 be fulfilled.

Let $z(t)$ be a complete solution of (1) defined on $[t_1, \omega)$, where $t_1 \geq t_0$.

If $\omega = \infty$, then

$$\lim_{t \to \infty} z(t) = 0.$$  \hspace{1cm} (23)

If $\omega < \infty$, then $\text{Re}\left[az(t)\right] < 0$ for $t \in [t_1, \omega)$ and

$$\lim_{t \to \omega} \mid z(t) \mid = \infty.$$

**Proof.** Let $z(t)$ be any solution of (1) defined on $[t_1, \omega)$. If $z(t)$ satisfies at $T \geq t_1$ the condition $\text{Re}\left[az(T)\right] \geq 0$, then by Theorem 1 there hold $\omega = \infty$ and (23).

Now, let $\text{Re}\left[az(t)\right] < 0$ be for $t \in [t_1, \omega)$. If $\omega < \infty$, then $\lim_{t \to \omega} \mid z(t) \mid = \infty$. 

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Let $\omega = \infty$. Suppose by contradiction that (23) is not satisfied. Then, there exists a $K > 0$ such that $\limsup_{t \to \infty} |z(t)| \geq 3K$. From (12), (13) it follows that there exists $T_1(K) = T_1 \geq t_1$ such that

$$
|q(t)| \leq \frac{|a|K^2}{3}
$$

$$
\text{Re}[(1 \pm i) \bar{p}(t)] \geq \frac{2}{3} |a|^2
$$

$$
\text{Re}[\bar{p}(t)] \geq \frac{2}{3} |a|^2
$$

for $t \geq T_1$. From the definition of the superior limit it follows that there exists $T_2 \geq T_1$ such that

$$
|z(T_2)| \geq 2K.
$$

Using Lemma 2 it is not difficult to see that

$$
(24) \quad |z(t)| \geq K \quad \text{for } t \geq T_2.
$$

Finally, choose the pencil of circles $W(z) = \gamma, \gamma < 0$ covering the half-plane $\text{Re}[az] < 0$. With respect to (24) there exists $\gamma_0 < 0$ so that $W(z(t)) \geq \gamma_0$ for $t \geq T$. To each point of the domain $\text{Re}[az] < 0$, $W(z) \geq \gamma_0$ there exists a unique circle $W(z) = \gamma, \gamma \in [\gamma_0, 0)$ passing through it.

According to (6) it holds

$$
D_t W(t, z(t)) \geq \frac{2}{3} |a|^2 - \frac{|a|^2 K^2}{3K^2} = \frac{1}{3} |a|^2.
$$

Integrating this inequality from $T \geq T_2$ to $t$ we get

$$
W(z(t)) \geq W(z(T)) + \frac{1}{3} |a|^2 (t - T) \to \infty \quad \text{as } t \to \infty
$$

which contradicts the fact that $\text{Re}[az(t)] < 0$ for $t \in [t_1, \infty)$.

Since in the case $\omega = \infty$ it holds (23) and the proof is complete.

**Theorem 4.** Let the assumptions of Theorem 2 be fulfilled.

Let $z(t)$ be a complete solution of (1) defined on $[t_1, \omega)$, where $t_1 \geq t_0$.

Then, the conclusion of Theorem 3 is valid.

**Proof.** The scheme of the proof is in the main the same as that used in the proof of Theorem 3 and thus it will be omitted here.

**Theorem 5.** Suppose in addition to the assumptions stated in Theorem 2 that

$\text{Re}p(t), \text{Im}p(t)$ are monotonic.

Then, each solution $z(t)$ of (1) defined for all $t \geq t_1 \geq t_0$ satisfies for $\alpha \geq 2$

$$
(25) \quad \int_{t_1}^{\infty} |z(t)|^\alpha \, dt < \infty.
$$
Proof. According to Theorem 4 it holds \( \lim_{t \to \infty} z(t) = 0 \). Consider circles \( V_1(z) = \gamma; \gamma > 0 \). Put \( \mathcal{M} = \{ t \geq t_1, z(t) \neq 0 \}, \mathcal{M}_0 = [t_1, \infty) \). According to (8) for \( t \in \mathcal{M} \) it holds

\[
- | q(t) | - | z(t) | \Re [p(t) z(t)] \leq D_f V_1(t, z(t)) = V_1'(z(t)) \leq | q(t) | - | z(t) | \Re [p(t) z(t)].
\]

