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FORCED OSCILLATIONS IN FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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1. INTRODUCTION

Our main purpose in this paper is to study the oscillatory phenomenon associated with the equation

$$L_n y(t) + F(t, y(g(t))) = f(t), \tag{1}$$

where $n \geqq 2$ and L_n is a disconjugate differential operator defined by

$$L_n y(t) = \frac{1}{p_n(t)} \frac{d}{dt} \frac{1}{p_{n-1}(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{1}{p_1(t)} \frac{d}{dt} \frac{y(t)}{p_0(t)}. \tag{2}$$

It is assumed throughout that:

(i) $p_i : [a, \infty) \to (0, \infty)$, $0 \leqq i \leqq n$, are continuous;

(3) (ii) $f, g : [a, \infty) \to R$ are continuous, and $\lim_{t \to \infty} g(t) = \infty$;

(iii) $F : [a, \infty) \times R \to R$ is continuous, and there are continuous functions $\varphi, \psi : [a, \infty) \to R$ such that for each $t \in [a, \infty)$ $F(t, y) \geqq \varphi(t)$ if $y \geqq 0$ and $F(t, y) \leqq \psi(t)$ if $y \leqq 0$.

We introduce the notation:

$$D^0(y; p_0)(t) = \frac{y(t)}{p_0(t)}, \tag{4}$$

$$D^i(y; p_0, \ldots, p_i)(t) = \frac{1}{p_i(t)} \frac{d}{dt} D^{i-1}(y; p_0, \ldots, p_{i-1})(t), \qquad 1 \leqq i \leqq n.$$

Equation (1) can then be rewritten as

$$D^n(y; p_0, \ldots, p_n)(t) + F(t, y(g(t))) = f(t).$$

The domain $\mathscr{D}(L_n)$ of L_n is defined to be the set of all functions $y : [T_y, \infty) \to R$ such that $D^i(y; p_0, \ldots, p_i)(t)$, $0 \leqq i \leqq n$, exist and are continuous on $[T_y, \infty)$.

In what follows by a "solution" of equation (1) we mean a function $y \in \mathscr{D}(L_n)$ which is nontrivial in any neighborhood of ∞ and satisfies (1) for all sufficiently large t. We make the standing hypothesis that equation (1) does possess solutions in this sense. A solution of (1) is called oscillatory if it has arbitiarily large zeros; otherwise the solution is called nonoscillatory.

A great many oscillation criteria are known for unforced equations of the form

$$\left(\frac{1}{p_k(t)} y^{(k)}(t)\right)^{(n-k)} + F(t, y(g(t))) = 0. \tag{5}$$

For this we refer the reader to Ševelo and Vareh [8], Onose [7], Singh [10], Kusano and Onose [5] and Naito [6]. A recently published Russian book by Ševelo [9] gives a detailed list of references on the subject. Extensions of these results to more general unforced equations of the form $L_n y(t) + F(t, y(g(t))) = 0$ seem to be in progress; see, for example, Kitamura and Kusano [4].

Obtaining an oscillation criterion for the forced equation (1) ($f(t) \neq 0$) is not so simple. To the best of the authors' knowledge, the first attempt in this direction was made by Kusano and Onose [5] who studied the equation $y^{(n)}(t) + q(t)h(y(g(t))) = f(t)$, and later by other authors including Singh [11]. The main technique rendered the forced equation into an almost unforced equation by employing a function $\lambda(t)$ such that $\lambda^{(n)}(t) = f(t)$ and $\lambda^{(i)}(t) \to 0$ as $t \to \infty$ for $i = 0, 1, \ldots, n-1$.

In this work we shall present an elementary but new technique to obtain oscillation criteria for equation (1). The result roughly asserts that if the forcing term $f(t)$ oscillates with amplitude sufficiently large, then all solutions of (1) are oscillatory regardless of the oscillation or nonoscillation of the associated unforced equation. Related results can be found in Graef, Grammatikopoulos and Spikes [1], and Graef and Kusano [2].

2. PRELIMINARIES

We introduce the notation for repeated integrals which will be extensively utilized in the formulations and the proofs of our theorems.

