B. L. Shekhter

On singular boundary value problems for two-dimensional differential systems

Archivum Mathematicum, Vol. 19 (1983), No. 1, 19--41

ARCH. MATH. 1, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIX: 19—42, 1983

ON SINGULAR BOUNDARY VALUE PROBLEMS FOR TWO-DIMENSIONAL DIFFERENTIAL SYSTEMS

B. L. SHEKHTER, Tbilisi

(Received December 20, 1981)

§ 1. Introduction

This paper deals with the two-dimensional system

(1) $$x' = f_1(t, x, y), \qquad y' = f_2(t, x, y)$$

under the boundary conditions

(2) $$x(a+) = 0, \qquad x(b-) = 0$$

or

(3) $$x(a+) = 0, \qquad y(b-) = 0.$$

The case is considered when the functions $f_i :]\, a, b[\times R^2 \to R$ $(i = 1, 2)$ may be nonsummable with respect to the first variable having singularities at the end points of the interval $]\, a, b[$.

Let I be an open or half-open interval. By $L_{loc}(I)$ we denote the set of functions $x : I \to R$ which are summable on every (closed) segment contained within I.

In what follows we assume that

1. $]\, a, b[$ is a finite interval;
2. the functions $f_i(., x, y) :]\, a, b[\to R$ are measurable for $x, y \in R$;
3. the functions $f_i(t, ., .) : R^2 \to R$ are continuous for $t \in]\, a, b[$;
4. $\sup \{| f_i(., x, y) | : | x | + | y | \leqq \varrho\} \in L_{loc}(]\, a, b[)$ for $\varrho > 0$ $(i = 1, 2)$.

(x, y) is said to be a solution of the system (1) if $x, y :]\, a, b[\to R$ are absolutely continuous on each segment contained within the interval $]\, a, b[$ and satisfy (1) almost everywhere in this interval. The theorems which are proved here allow to reduce the question on existence and uniqueness for the problems (1), (2) and (1), (3) to the question on unique solvability of the corresponding boundary value problems for some classes of linear differential systems. Such method goes back to the works by L. Tonelli [1] and H. Epheser [2] and was used in [3] where, in particular, the second order singular differential equation was investigated under the boundary conditions of the type (2) (for more detailed bibliographical remarks see [4] which is devoted to the regular problems). Other approaches in study of the singular problems (1), (2) and (1), (3) were realized in [3, 5].

With a view to indicate the class of linear systems

$$u' = g_1(t)\,u + h_1(t)\,v, \qquad v' = h_2(t)\,u + g_2(t)\,v, \tag{4}$$

we are interested in, put

$$\mu_i(t) = \int_a^t |h_i(\tau)|\,d\tau, \qquad \nu_i(t) = \int_t^b |h_i(\tau)|\,d\tau \tag{5}$$

and introduce the following definitions (cf. [3, 4, 6]).

Definition 1. *Let k be an integer and*

$$h_1 \in L([a, b]), \qquad h_2 \in L_{loc}(]a, b[),$$

$$h_2(t)\,\mu_1(t)\,\nu_1(t) \in L([a, b]), \tag{6}$$

$$h_3 \in L([a, b]), \qquad h_3(t) \geqq 0 \qquad \text{for } a \leqq t \leqq b. \tag{7}$$

Then $(h_1, h_2, h_3) \in \mathscr{P}_{k1}(a, b)$ if and only if for all $g_i \in L([a, b])$ $(i = 1, 2)$ satisfying the condition

$$|g_2(t) - g_1(t)| \leqq h_3(t) \qquad \text{for } a \leqq t \leqq b, \tag{8}$$

we have $u(b-) \neq 0$ and there exists $\delta > 0$ such that

$$-\frac{\pi}{2} - \pi k < \varphi(t) < \frac{\pi}{2} - \pi k \qquad \text{for } b - \delta < t < b,$$

where $\varphi : [a, b] \to R$ is continuous,

$$\operatorname{tg} \varphi(t) = \frac{v(t)}{u(t)} \quad \text{when} \quad u(t) \neq 0, \qquad \varphi(a) = \frac{\pi}{2} \tag{9}$$

and (u, v) is a solution of the system (4) under the initial conditions

$$u(a+) = 0, \qquad v(a+) = 1. \text{ [1]} \tag{10}$$

Definition 2. *Let k be an integer,*

$$h_1 \in L_{loc}([a, b[), \qquad h_2 \in L_{loc}(]a, b]),$$

$$h_2(t)\,\mu_1(t) \in L_{loc}([a, b[), \qquad h_1(t)\,\nu_2(t) \in L_{loc}(]a, b]), \tag{11}$$

and let (7) be observed. Then $(h_1, h_2, h_3) \in \mathscr{P}_{k2}(a, b)$ if and only if for all $g_i \in L([a, b])$

[1]) Lemma 1 stated below implies that such a solution exists.

$(i = 1, 2)$ *satisfying* (8) $v(b-) \neq 0$ *and there exists* $\delta > 0$ *such that*

$$-\pi k < \varphi(t) < \pi - \pi k \qquad \text{for } b - \delta < t < b,$$

where φ *and* (u, v) *are the same as in Definition 1.*

§ 2. Lemmas

This section is concerned with the linear singular systems (4). First of all we study behavior of solutions at the point of singularity (see also [3, p. 222] and [7, p. 443]).

Lemma 1. *Let* $h_1, g_i \in L([a, b])$ $(i = 1, 2)$, $h_2 \in L_{loc}(]a, b])$, $h_2(t)\,\mu_1(t) \in L([a, b])$ *where* μ_1 *is given by* (5), *and let* (u, v) *be a solution of the system* (4). *Then*

(12) $$\lim_{t \to a+} v(t)\,\mu_1(t) = 0$$

and the limit $u(a+)$ *exists and is finite. Moreover, if this limit is zero, then there exists finite* $v(a+)$.