Let \( \tau \geq t_1 \) be such that \( z(\tau) = 0 \). Then

\[
D^+ V_1'(z(\tau)) = \lim_{t \to \tau^+} \frac{|z(t)|}{t - \tau} = | z'(\tau) | = | q(\tau) |,
\]

\[
D^- V_1'(z(\tau)) = \lim_{t \to \tau^-} \frac{|z(t)|}{t - \tau} = - | q(\tau) |,
\]

e.g. \( V_1'(z(\tau)) \) does not exist, as \( q(t) \neq 0 \) for \( t > t_0 \). The set \( \mathcal{M}_0 \setminus \mathcal{M} \) is, as known, at most countable.

Define

\[
B(t) = \begin{cases} 
V_1'(z(t)) & t \in \mathcal{M} \\
0 & t \in \mathcal{M}_0 \setminus \mathcal{M}.
\end{cases}
\]

For \( s \in \mathcal{M}_0 \) it holds

\[
(26) \quad - | q(t) | - | z(t) | \Re [p(t) z(t)] \leq B(t) \leq \Re \left[ \frac{p(t)}{z(t)} \right].
\]

The function \( B(t) \) is continuous on \( \mathcal{M} \). Denote \( \mathcal{M}_1 = \{ t \geq t_1 : B(t) \) is not continuous \}. Since \( \mathcal{M}_1 \subset \mathcal{M}_0 \setminus \mathcal{M} \) is valid, \( \mathcal{M}_1 \) is at most countable and thus

\[
\int_{t_1}^t B(s) \, ds = V_1(z(t)) - V_1(z(t_1)), \quad t \geq t_1.
\]

Consequently integrating the inequality (26) we get

\[
- \int_{t_1}^t | q(s) | \, ds - \int_{t_1}^t | z(s) | \Re [p(s) z(s)] \, ds \leq V_1(z(t)) - V_1(z(t_1)) \leq \int_{t_1}^t | q(s) | \, ds - \int_{t_1}^t | z(s) | \Re [p(s) z(s)] \, ds.
\]

From the proof of Theorem 2 it follows either that \( \Re [p(t) z(t)] < 0 \) for \( t \geq t_1 \), or there exists \( \tau \geq t_1 \) such that \( \Re [p(t) z(t)] > 0 \) for \( t \geq \tau \). Hence,

\[
(27) \quad \int_{t_1}^\infty | z(t) | | \Re [p(t) z(t)] | \, dt < \infty.
\]

According to (13), (15) it follows from (27)

\[
(28) \quad \int_{t_1}^\infty \Re^2 [p(t) z(t)] \, dt < \infty.
\]

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Integration the equation (1) from \( t_1 \) to \( t \), \( t \to \infty \), we receive

\[
\left| \int_{t_1}^{\infty} p(t) z^2(t) \, dt \right| < \infty.
\]

Hence there exist integrals

\[
\begin{align*}
\int_{t_1}^{\infty} \text{Re} \, p(t) \text{Re} \, [p(t) z^2(t)] \, dt, \\
\int_{t_1}^{\infty} \text{Im} \, p(t) \text{Im} \, [p(t) z^2(t)] \, dt.
\end{align*}
\]

It holds \( \text{Re} \, [u] \, \text{Re} \, [uz^2] - \text{Re}^2 \, [uz] = -|u|^2 \text{Im}^2 z, \) \( \text{Im} \, [u] \, \text{Im} \, [uz^2] + \text{Re}^2 \, [uz] = |u|^2 \text{Re}^2 z \). Using (28), (29) we get

\[
\begin{align*}
\int_{t_1}^{\infty} |p(t)|^2 \text{Im}^2 z(t) \, dt < \infty, \\
\int_{t_1}^{\infty} |p(t)|^2 \text{Re}^2 z(t) \, dt < \infty,
\end{align*}
\]

therefore

\[
\int_{t_1}^{\infty} |p(t)|^2 |z(t)|^2 \, dt < \infty.
\]

Thus, with respect to (13), (15) it holds

\[
\int_{t_1}^{\infty} |z(t)|^2 \, dt < \infty,
\]

and with respect to (17) the inequality (25) is proved. The proof is complete.