Let $h_i : [a, \infty) \to R$, $1 \leqq i \leqq N$, be continuous functions. We put for $t, s \in [a, \infty)$

$$\begin{aligned} & I_0 = 1 \\ & I_i(t, s; h_1, \ldots, h_i) = \int_s^t h_1(r)\, I_{i-1}(r, s; h_2, \ldots, h_i)\, \mathrm{d}r, \qquad 1 \leqq i \leqq N. \end{aligned} \tag{6}$$

The following identities are easily verified:

$$I_i(t, s; h_1, \ldots, h_i) = (-1)^i I_i(s, t; h_i, \ldots, h_1), \tag{7}$$

$$I_i(t, s; h_1, \ldots, h_i) = \int_s^t h_i(r)\, I_{i-1}(t, r; h_1, \ldots, h_{i-1})\, \mathrm{d}r. \tag{8}$$

Lemma 1. *Suppose that $h_i(t)$, $1 \leqq i \leqq N$, are positive on $[a, \infty)$. If $I_N(t, a; h_1, \ldots, h_N)$ is bounded on $[a, \infty)$, then so are the functions $I_i(t, a; h_1, \ldots, h_i)$ for $1 \leqq i \leqq N - 1$.*
Proof. Let $b > a$ be fixed. Using (8), we have for $t \geqq b$

$$I_N(t, a; h_1, \ldots, h_N) = \int_a^t h_N(r)\, I_{N-1}(t, r; h_1, \ldots, h_{N-1})\, dr$$

$$\geqq \int_a^b h_N(r)\, I_{N-1}(t, r; h_1, \ldots, h_{N-1})\, dr$$

$$\geqq I_{N-1}(t, b; h_1, \ldots, h_{N-1}) \int_a^b h_N(r)\, dr,$$

from which it follows that $I_{N-1}(t, b; h_1, \ldots, h_{N-1})$ is bounded on $[b, \infty)$. Hence $I_{N-1}(t, a; h_1, \ldots, h_{N-1})$ is bounded on $[a, \infty)$. The boundedness of $I_i(t, a; h_1, \ldots, h_i)$ for $1 \leqq i \leqq N - 2$ follows by induction.

Lemma 2. *If $y \in \mathscr{D}(L_n)$, then for $0 \leqq i \leqq n - 1$ and $t, s \in [T_y, \infty)$ we have*

$$D^i(y; p_0, \ldots, p_i)(t) = \sum_{j=i}^{n-1} D^j(y; p_0, \ldots, p_j)(s)\, I_{j-i}(t, s; p_{i+1}, \ldots, p_j) + \tag{9}$$
$$+ \int_s^t I_{n-i-1}(t, r; p_{i+1}, \ldots, p_{n-1})\, p_n(r)\, D^n(y; p_0, \ldots, p_n)(r)\, dr.$$

This lemma is a generalization of Taylor's formula with remainder encountered in calculus. The proof is straightforward. Note that the last integral in (9) may be rewritten as

$$I_n(t, s; p_1, \ldots, p_{n-1}, p_n D^n(y; p_0, \ldots, p_n))$$
$$= \int_s^t p_1(r_1) \int_s^{r_1} p_2(r_2) \int_s^{r_2} \ldots \int_s^{r_{n-2}} p_{n-1}(r_{n-1}) \int_s^{r_{n-1}} p_n(r_n) D^n(y; p_0, \ldots, p_n)(r_n)\, dr_n\, dr_{n-1} \ldots dr_2\, dr_1.$$

3. MAIN RESULTS

Theorem 1. *Suppose there is a function $\varrho : [a, \infty) \to (0, \infty)$ such that $\varrho'(t) \leqq 0$ on $[a, \infty)$ and the following conditions are satisfied for all $T \geqq a$:*

$$\lim_{t \to \infty} I_{n-1}(t, T; \varrho p_1, p_2, \ldots, p_{n-1}) < \infty, \tag{10}$$

$$\liminf_{t \to \infty} I_n(t, T; \varrho p_1, p_2, \ldots, p_{n-1}, p_n(f - \varphi)) = -\infty, \tag{11}$$

$$\limsup_{t \to \infty} I_n(t, T; \varrho p_1, p_2, \ldots, p_{n-1}, p_n(f - \psi)) = \infty. \tag{12}$$