Proof. Suppose that $a_n \in]a, b]$ $(n = 1, 2, \ldots)$ and $a_n \to a$ when $n \to \infty$. If (u_n, v_n) $(n = 1, 2, \ldots)$ are the solutions of (4) under the initial conditions

(13) $$u(a_n) = 0, \qquad v(a_n) = 1,$$

then

(14) $$|u_n(t)| \leqq \lambda \int_{a_n}^{t} |h_1(\tau)|\,d\tau \Big(1 + \int_{a_n}^{t} |h_2(\tau)\,u_n(\tau)|\,d\tau\Big) \qquad \text{for } a_n \leqq t \leqq b,$$

where

(15) $$\lambda = \exp\Big(\int_a^b [|g_1(\tau)| + |g_2(\tau)|]\,d\tau\Big).$$

Using this inequality and setting

$$w_n(t) = \lambda\Big(1 + \int_{a_n}^{t} |h_2(\tau)\,u_n(\tau)|\,d\tau\Big),$$

we obtain

$$w_n(t) \leqq \lambda\Big(1 + \int_{a_n}^{t} |h_2(\tau)|\,\mu_1(\tau)\,w_n(\tau)\,d\tau\Big) \qquad \text{for } a_n \leqq t \leqq b.$$

By Gronwall–Bellman lemma (see e.g. [3, p. 49]) $w_n(t) \leqq A_0$ on $[a_n, b]$ where

(16) $$A_0 = \lambda \exp\Big(\lambda \int_a^b |h_2(\tau)|\,\mu_1(\tau)\,d\tau\Big),$$

and according to (14)

(17) $$|u_n(t)| \leqq A_0 \mu_1(t) \qquad \text{for } a_n \leqq t \leqq b.$$

In addition, (4), (13) and (17) imply

$$|v_n(t)| \leqq \lambda(1 + A_0 \int_a^b |h_2(\tau)| \mu_1(\tau)\, d\tau) \qquad \text{for } a_n \leqq t \leqq b.$$

Thus the sequences $(u_n^*)_{n=1}^{\infty}$ and $(v_n^*)_{n=1}^{\infty}$, where

$$u_n^*(t) = u_n(t), \qquad v_n^*(t) = v_n(t) \qquad \text{for } a_n \leqq t \leqq b,$$
$$u_n^*(t) = 0, \qquad v_n^*(t) = 1 \qquad \text{for } a \leqq t < a_n,$$

are uniformly bounded and equicontinuous on $[a, b]$. Therefore, they contain certain consequences which uniformly on $[a, b]$ converge to the functions u_0 and v_0 such that (u_0, v_0) is the solution of the problem (4), (10).

Let $c \in]a, b]$,

$$\exp(-\int_a^b |g_1(\tau)|\, d\tau) > A_0\lambda \int_a^c |h_2(\tau)| \mu_1(\tau)\, d\tau, \tag{18}$$

and let $(\tilde{u}, \tilde{v})$ be the solution of the system (4) satisfying the initial conditions $u(c) = 1$, $v(c) = 0$. Then

$$\tilde{u}(t) = \exp(-\int_t^c g_1(\tau)\, d\tau) +$$
$$+ \int_t^c h_2(\tau)\tilde{u}(\tau) \int_t^s h_1(s) \exp(\int_s^\tau g_1(p)\, dp + \int_\tau^s g_2(p)\, dp)\, ds\, d\tau. \tag{19}$$

Hence

$$|\tilde{u}(t)| \leqq \lambda + \lambda \int_t^c |h_2(\tau)\tilde{u}(\tau)| \mu_1(\tau)\, d\tau \qquad \text{for } a < t \leqq c,$$

and taking into account (16) by Gronwall–Bellman lemma we obtain

$$|\tilde{u}(t)| \leqq A_0 \qquad \text{for } a < t \leqq c. \tag{20}$$

As it follows from this inequality and from the conditions of the lemma, the right-hand side of (19) tends to a finite limit when $t \to a$. Thus $\tilde{u}(a+)$ exists and according to (18) is not zero.

Furthermore, (20) yields

$$|\tilde{v}(t)| \leqq A_0\lambda \int_t^c |h_2(\tau)|\, d\tau \qquad \text{for } a < t \leqq c.$$

If $t_0 \in]a, c]$, then

$$\mu_1(t)|\tilde{v}(t)| \leqq A_0\lambda(\mu_1(t) \int_{t_0}^c |h_2(\tau)|\, d\tau + \int_a^{t_0} |h_2(\tau)| \mu_1(\tau)\, d\tau) \qquad \text{for } a < t \leqq c. \tag{21}$$

Now taking into account that t_0 is arbitrarily close to a, we obtain

$$\lim_{t \to a+} \tilde{v}(t)\mu_1(t) = 0.$$

Since (u_0, v_0) and $(\tilde{u}, \tilde{v})$ are linearly independent solutions of the system (4), each solution (u, v) of this system may be represented in the form

$$u(t) = d_1 u_0(t) + d_2 \tilde{u}(t), \qquad v(t) = d_1 v_0(t) + d_2 \tilde{v}(t),$$

where d_1 and d_2 are certain constants. Thus there exists a finite limit $u(a+)$ and (12) is fulfilled. If, in addition, $u(a+) = 0$, then $d_2 = 0$, and so $v(a+) = d_1$. This completes the proof.

Remark. (21) is valid not only for μ_1 defined by (5), but for any continuous nondecreasing function $\mu_1 : [a, b] \to [0, +\infty[$ such that $h_2(t)\, \mu_1(t) \in L([a, b])$. Therefore, if, in addition, $\mu_1(a) = 0$, then (12) holds for all solutions of the system (4).