Remark 2. Choose in the equation (1) the functions

\[
p(t) \equiv 1, \quad q(t) = \frac{1}{\sqrt{t}} - \frac{1}{\alpha t \sqrt{t}}, \quad t \geq t_0 > \frac{1}{\alpha},
\]

where if \( \alpha \geq 2 \) or \( 1 < \alpha < 2 \), then the assumptions of Theorem 1 or Theorem 5, respectively, are fulfilled. Thus the solution \( z(t) = \frac{1}{\sqrt{t}} \) for \( t > \frac{1}{\alpha} \) does not satisfy (25).

This example shows the invalidity of the assertion of Theorem 5 under the assumptions of Theorem 1 and the invalidity of Theorem 5 for \( 1 < \alpha < 2 \).

### 3. APPLICATIONS

Using some results concerning solutions of the Riccati differential equation we establish asymptotic behaviour of the equation

\[
x'' + P(t) x' + Q(t) x = 0,
\]

where \( P(t) \) and \( Q(t) \) are complex functions of the real variable \( t \in J = [t_0, \infty) \) and \( x \) is the complex variable.
Remark 3. Let

\[ P(t) \in C^1(J), \quad Q(t) \in C^0(J). \]

(i) If \( x(t) \) is a solution of (30) on an interval \( J_0 \subset J \) and \( x(t) \neq 0 \) on \( J_0 \), then the function

\[ z(t) = x'(t) x^{-1}(t) + \frac{1}{2} P(t) \]

is a solution of the equation

\[ z' = \frac{1}{4} P^2(t) - Q(t) + \frac{1}{2} P'(t) - z^2 \]

on \( J_0 \).

(ii) If \( z(t) \) is a solution of (32) on \( J_0 \subset J \) and \( \beta \in J_0 \) then the function

\[ x(t) = \exp \int_{\beta}^{t} \left( z(s) - \frac{1}{2} P(s) \right) ds \]

is a solution of (30) on \( J_0 \).

Successive corollaries immediately follow from Theorem 1–5 and Remark 4.

Corollary 1. Suppose (31) and

\[ \lim_{t \to \infty} (P^2(t) - 4Q(t) + 2P'(t)) = 0, \]

\[ \Re \left[ P^2(t) - 4Q(t) + 2P'(t) \right] \geq 0, \quad P^2(t) - 4Q(t) + 2P'(t) \neq 0. \]

Then each solution \( x(t) \) of (30) satisfying \( t_1 \) initial conditions

\[ \Re \left[ x'(t_1) x^{-1}(t_1) + \frac{1}{2} P(t_1) \right] \geq 0, \quad x(t_1) \neq 0, \]

exists for \( t \geq t_1 \) and it holds

\[ \lim_{t \to \infty} \left[ 2x'(t) x^{-1}(t) + (P(t)) \right] = 0. \]

Corollary 2. Let us assume (31), (34) and

\[ \int_{t_0}^{\infty} \left| P^2(t) - 4Q(t) + 2P'(t) \right| dt < \infty. \]

Then, the conclusion of Corollary 1 is valid.

Corollary 3. Let us assume (31), (33), (34) and let \( x(t) \) be a complete solution of (30) defined on \([t_1, \omega)\), \( t_1 \geq t_0 \).

If \( \omega = \infty \), then

\[ \lim_{t \to \infty} \left[ 2x'(t) x^{-1}(t) + P(t) \right] = 0. \]

If \( \omega < \infty \), then \( \Re \left[ x'(t) x^{-1}(t) + \frac{1}{2} P(t) \right] < 0 \) for \( t \in [t_1, \omega) \)
\[
\lim_{t \to \infty} \left| x'(t) x^{-1}(t) + \frac{1}{2} P(t) \right| = \infty.
\]

**Corollary 4.** Let us assume (31), (34), (35) and let \( x(t) \) be a complete solution of (30) defined on \([t_1, \infty)\), \( t_1 \geq t_0 \). Then, the conclusion of Corollary 3 is valid.

**Corollary 5.** Let us suppose (31), (34), (35). Then, each solution \( x(t) \) of (30) defined for all \( t \geq t_1 \geq t_0 \) and \( x(t) \neq 0 \), satisfies for \( \alpha \geq 2 \)

\[
\int_{t_1}^{\infty} \left| x'(t) x^{-1}(t) + \frac{1}{2} P(t) \right|^\alpha \, dt < \infty.
\]

**REFERENCES**

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