Then all solutions of equation (1) *are oscillatory.*

Proof. Let $y(t)$ be an eventually positive solution of (1). Let T be such that

$y(g(t)) > 0$ for $t \geqq T$. From Lemma 2 (with $i = 1$ and $s = T$) it follows that

(13) $$D^1(y; p_0, p_1)(t) = \sum_{j=1}^{n-1} c_j I_{j-1}(t, T; p_2, \ldots\ldots, p_j) + + I_{n-1}(t, T; p_2, \ldots, p_{n-1}, p_n D^n(y; p_0, \ldots, p_n))$$

for $t \geqq T$, where $c_j = D^j(y; p_0, \ldots, p_j)(T)$. In view of condition (3) – (iii) we have

(14) $$p_n(t)\, D^n(y; p_0, \ldots, p_n)(t) = p_n(t)\, f(t) - p_n(t)\, F(t, y(g(t))) \leqq p_n(t)\,(f(t) - \varphi(t)), \qquad t \geqq T.$$

Combining (13) with (14), we obtain

(15) $$D^1(y; p_0, p_1)(t) \leqq \sum_{j=1}^{n-1} c_j I_{j-1}(t, T; p_2, \ldots, p_j) + + I_{n-1}(t, T; p_2, \ldots, p_{n-1}, p_n(f - \varphi)), \qquad t \geqq T.$$

We multiply (15) by $\varrho(t)\, p_1(t)$ and integrate it over $[T, t]$. Noting that

$$\int_T^t \varrho(s)\, p_1(s)\, D^1(y; p_0, p_1)(s)\, ds = \int_T^t \varrho(s)\,(D^0(y; p_0)(s))'\, ds$$

$$= \varrho(t)\, D^0(y; p_0)(t) - \varrho(T)\, D^0(y; p_0)(T) - \int_T^t \varrho'(s)\, D^0(y; p_0)(s)\, ds$$

$$\geqq \varrho(t)\, D^0(y; p_0)(t) - \varrho(T)\, D^0(y; p_0)(T), \qquad t \geqq T,$$

we see that

(16) $$\varrho(t)\, D^0(y; p_0)(t) \leqq \sum_{j=0}^{n-1} c_j I_j(t, T; \varrho p_1, p_2, \ldots, p_j) + + I_n(t, T; \varrho p_1, p_2, \ldots, p_{n-1}, p_n(f - \varphi))$$

for $t \geqq T$, where $c_0 = \varrho(T)\, D^0(y; p_0)(T)$. Condition (10) guarantees that $I_j(t, T; \varrho p_1, p_2, \ldots, p_j)$, $0 \leqq j \leqq n - 2$, are bounded on $[T, \infty)$ (see Lemma 1). Using this fact and (11), we conclude from (16) that

(17) $$\liminf_{t\to\infty} \varrho(t) D^0(yp; p_0)(t) = \liminf_{t\to\infty} \frac{\varrho(t)\, y(t)}{p_0(t)} = -\infty,$$

which contradicts the assumption that $y(t)$ is eventually positive.

Likewise, if $y(t)$ is an eventually negative solution of (1), we are led to a contradiction with the help of (12). This completes the proof.