Lemma 2. *Let* $h_1, g_i \in L([a, b])$, $h_2 \in L_{loc}(]\, a, b[)$,

$$h_i(t) \geqq 0, \qquad g_i(t) \geqq 0 \qquad \textit{for } a < t < b (i = 1, 2),$$

and let (6) *be fulfilled where* μ_1 *and* ν_1 *are defined by* (5). *Then there exists a constant* A *such that for any point* $t_0 \in [a, b]$ *and any measurable functions* $h_{i0}, g_{i0} :]\, a, b[\to R$ *satisfying the inequalities*

(22) $$| h_{i0}(t) | \leqq h_i(t), \qquad | g_{i0}(t) | \leqq g_i(t) \qquad \textit{for } a < t < b (i = 1, 2)$$

we have

$$| u(t) | \leqq A\, | I(t_0, t) | \qquad \textit{for } a \leqq t \leqq b, \qquad I(s, t) = \int_s^t | h_{10}(\tau) |\, d\tau,$$

(23) $$| v(t) | \leqq A \max\left\{1, \frac{I(t, t_0)}{\mu_1(t)}\right\} \qquad \textit{if} \quad a < t \leqq t_0 \quad \textit{and} \quad \mu_1(t) \neq 0,$$

$$| v(t) | \leqq A \max\left\{1, \frac{I(t_0, t)}{\nu_1(t)}\right\} \qquad \textit{if} \quad t_0 \leqq t < b \quad \textit{and} \quad \nu_1(t) \neq 0,$$

where (u, v) *is the solution of the initial value problem*

(24) $$u' = g_{10}(t)\, u + h_{10}(t)\, v, \qquad v' = h_{20}(t)\, u + g_{20}(t)\, v,$$

(25) $$u(t_0) = 0, \qquad v(t_0) = 1.$$

Proof. By Lemma 1 the problem (24), (25) has a solution (u, v) for any $t_0 \in \in [a, b]$.

Set (15). If $t_0 \geqq (a + b)/2$, then applying the Gronwall–Bellman lemma to the inequality

(26) $$| u(t) | \leqq \lambda\left(I(t_0, t) + \int_{t_0}^t | h_{20}(\tau)\, u(\tau) |\, I(\tau, t)\, d\tau\right) \qquad \text{for } t_0 \leqq t \leqq b,$$

we obtain

$$(27)\qquad |u(t)| \leqq \lambda I(t_0, t)\exp\Big(\lambda \int_{\frac{a+b}{2}}^{b} h_2(\tau)\,\nu_1(\tau)\,d\tau\Big) \qquad \text{for } t_0 \leqq t \leqq b.$$

Now let $t_0 < (a + b)/2$. Then the argument by which the estimate (16), (17) was established yields

$$(28)\qquad |u(t)| \leqq A_0 I(t_0, t) \qquad \text{for } t_0 \leqq t \leqq \frac{a+b}{2},$$

where

$$A_0 = \lambda \exp\Big(\lambda \int_{a}^{\frac{a+b}{2}} h_2(\tau)\,\mu_1(\tau)\,d\tau\Big).$$

Hence it follows from (26) that for $(a + b)/2 \leqq t \leqq b$

$$|u(t)| \leqq \lambda I(t_0, t)\Big(1 + A_0 \int_{a}^{\frac{a+b}{2}} h_2(\tau)\,\mu_1(\tau)\,d\tau\Big) + \lambda \int_{\frac{a+b}{2}}^{t} h_2(\tau)\,\nu_1(\tau)\,|u(\tau)|\,d\tau,$$

and using the Gronwall – Bellman lemma once again, we derive

$$|u(t)| \leqq \lambda I(t_0, t)\Big(1 + A_0 \int_{a}^{\frac{a+b}{2}} h_2(\tau)\,\mu_1(\tau)\,d\tau\Big)\exp\Big(\lambda \int_{\frac{a+b}{2}}^{b} h_2(\tau)\,\nu_1(\tau)\,d\tau\Big).$$

This inequality along with (27) and (28) implies

$$|u(t)| \leqq A^* I(t_0, t) \qquad \text{for } t_0 \leqq t \leqq b,$$

where the constant A^* does not depend on the choice of h_{i0}, g_{i0} and t_0.

Let $t \in [t_0, b[$ and $\nu_1(t) \neq 0$. Note that

$$(29)\qquad |v(t)| \leqq \lambda\Big(1 + A^* \int_{t_0}^{t} |h_{20}(\tau)|\, I(t_0, \tau)\,d\tau\Big).$$

Thus

$$|v(t)| \leqq \lambda\left(1 + A^* \frac{I(t_0, t)}{\nu_1(t)} \int_{\frac{a+b}{2}}^{b} h_2(\tau)\,\nu_1(\tau)\,d\tau\right) \qquad \text{if} \quad t_0 \geqq \frac{a+b}{2},$$

$$|v(t)| \leqq \lambda\Big(1 + A^* \int_{a}^{\frac{a+b}{2}} h_2(\tau)\,\mu_1(\tau)\,d\tau\Big) \qquad \text{if} \quad t_0 < \frac{a+b}{2} \quad \text{and} \quad t < \frac{a+b}{2},$$

$$|v(t)| \leqq \lambda\left(1 + A^* \int_{a}^{\frac{a+b}{2}} h_2(\tau)\,\mu_1(\tau)\,d\tau + A^* \frac{I(t_0, t)}{\nu_1(t)} \int_{\frac{a+b}{2}}^{b} h_2(\tau)\,\nu_1(\tau)\,d\tau\right)$$

$$\text{if} \qquad t_0 < \frac{a+b}{2} \leqq t.$$

It becomes evident from the obtained relations that there exists an independent on h_{i0}, g_{i0} and t_0 constant A for which the inequalities in question are valid in $[t_0, b[$. The case of $]a, t_0]$ may be treated in the similar way.

Lemma 2 establishes in $]a, b[$ an a priori estimate for v providing that $\mu_1(t)\, \nu_1(t) > 0$ in this interval. In the general case (29) implies the following statement.

Lemma 2'. *Let the conditions of Lemma 2 be fulfilled. Then for any $\varepsilon \in]0, b-a[$ there exists a constant $A = A(\varepsilon)$ such that if measurable functions $h_{i0}, g_{i0}:]a, b[\to R$ satisfy* (22) *and $t_0 \in [a, b]$, then*

$$|v(t)| \leqq A \qquad \textit{for } t \in [a+\varepsilon, t_0] \cup [t_0, b-\varepsilon],$$

where (u, v) is the solution of (24), (25).

Lemmas 3–5 are essentially of comparison type.