Theorem 2. *Suppose there is a function* $\sigma : [a, \infty) \to (0, \infty)$ *such that* $(\sigma'(t)/p_1(t))' \geqq 0$ *on* $[a, \infty)$ *and the following conditions are satisfied for all* $T \geqq a$:

(18) $$\lim_{t\to\infty} I_{n-1}(t, T; \sigma^{-2} p_1, \sigma p_2, p_3, \ldots, p_{n-1}) < \infty,$$

(19) $$\liminf_{t\to\infty} I_n(t, T; \sigma^{-2} p_1, \sigma p_2, p_3, \ldots, p_{n-1}, p_n(f - \varphi)) = -\infty,$$

$$\liminf_{t\to\infty} I_n(t, T; \sigma^{-2}p_1, \sigma p_2, p_3, \dots, p_{n-1}, p_n(f-\psi)) = \infty. \tag{20}$$

Then all solutions of equation (1) *are oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution such that $y(g(t)) > 0$ for $t \geqq T$. By Lemma 2 (with $i = 2$ and $s = T$) and (14) we get

$$D^2(y; p_0, p_1, p_2)(t) \leqq \sum_{j=2}^{n-1} c_j I_{j-2}(t, T; p_3, \dots, p_j) + I_{n-2}(t, T; p_3, \dots, p_{n-1}, p_n(f-\varphi)) \tag{21}$$

for $t \geqq T$, where $c_j = D^j(y; p_0, \dots, p_j)(T)$. Multiplying (21) by $\sigma(t)p_2(t)$ and integrating over $[T, t]$, we have

$$\int_T^t \sigma(s) p_2(s) D^2(y; p_0, p_1, p_2)(s)\, ds \leqq \sum_{j=2}^{n-1} c_j I_{j-1}(t, T; \sigma p_2, p_3, \dots, p_j) + I_{n-1}(t, T; \sigma p_2, p_3, \dots, p_{n-1}, p_n(f-\varphi)) \tag{22}$$

for $t \geqq T$. Integrating the left hand side of (22) by parts, we have

$$\begin{aligned} &\int_T^t \sigma(s)(D^1(y; p_0, p_1)(s))'\, ds \\ &= \sigma(t) D^1(y, p_0, p_1)(t) - \sigma(T) D^1(y; p_0, p_1)(T) - \int_T^t \frac{\sigma'(s)}{p_1(s)} (D^0(y, p_0)(s))'\, ds \\ &= \sigma(t) D^1(y; p_0, p_1)(t) - \frac{\sigma'(t)}{p_1(t)} D^0(y; p_0)(t) - c_1 + \int_T^t \left(\frac{\sigma'(s)}{p_1(s)}\right)' D^0(y; p_0)(s)'\, ds \\ &\geqq \frac{\sigma^2(t)}{p_1(t)} \left(\frac{1}{\sigma(t)} D^0(y; p_0)(t)\right)' - c_1, \qquad t \geqq T, \end{aligned} \tag{23}$$

where $c_1 = \sigma(T) D^1(y; p_0, p_1)(T) + \sigma(T) D^0(y; p_0)(T)/p_1(T)$. From (22) and (23) it follows that

$$\frac{\sigma^2(t)}{p_1(t)} \left(\frac{1}{\sigma(t)} D^0(y; p_0)(t)\right)' \leqq \sum_{j=1}^{n-1} c_j I_{j-1}(t, T; \sigma p_2, p_3, \dots, p_j) + I_{n-1}(t, T; \sigma p_2, p_3, \dots, p_{n-1}, p_n(f-\varphi)) \tag{24}$$

for $t \geqq T$. Dividing (24) by $\sigma^2(t)/p_1(t)$ and integrating, we obtain

$$\frac{1}{\sigma(t)} D^0(y; p_0)(t) \leqq \sum_{j=0}^{n-1} c_j I_j(t, T; \sigma^{-2}p_1, \sigma p_2, p_3, \dots, p_j) + I_n(t, T; \sigma^{-2}p_1, \sigma p_2, p_3, \dots, p_{n-1}, p_n(f-\varphi)) \tag{25}$$

for $t \geqq T$, where $c_0 = D^0(y; p_0)(T)/\sigma(T)$. From (25), (18) and (19) we see that

$$\liminf_{t\to\infty} \frac{1}{\sigma(t)} D^0(y; p_0)(t) = \liminf_{t\to\infty} \frac{y(t)}{\sigma(t) p_0(t)} = -\infty,$$

which contradicts the eventual positivity of $y(t)$. A similar argument holds if $y(t)$ is an eventually negative solution of (1), and the proof is complete.

Remark 1. It is often assumed that the function $F(t, y)$ satisfies $yF(t, y) \geqq 0$ for all y. In this case we can take $\varphi(t) \equiv \psi(t) \equiv 0$ in Theorems 1 and 2.

Remark 2. Suppose in particular that $L_n y(t) = y^{(n)}(t)$. Then conditions (10), (11) and (12) respectively reduce to

$$\lim_{t\to\infty} \int_N^t \varrho(s)\, s^{n-2}\, ds < \infty, \tag{26}$$

$$\liminf_{t\to\infty} \int_T^t \varrho(s) \int_T^s (s-r)^{n-2}\, [f(r) - \varphi(r)]\, dr = -\infty, \tag{27}$$

$$\limsup_{j\to\infty} \int_T^t \varrho(s) \int_T^t (s-r)^{n-2}\, [f(r) - \psi(r)]\, dr = \infty, \tag{28}$$

and conditions (18), (19) and (20) respectively reduce to

$$\lim_{t\to\infty} \int_T^t \sigma^{-2}(s) \int_T^s \sigma(r)\, r^{n-3}\, dr\, ds < \infty, \tag{29}$$

$$\liminf_{t\to\infty} \int_T^t \sigma^{-2}(s) \int_T^s \sigma(r) \int_T^r (r-u)^{n-3}[f(u) - \varphi(u)]\, du\, dr\, ds = -\infty, \tag{30}$$

and

$$\limsup_{t\to\infty} \int_T^t \sigma^{-2}(s) \int_T^s \sigma(r) \int_T^r (r-u)^{n-3}\, [f(u) - \psi(u)]\, du\, dr\, ds = \infty. \tag{31}$$

Example 1. Consider the equation

$$y^{(iv)}(t) + e^{2\pi} y(t-\pi) = -63\, e^{2t} \sin 2t, \qquad t \geqq \pi. \tag{32}$$

If we choose $\varrho(t) = e^{-t}$ (or $\sigma(t) = e^t$), then the conditions (26) – (28) (or (29) – (31)) can easily be verified with $\varphi(t) \equiv \psi(t) \equiv 0$. Hence all solutions of equation (32) are oscillatory by Theorem 1 (or Theorem 2) and Remark 2. In fact $y(t) = e^{2t} \sin 2t$ is one such solution. We note that all solutions of the unforced equation

$$y^{(iv)}(t) + e^{2\pi} y(t-\pi) = 0, \qquad t \geqq \pi,$$

are oscillatory.

Example 2. Consider the equation

$$(t^2 y')' + t\, e^y = t^2 \sin(\log t), \qquad t \geqq 1. \tag{33}$$

The associated unforced equation

$$(t^2 y')' + t\, e^y = 0$$

has a nonoscillatory solution $y(t) = -\log t$. It is easy to verify that the conditions (10) – (12) of Theorem 1 are satisfied with $\varrho(t) = 1$, $\varphi(t) = 0$ and $\psi(t) = t$, and so all solutions of equation (33) are oscillatory.

We now obtain conditions which guarantee the oscillation of all bounded solutions of equations of the form

$$(34) \qquad L_n y(t) + q(t)\, h(y(g(t))) = f(t),$$

where L_n, f and g are as above and $q : [a, \infty) \to R$ and $h : R \to R$ are continuous functions.

Theorem 3. *All bounded solutions of equation* (34) *are oscillatory if there is a function* $\varrho : [a, \infty) \to (0, \infty)$ *such that* $\varrho'(t) \leqq 0$ *on* $[a, \infty)$ *and, in addition to* (10), *the following conditions are satisfied for all* $k > 0$ *and all* $T \geqq a$;

$$(35) \qquad \liminf_{t \to \infty} I_n(t, T; \varrho p_1, p_2, \ldots, p_{n-1}, p_n(f + k\,|\,q\,|)) = -\infty,$$

$$(36) \qquad \limsup_{t \to \infty} I_n(t, T; \varrho p_1, p_2, \ldots, p_{n-1}, p_n(f - k\,|\,q\,|)) = \infty.$$