Lemma 3. *Let k be an integer,*

(30) $$(h_{1i}, h_{2i}, h_3) \in \mathscr{P}_{k1}(a, b)\ ((h_{1i}, h_{2i}, h_3) \in \mathscr{P}_{k2}(a, b)) \qquad (i = 1, 2),$$

(31) $$h_{ii}(t) \leqq h_{i3-i}(t) \qquad \textit{for } a < t < b (i = 1, 2),$$

and let the condition (6) *(the conditions* (11)*) be fulfilled where the functions $\mu_1, \nu_i (i = 1, 2)$ are defined by* (5) *and*

(32) $$h_i(t) \equiv |h_{i1}(t)| + |h_{i2}(t)|.$$

Then

$$(h_{10}, h_{20}, h_3) \in \mathscr{P}_{k1}(a, b) \qquad ((h_{10}, h_{20}, h_3) \in \mathscr{P}_{k2}(a, b))$$

for any measurable $h_{i0}:]a, b[\to R$ satisfying the inequalities

$$h_{ii}(t) \leqq h_{i0}(t) \leqq h_{i3-i}(t) \qquad \textit{for } a < t < b \quad (i = 1, 2).$$

Proof. We shall carry out the proof for the set $\mathscr{P}_{k1}(a, b)$. For $\mathscr{P}_{k2}(a, b)$ the argument is similar.

Let $g_1, g_2 \in L([a, b])$ satisfy (8), and let (u_i, v_i) $(i = 0, 1, 2)$ be solutions of the systems

(33) $$u' = g_1(t)\, u + h_{1i}(t)\, v, \qquad v' = h_{2i}(t)\, u + g_2(t)\, v$$

under the initial conditions (10). Assuming that $a_n \in]a, b[$ $(n = 1, 2, \ldots)$ and $a_n \to a$ when $n \to \infty$, approximating (u_i, v_i) by the solutions of the problems (33), (13) (cf. Proof of Lemma 1) and using Lemma 15.2 and Theorem 14.5 of [6], we can easily verify that

$$\varphi_2(t) \leqq \varphi_0(t) \leqq \varphi_1(t) \qquad \text{for } a \leqq t < b,$$

where $\varphi_i (i = 0, 1, 2)$ are the angular functions of the solutions (u_i, v_i) defined by the conditions $\varphi_i(a) = \pi/2$ (i.e. $\varphi_i : [a, b[\to R$ are continuous functions satisfying the equalities similar to (9)).

Hence it follows from (30) that for all $t \in]a, b[$ sufficiently close to b

$$-\frac{\pi}{2} - \pi k < \varphi_0(t) < \frac{\pi}{2} - \pi k, \qquad \frac{v_2(t)}{u_2(t)} \leqq \frac{v_0(t)}{u_0(t)} \leqq \frac{v_1(t)}{u_1(t)},$$

and so

$$(34) \qquad |v_0(t)| \leqq \left[\left|\frac{v_1(t)}{u_1(t)}\right| + \left|\frac{v_2(t)}{u_2(t)}\right|\right] |u_0(t)|.$$

If $u_0(b-) = 0$, then by (6) and Lemma 2

$$|u_0(t)| \leqq Av_1(t) \qquad \text{for } a \leqq t \leqq b,$$

where A is a constant. Considering this inequality, Remark to Lemma 1 and (34), we derive that $v_0(b-) = 0$, i.e. (u_0, v_0) is the trivial solution, but it is not the case. Thus $u_0(b-) \neq 0$ and the proof is completed.

The following two lemmas may be proved in the similar way.

Lemma 4. *Let* $(h_1, h_2, h_3) \in \mathscr{P}_{01}(a, b)$ *and* $h_1(t) \geqq 0$ *for* $a \leqq t \leqq b$. *Then* $(h_{10}, h_{20}, h_3) \in \mathscr{P}_{01}(t_1, t_2)$ *for any segment* $[t_1, t_2] \subset [a, b]$ *and any measurable functions* $h_{i0} :]a, b[\to R$ $(i = 1, 2)$ *satisfying the conditions*

$$h_{20}(t) \int_a^t h_{10}(\tau)\,d\tau \int_t^b h_{10}(\tau)\,d\tau \in L([a, b]), \qquad \int_{t_1}^{t_2} h_{10}(\tau)\,d\tau > 0,$$

$$0 \leqq h_{10}(t) \leqq h_1(t), \qquad h_{20}(t) \geqq h_2(t) \qquad \text{for } a < t < b.$$

Lemma 5. *Let* $c \in]a, b[$, $(h_1, h_2, h_3) \in \mathscr{P}_{02}(a, c) \cap \mathscr{P}_{02}(c, b)$ *and*

$$h_1(t) \geqq 0 \qquad \text{for } a \leqq t \leqq c, \qquad h_2(t) \geqq 0 \qquad \text{for } c \leqq t \leqq b.$$

Then $(h_{10}, h_{20}, h_3) \in \mathscr{P}_{02}(t_1, t_2)$ *for any* $t_1 \in [a, c]$, $t_2 \in [c, b]$ $(t_1 < t_2)$ *and any functions* $h_{i0} \in L_{loc}(]a, b[)$ $(i = 1, 2)$ *satisfying the conditions*

$$h_{20}(t) \int_a^t h_{10}(\tau)\,d\tau \in L([a, c]), \qquad h_{10}(t) \int_t^b h_{20}(\tau)\,d\tau \in L([c, b]),$$

$$0 \leqq h_{10}(t) \leqq h_1(t), \qquad h_{20}(t) \geqq h_2(t) \qquad \text{for } a < t \leqq c,$$

$$h_{10}(t) \geqq h_1(t), \qquad 0 \leqq h_{20}(t) \leqq h_2(t) \qquad \text{for } c \leqq t < b.$$

Lemma 6. *Let the functions* $g_i : [a, b] \to [0, +\infty[$ *be summable,*

$$(35) \qquad (h_{1i}, h_{2i}, g_1 + g_2) \in \mathscr{P}_{k1}(a, b) \; ((h_{1i}, h_{2i}, g_1 + g_2) \in \mathscr{P}_{k2}(a, b)) \qquad (i = 1, 2)$$

for a certain integer k, and let the inequality (31) *and the condition* (6) (*the condition* (11)) *be fulfilled where the functions* μ_1, v_i, h_i *are defined by* (5) *and* (32). *Then there* *exists a positive constant B such that for any measurable functions* h_{i0}, g_{i0} : $]a, b[\to R$ $(i = 1, 2)$ *satisfying the conditions*

$$(36) \qquad h_{ii}(t) \leqq h_{i0}(t) \leqq h_{i3-i}(t), \qquad |g_{i0}(t)| \leqq g_i(t) \qquad \textit{for } a < t < b$$

the inequality

$$|u(b)| \geqq B \qquad (|v(b)| \geqq B)$$

holds where (u, v) *is the solution of the problem* (24), (10).