Theorem 4. *All bounded solutions of equation* (34) *are oscillatory if there is a function* $\sigma : [a, \infty) \to (0, \infty)$ *such that* $(\sigma'(t)/p_1(t))' \geqq 0$ *on* $[a, \infty)$ *and, in addition to* (18), *the following conditions are satisfied for all* $k > 0$ *and all* $T \geqq a$;

$$(37) \qquad \liminf_{t \to \infty} I_n(t, T; \sigma^{-2} p_1, \sigma p_2, p_3, \ldots, p_{n-1}, p_n(f + k\,|\,q\,|)) = -\infty,$$

$$(38) \qquad \limsup_{t \to \infty} I_n(t, T; \sigma^{-2} p_1, \sigma p_2, p_3, \ldots, p_{n-1}, p_n(f - k\,|\,q\,|)) = \infty.$$

Proof of Theorem 3. Let $y(t)$ be a bounded nonoscillatory solution of (34). If $y(t)$ is eventually positive, then there are positive constants M and T such that $0 < y(g(t)) < M$ for $t \geqq T$. Let $K = \max\{|\,h(y)\,| : 0 \leqq y \leqq M\}$. Then $q(t)\, h(y(g(t))) \geqq -K\,|\,q(t)\,|$ for $t \geqq T$. Proceeding now exactly as in the proof of Theorem 1, we obtain

$$(39) \qquad \varrho(t)\, D^0(y; p_0)(t) \leqq \sum_{j=0}^{n-1} c_j I_j(t, T; \varrho p_1, p_2, \ldots, p_j) + I_n(t, T; \varrho p_1, p_2, \ldots, p_{n-1}, p_n(f + K\,|\,q\,|))$$

for $t \geqq T$, where c_j, $0 \leqq j \leqq n - 1$, are constants. Passing to the lower limit as $t \to \infty$ in (39), we see that (17) holds, a contradiction. A parallel argument applies if $y(t)$ is eventually negative.

Theorem 4 can be proved similarly.

Remark 3. Under the hypotheses of Theorem 3 (or Theorem 4) all nonoscillatory solutions of (34) are necessarily unbounded.

Remark 4. Conditions (35) and (36) hold if

$$(41) \qquad \liminf_{t \to \infty} I_n(t, T; \varrho p_1, p_2, \ldots, p_{n-1}, p_n f) = -\infty,$$

$$(42) \qquad \limsup_{t \to \infty} I_n(t, T; \varrho p_1, p_2, \ldots, p_{n-1}, p_n f) = \infty,$$

and

$$(43) \qquad \lim_{t \to \infty} I_n(t, T; \varrho p_1, p_2, \ldots, p_{n-1}, p_n\,|\,q\,|) < \infty.$$

Example 3. Consider the equation

$$(44) \qquad y'''(t) + e^{\pi} \sin(\log t) . \exp[y(e^{-\pi}t)] = t \sin(\log t) + 2t^{-3}, \qquad t \geqq 1.$$

It is not hard to check that the conditions of Theorem 3 are satisfied with $\varrho(t) = t^{-3}$. Therefore all bounded solutions of (44) are oscillatory. Equation (44) has an unbounded nonoscillatory solution $y(t) = \log t$. Notice that condition (43) is not satisfied for this choice of $\varrho(t)$.

Remark 5. A disconjugate differential operator L_n of the form (2) is said to be in the first canonical form if

$$(45) \qquad \int_a^{\infty} p_i(t)\,dt = \infty \qquad \text{for } 1 \leqq i \leqq n-1.$$

It has been shown by Trench [12] that every disconjugate operator can be put in the first canonical form in an essentially unique way. Oscillation of equation (1) with L_n in the first canonical form has been studied by Graef and Kusano [2].

We say that an operator L_n of the form (2) is in the second canonical form if

$$(46) \qquad \int_a^{\infty} p_i(t)\,dt < \infty \qquad \text{for } 1 \leqq i \leqq n-1.$$

Very recently Granata [3] has proved that every disconjugate operator can be represented in the second canonical form. If L_n is in the second canonical form, then it is easy to see that

$$\lim_{t\to\infty} I_{n-1}(t, T; p_1, p_2, \dots, p_{n-1}) < \infty$$

for all $T \geqq a$, and we have the following results as corollaries to Theorems 1 and 3.

Theorem 5. *Suppose* (46) *holds. If*

$$(47) \qquad \liminf_{t\to\infty} I_n(t, T; p_1, \dots, p_{n-1}, p_n(f - \varphi)) = -\infty,$$

$$(48) \qquad \limsup_{t\to\infty} I_n(t, T; p_1, \dots, p_{n-1}, p_n(f - \psi)) = \infty,$$

for all $T \geqq a$, *then all solutions of* (1) *are oscillatory.*

Theorem 6. *Suppose* (46) *holds. If*

$$(49) \qquad \liminf_{t\to\infty} I_n(t, T; p_1, \dots, p_{n-1}, p_n(f + k|q|)) = -\infty,$$

$$(50) \qquad \limsup_{t\to\infty} I_n(t, T; p_1, \dots, p_{n-1}, p_n(f - k|q|)) = \infty,$$

for all $k > 0$ *and* $T \geqq a$, *then all bounded solutions of* (34) *are oscillatory.*

We re-examine the example 1. The second canonical form of the operator d^4/dt^4 is

$$\frac{d^4}{dt^4} = \frac{1}{t^3}\frac{d}{dt}t^2\frac{d}{dt}t^2\frac{d}{dt}t^2\frac{d}{dt}\frac{\cdot}{t^3},$$

and so equation (32) is equivalent to

$$(t^2(t^2(t^2(t^{-3}y(t))')')')' + e^{2\pi}t^3y(t-\pi) = -63t^3 e^{2t}\sin 2t. \tag{51}$$

Conditions (47) and (48) are satisfied for (51), and it follows from Theorem 5 that all solutions of (32) are oscillatory.

The actual computations leading a general operator L_n to its first and/or second canonical form are tedious.

REFERENCES

[1] J. R. Graef, M. K. Grammatikopoulos and P. W Spikes, *Growth and oscillatory behavior of solutions of a differential equation with a deviating argument*, to appear.

[2] J. R. Graef and T. Kusano, *Oscillation and nonoscillation criteria for a class of forced differential equations with deviating arguments*, in preparation.

[3] A. Granata, *Canonical factorizations of disconjugate differential operators*, SIAM J. Math. Anal. 11 (1980), 160—172.

[4] Y. Kitamura and T. Kusano, *Nonlinear oscillation of higher order functional differential equations with deviating arguments*, J. Math. Anal. Appl. 77 (1980), 100—119.

[5] T. Kusano and H. Onose, *Oscillations of functional differential equations with retarded argument*, J. Differential Equations 15 (1974), 269—277.

[6] M. Naito, *Existence and asymptotic behavior of positive solutions of differential inequalities with deviating argument*, Funkcial. Ekvac. 22 (1979), 127—142.

[7] H. Onose, *Oscillation and asymptotic behavior of solutions of retarded differential equations of arbitrary order*, Hiroshima Math. J. 3 (1973), 333—360.

[8] V. N. Ševelo and N. V. Vareh, *On the oscillation of solutions of the equation* $[r(t)\,y^{(n-1)}(t)]' + p(t)\,f(y(\tau(t))) = 0$, Ukrain. Mat. Ž. 25 (1973), 724—731. (Russian).

[9] V. N. Sevelo, *Oscillation of Solutions of Differential Equations with Deviating Arguments*, Naukova Dumka, Kiev, 1978. (Russian).

[10] B. Singh, *A necessary and sufficient condition for the oscillation of an even order nonlinear delay differential equation*, Canad. J. Math. 25 (1973), 1078—1089.

[11] B. Singh, *Impact of delays on oscillations in general functional equations*, Hiroshima Math. J. 5 (1975), 351—361.

[12] W. F. Trench, *Canonical forms and principal systems for general disconjugate equations*, Trans. Amer. Math. Soc. 189 (1974), 319—327.

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