Proof. With a view to fix the idea, we shall carry out the proof for the set $\mathscr{P}_{k1}(a, b)$.

Assume that the lemma is not true. Then there exist measurable functions $\xi_{in}, \zeta_{in} :]a, b[\to R$ $(i = 1, 2; n = 1, 2, \ldots)$ such that

$$h_{ii}(t) \leqq \xi_{in}(t) \leqq h_{i3-i}(t), \qquad |\zeta_{in}(t)| \leqq g_i(t) \qquad \text{for } a < t < b,$$

$$(37) \qquad |u_n(b)| \leqq \frac{1}{n},$$

where (u_n, v_n) are solutions of the systems

$$u' = \zeta_{1n}(t)\, u + \xi_{1n}(t)\, v, \qquad v' = \xi_{2n}(t)\, u + \zeta_{2n}(t)\, v$$

under the conditions (10). The sequences $(\int_a^t \xi_{1n}(\tau)\, d\tau)_{n=1}^{\infty}$ and $(\int_a^t \zeta_{in}(\tau)\, d\tau)_{n=1}^{\infty}$ $(i = 1, 2)$ are uniformly bounded and equicontinuous on the segment $[a, b]$ and, hence, without loss of generality we may hold that they are uniformly convergent on this segment. Furthermore, according to Lemma 2 there exists a constant A such that

$$|u_n(t)| \leqq A \int_a^t |\xi_{1n}(\tau)|\, d\tau \qquad \text{for} \qquad a \leqq t \leqq b \quad (n = 1, 2, \ldots).$$

This implies that we may assume the sequences $(v_n)_{n=1}^{\infty}$ and

$$\Big(\int_a^t \xi_{2n}(\tau)\, u_n(\tau) \exp\Big(-\int_a^{\tau} \zeta_{2n}(s)\, ds\Big)\, d\tau\Big)_{n=1}^{\infty}.$$

to be uniformly convergent on each segment contained in $[a, b[$ and the sequence $(u_n)_{n=1}^{\infty}$ – uniformly convergent on $[a, b]$. The latter becomes evident when apply the inequalities

$$\begin{aligned} |u_n'(t)| &\leqq A\, |\zeta_{1n}(t)| \int_a^t |\xi_{1n}(\tau)|\, d\tau + |\xi_{1n}(t)\, v_n(t)| \\ |v_n(t)| &\leqq \lambda\Big(1 + A \int_a^t |\xi_{2n}(\tau)| \int_a^{\tau} |\xi_{1n}(s)|\, ds\, d\tau\Big) \end{aligned} \qquad \text{for } a \leqq t < b,$$

where λ is defined by (15).

Let $u_n \to u$, $v_n \to v$ when $n \to \infty$. According to Lemma 2.6 of [3] (u, v) is a solution of a certain system (24) with coefficients satisfying (36). Thus (35) and Lemma 3 yield $u(b) \neq 0$ which contradicts to (37). This completes the proof.

The method of the proof of Lemmas 7 and 8 is essentially the same as that of Lemma 6, but instead of Lemma 3 one must use Lemmas 4 and 5 respectively (see also [4]).

Lemma 7. *Let the functions* $g_i : [a, b] \to [0, +\infty[$ $(i = 1, 2)$ *be summable,* $(h_1, h_2, g_1 + g_2) \in \mathscr{P}_{01}(a, b)$, $h \in L_{loc}(]a, b[)$,

$$h_1(t) \geqq 0, \qquad h_2(t) \leqq h(t) \qquad \text{for } a < t < b,$$

$h(t)\, \mu_1(t)\, \nu_1(t) \in L([a, b])$ *where* μ_1 *and* ν_1 *are defined by* (5). *Then there exists a positive constant* B *such that for any* $t_0 \in [a, b]$ *and any measurable functions* $h_{i0}, g_{i0} :]a, b[\to R$ $(i = 1, 2)$ *satisfying the conditions*

$$0 \leqq h_{10}(t) \leqq h_1(t), \qquad h_2(t) \leqq h_{20}(t) \leqq h(t), \qquad |g_{i0}(t)| \leqq g_i(t) \qquad \text{for } a < t < b$$

the inequality

$$|u(t)| \geqq B \left| \int_{t_0}^{t} h_{10}(\tau)\, d\tau \right| \qquad \text{for } a \leqq t \leqq b \tag{38}$$

holds where (u, v) *is the solution of the problem* (24), (25).

Lemma 8. *Let the functions* $g_i : [a, b] \to [0, +\infty[$ *be summable,* $h, h_i \in L_{loc}(]a, b[)$ $(i = 1, 2)$, $c \in]a, b[$,

$$(h_1, h_2, g_1 + g_2) \in \mathscr{P}_{02}(a, c) \cap \mathscr{P}_{02}(c, b),$$
$$h_1(t) \geqq 0, \qquad h_2(t) \leqq h(t) \qquad \text{for } a < t \leqq c,$$
$$h_1(t) \leqq h(t), \qquad h_2(t) \geqq 0 \qquad \text{for } c \leqq t < b,$$

$h(t)\, \mu_1(t) \in L([a, c])$, $h(t)\, \nu_2(t) \in L([c, b)])$ *where* μ_1 *and* ν_2 *are defined by* (5). *Then there exists a positive constant* B *such that for any* $t_0 \in [a, c]$ *and any measurable functions* $h_{i0}, g_{i0} :]a, b[\to R$ $(i = 1, 2)$ *satisfying the conditions*

$$0 \leqq h_{10}(t) \leqq h_1(t), \qquad h_2(t) \leqq h_{20}(t) \leqq h(t) \qquad \text{for } a < t \leqq c,$$
$$h_1(t) \leqq h_{10}(t) \leqq h(t), \qquad 0 \leqq h_{20}(t) \leqq h_2(t) \qquad \text{for } c \leqq t < b,$$
$$|g_{i0}(t)| \leqq g_i(t) \qquad \text{for } a < t < b$$

the inequality

$$v(t) \geqq B \qquad \text{for } c \leqq t \leqq b$$

holds where (u, v) *is the solution of the problem* (24), (25).

§ 3. Main results

In this section we shall prove existence and uniqueness theorems for the problems (1), (2) and (1), (3). Remind that the class of functions f_i under consideration as well as the idea of solutions of the system (1) were defined in § 1.

1. Existence theorems.

Theorem 1. *Let in* $]a, b[\times R^2$ *the inequalities*

$$
\begin{aligned}
(39)\qquad & -g_1(t) \mid x \mid + h_{11}(t) \mid y \mid - \eta_1(t) \leqq f_1(t, x, y) \operatorname{sign} y \leqq \\
& \leqq g_1(t) \mid x \mid + h_{12}(t) \mid y \mid + \eta_1(t), \\
& h_{22}(t) \mid x \mid - g_2(t) \mid y \mid - \eta_2(t) \leqq f_2(t, x, y) \operatorname{sign} x \leqq \\
& \leqq h_{21}(t) \mid x \mid + g_2(t) \mid y \mid + \eta_2(t)
\end{aligned}
$$

hold where $\eta_1, g_i \in L([a, b])$, (6) *is fulfilled,* $\eta_2 \in L_{loc}(]a, b[)$, $\eta_2(t)\, \mu_1(t)\, \nu_1(t) \in L([a, b])$, μ_1, ν_1, h_i *are defined by* (5) *and* (32) *and for a certain integer* k

$$(40)\qquad (h_{1i}, h_{2i}, g_1 + g_2) \in \mathscr{P}_{k1}(a, b) \qquad (i = 1, 2).$$

Then the problem (1), (2) *has at least one solution.*

Theorem 2. *Let in* $]a, b[\times R^2$ *the inequalities* (39) *hold where* $g_i \in L([a, b])$, (11) *is fulfilled,*

$$
\begin{aligned}
&\eta_1 \in L_{loc}([a, b[), \qquad \eta_1(t)\, \nu_2(t) \in L_{loc}(]a, b]), \\
&\eta_2 \in L_{loc}(]a, b]), \qquad \eta_2(t)\, \mu_1(t) \in L_{loc}([a, b[),
\end{aligned}
$$

μ_1, ν_2, h_i *are defined by* (5) *and* (32) *and for a certain integer* k

$$(41)\qquad (h_{1i}, h_{2i}, g_1 + g_2) \in \mathscr{P}_{k2}(a, b) \qquad (i = 1, 2).$$

Then the problem (1), (3) *has at least one solution.*

Theorem 3. *Let in* $]a, b[\times R^2$ *the inequalities*

$$
\begin{aligned}
(42)\qquad & -g_1(t) \mid x \mid + h_0(t) \mid y \mid - h_0(t)\, \eta_0 \leqq f_1(t, x, y) \operatorname{sign} y \leqq \\
& \leqq g_1(t) \mid x \mid + h_1(t) \mid y \mid + h_0(t)\, \eta_0, \\
& f_2(t, x, y) \operatorname{sign} x \geqq h_2(t) \mid x \mid - g_2(t) \mid y \mid - \eta(t)
\end{aligned}
$$

hold where $\eta_0 \in [0, +\infty[$, *the functions* $h_0, g_i :]a, b[\to [0, +\infty[$ $(i = 1, 2)$ *are summable,*

$$(43)\qquad I_0(a, t)\, I_0(t, b) > 0 \quad \text{for } a < t < b, \qquad I_0(s, t) = \int_s^t h_0(\tau)\, d\tau,$$

$\eta(t)\, \mu_1(t)\, \nu_1(t) \in L([a, b])$, μ_1 *and* ν_1 *are defined by* (5) *and*

$$(44)\qquad (h_1, h_2, g_1 + g_2) \in \mathscr{P}_{01}(a, b).$$

Then the problem (1), (2) *has at least one solution.*

Theorem 4. *Let* $c \in]a, b[$, *and let the inequalities* (42) *be valid in* $]a, c[\times R^2$ *and the inequalities*

$$f_1(t, x, y) \operatorname{sign} y \geqq -g_1(t) \mid x \mid + h_1(t) \mid y \mid - \eta(t),$$

$$h_0(t)\,|x| - g_2(t)\,|y| - h_0(t)\,\eta_0 \leqq f_2(t,x,y)\operatorname{sign} x \leqq$$
$$\leqq h_2(t)\,|x| + g_2(t)\,|y| + h_0(t)\,\eta_0$$

hold in $]c, b[\times R^2$ *where* $\eta_2 \in [0, +\infty[$, $h_i \in L_{loc}(]a, b[)$, *the functions* $h_0, g_i :]a,b[\to [0, +\infty[$ $(i = 1, 2)$ *are summable,* (43) *is fulfilled,*

$$\eta(t)\,\mu_1(t) \in L([a, c]), \qquad \eta(t)\,\nu_2(t) \in L\,[c, b]),$$

μ_1 *and* ν_2 *are defined by* (5) *and*

$$(h_1, h_2, g_1 + g_2) \in \mathscr{P}_{02}(a, c) \cap \mathscr{P}_{02}(c, b). \tag{45}$$

Then the problem (1), (3) *has at least one solution.*

Proof of Theorem 1. Let $a_n \in]a, b[$, $b_n \in]a_n, b[$ $(n = 1, 2, \dots)$ and $a_n \to a$, $b_n \to b$ when $n \to \infty$. According to Lemmas 2, 2′ and 6 there exist positive constants A, A_n $(n = 1, 2, \dots)$ and B such that for any $t_0 \in [a, b]$ and any measurable functions $h_{i0}, g_{i0} :]a, b[\to R$ $(i = 1, 2)$ satisfying (36) the inequalities

$$|u_1(t)| \leqq A\mu_1(t), \quad |u_2(t)| \leqq A\nu_1(t) \quad \text{for } a \leqq t \leqq b, \quad |u_1(b)| \geqq B,$$
$$|v_1(t)| \leqq A_n \quad \text{for } a \leqq t \leqq b_n, \quad |v_2(t)| \leqq A_n \quad \text{for } a_n \leqq t \leqq b$$

hold and on $]a, b[$, in addition,

$$|v_1(t)| \leqq \frac{A}{\nu_1(t)} \quad \text{if} \quad \nu_1(t) \neq 0, \qquad |v_2(t)| \leqq \frac{A}{\mu_1(t)} \quad \text{if} \quad \mu_1(t) \neq 0,$$

where (u_1, v_1) and (u_2, v_2) are solutions of (24) under the conditions

$$u_1(a) = 0,\ v_1(a) = 1; \qquad u_2(b) = 0,\ v_2(b) = 1. \tag{46}$$

Set (15) and

$$\varrho_n = \frac{(A\mu_1(b) + A_n)\,\lambda}{B}\Big[A_n \int_a^b \eta_1(\tau)\,d\tau + A\int_a^{b_n} \mu_1(\tau)\,\eta_2(\tau)\,d\tau + A\int_{a_n}^b \nu_1(\tau)\,\eta_2(\tau)\,d\tau\Big],$$

$$\sigma_{0n}(t) = \begin{cases} 1 & \text{for } t \in [a_n, b_n], \\ 0 & \text{for } t \in [a, b] \setminus [a_n, b_n], \end{cases} \qquad \sigma_n^*(t) = \begin{cases} 1 & \text{for } 0 \leqq t \leqq \varrho_n, \\ 2 - \dfrac{t}{\varrho_n} & \text{for } \varrho_n < t < 2\varrho_n, \\ 0 & \text{for } t \geqq 2\varrho_n, \end{cases} \tag{47}$$

$$\begin{gathered} \sigma_n(t, x, y) = \sigma_{0n}(t)\,\sigma_n^*(|x| + |y|), \\ f_{1n}(t, x, y) = h_{11}(t)\,y + \sigma_n(t, x, y)\,[f_1(t, x, y) - h_{11}(t)\,y], \\ f_{2n}(t, x, y) = h_{22}(t)\,x + \sigma_n(t, x, y)\,[f_2(t, x, y) - h_{22}(t)\,x], \\ (n = 1, 2, \dots). \end{gathered} \tag{48}$$

Let n be a natural number. Suppose that (u_{10}, v_{10}) and (u_{20}, v_{20}) are nontrivial solutions of the system

$$u' = h_{11}(t)\,v, \qquad v' = h_{22}(t)\,u \tag{49}$$

and $u_{10}(a) = 0$, $u_{20}(b) = 0$. For certain $\alpha, \beta \in [-\pi/2, \pi/2]$ we have

$$u_{10}(a_n) \sin \alpha - v_{10}(a_n) \cos \alpha = 0, \qquad u_{20}(b_n) \sin \beta - v_{20}(b_n) \cos \beta = 0.$$

If j is a sufficiently large positive number, then (ju_{10}, jv_{10}) and $(-ju_{10}, -jv_{10})$ are solutions of the system

$$x' = f_{1n}(t, x, y), \qquad y' = f_{2n}(t, x, y) \tag{50}$$

on $[a_n, b_n]$. The points $(ju_{10}(b_n), jv_{10}(b_n))$ and $(-ju_{10}(b_n), -jv_{10}(b_n))$ lie either in distinct half planes with respect to the straight line $x \sin \beta - y \cos \beta = 0$ or directly on this line. In any case, by the Kneser theorem ([7], p. 28) (50) has a solution (x_n, y_n) on $]a, b[$ such that

$$x(a_n) \sin \alpha - y(a_n) \cos \alpha = 0, \qquad x(b_n) \sin \beta - y(b_n) \cos \beta = 0.$$

Obviously, (x_n, y_n) satisfies (2).

Using (39), it is easy to verify that (x_n, y_n) is a solution of a certain system

$$x' = g_{10}(t)\, x + h_{10}(t)\, y + \eta_{10}(t), \qquad y' = h_{20}(t)\, x + g_{20}(t)\, y + \eta_{20}(t),$$

where the functions $g_{i0}, h_{i0}, \eta_{i0} :]a, b[\to R$ are measurable, the inequalities (36) hold and

$$|\eta_{i0}(t)| \leqq \eta_i(t) \quad \text{for} \quad a < t < b \quad (i = 1, 2).$$

Now let (u_1, v_1) and (u_2, v_2) be the solutions of (24) satisfying (46) and let

$$A(t) = \exp\left(\int_t^b [g_{10}(\tau) + g_{20}(\tau)]\, d\tau\right), \qquad w = u_1(b).$$

Define in $]a, b[\times]a, b[$ a second order quadratic matrix $\mathscr{G}$ by the relations

$$\mathscr{G}(t, \tau) = \frac{A(\tau)}{w} \begin{pmatrix} -u_2(t)\, v_1(\tau) & u_2(t)\, u_1(\tau) \\ -v_2(t)\, v_1(\tau) & v_2(t)\, u_1(\tau) \end{pmatrix} \quad \text{for } \tau \leqq t, \tag{51}$$

$$\mathscr{G}(t, \tau) = \frac{A(\tau)}{w} \begin{pmatrix} -u_1(t)\, v_2(\tau) & u_1(t)\, u_2(\tau) \\ -v_1(t)\, v_2(\tau) & v_1(t)\, u_2(\tau) \end{pmatrix} \quad \text{for } t < \tau.$$

Lemma 3 implies[1])

$$\operatorname{col}(x_n(t), y_n(t)) = \int_a^b \mathscr{G}(t, \tau) \operatorname{col}(\eta_{10}(\tau), \eta_{20}(\tau))\, d\tau \quad \text{for } a < t < b, \tag{52}$$

and by the definition of the constants λ, A, A_m, B and ϱ_m we obtain

$$|x_n(t)| + |y_n(t)| \leqq \varrho_m \quad \text{for } a_m < t < b_m \quad (m = 1, 2, \ldots). \tag{53}$$

Thus the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are uniformly bounded on each segment $[a_m, b_m]$ $(m = 1, 2, \ldots)$. Simple arguments show that without loss of generality

---

[1]) Here and in what follows col (., .) denotes a column vector.

we may assume these sequences to be uniformly convergent on each segment contained within $]a, b[$.

Let $a^* \in]a, b[$. Then according to (52) in $]a, a^*[$ we have

$$|x_n(t)| \leqq \frac{A^2\lambda}{B}\left\{\int_a^{a^*}[\eta_1(\tau) + \mu_1(\tau)\nu_1(\tau)\eta_2(\tau)]\,d\tau + \mu_1(t)\int_{a^*}^{b}\left[\frac{\eta_1(\tau)}{\mu_1(\tau)} + \nu_1(\tau)\eta_2(\tau)\right]d\tau\right\}$$

$$\text{if} \qquad \mu_1(a^*) \neq 0,$$

$$|x_n(t)| \leqq \frac{A^2\lambda}{B}\int_a^t \eta_1(\tau)\,d\tau \qquad \text{if} \qquad \mu_1(a^*) = 0.$$

Since a^* is arbitrarily close to a, and since $\mu_1(t) \to 0$ when $t \to a$, these inequalities give

(54) $$\sup\{|x_n(t)| : n = 1, 2, \ldots\} \to 0 \qquad \text{when } t \to a.$$

Similarly,

(55) $$\sup\{|x_n(t)| : n = 1, 2, \ldots\} \to 0 \qquad \text{when } t \to b.$$

Hence, as it follows from Lemma 2.5 of [3], the sequence $(x_n)_{n=1}^{\infty}$ uniformly converges on $[a, b]$. Furthermore, (53) and the definition of f_{1n} and f_{2n} imply that (x_n, y_n) $(n = 1, 2, \ldots)$ are solutions of the system (1) on $[a_n, b_n]$. Thus, if

(56) $$x(t) = \lim_{n\to\infty} x_n(t), \qquad y(t) = \lim_{n\to\infty} y_n(t) \qquad \text{for } a < t < b,$$

then (x, y) is a solution of (1), (2). This completes the proof.

The proof of Theorem 2 is quite similar.

Proof of Theorem 3. According to Lemmas 2 and 7 there exist constants $A \in [1, +\infty[$ and $B \in]0, 1]$ such that if the measurable functions $h_{10}, g_{i0} : [a, b] \to R$ $(i = 1, 2)$ satisfy the inequalities

(57) $$h_0(t) \leqq h_{10}(t) \leqq h_1(t), \qquad |g_{i0}(t)| \leqq g_i(t) \qquad \text{for } a \leqq t \leqq b$$

and if (u, v) is a solution of the system

(58) $$u' = g_{10}(t)\,u + h_{10}(t)\,v, \qquad v' = h_2(t)\,u + g_{20}(t)\,v$$

under the initial conditions (25) where $t_0 \in [a, b]$, then (23) and (38) hold. (Note that (42) and (43) imply $\mu_1(t)\,\nu_1(t) > 0$ for $a < t < b$).

Let $a_0, b_0 \in]a, b[$, $a_n \in]a, a_0[$, $b_n \in]b_0, b[$ $(n = 1, 2, \ldots)$, $a_n \to a$, $b_n \to b$ when $n \to \infty$ and

$$I_0(a_0, b_0) > 0.$$

Set (15), (47), (48) and

(59) $$\varrho_n = \frac{7A^3\lambda^2}{B^2}(1 + \mu_1(\;))^2\left(1 + \frac{1}{I_0(a, a_n)\,I_0(b_n, b)}\right)[\eta_0 + \int_a^b \mu_1(\tau)\,\nu_1(\tau)\,\eta(\tau)\,d\tau],$$

$$f_{1n}(t, x, y) = h_1(t)\, y + \sigma_n(t, x, y)\,[f_1(t, x, y) - h_1(t)\, y],$$
(60)
$$f_{2n}(t, x, y) = h_2(t)\, x + \sigma_n(t, x, y)\,[f_2(t, x, y) - h_2(t)\, x],$$
$$(n = 1, 2, \ldots).$$

Let n be a natural number. Considering the system

$$u' = h_1(t)\, v, \qquad v' = h_2(t)\, u$$

instead of (49) and using the arguments carried out in Proof of Theorem 1, we verify that the problem (50), (2) has a solution (x_n, y_n). At the same time, by (42) (x_n, y_n) is a solution of a certain system

$$x' = g_{10}(t)\, x + h_{10}(t)\, y + \eta_{10}(t), \qquad y' = h_2(t)\, x + g_{20}(t)\, y + \eta_{20}(t) + \xi(t),$$

where the functions $h_{10}, \xi, g_{i0}, \eta_{i0} : [a, b] \to R$ $(i = 1, 2)$ are summable, satisfy (57) and

(61) $$|\eta_{10}(t)| \leqq h_0(t)\, \eta_0, \quad |\eta_{20}(t)| \leqq \eta(t), \quad \xi(t)\, x_n(t) \geqq 0 \quad \text{for } a \leqq t \leqq b.$$

Let $s \in\,]a, b[$ and $x_n(s) \neq 0$. Then there exist $t_1 \in [a, s[$ and $t_2 \in\,]s, b]$ such that

(62) $$x_n(t_1) = x_n(t_2) = 0, \qquad x_n(t) \neq 0 \quad \text{in }]t_1, t_2[.$$

Lemma 4 implies

(63) $$\operatorname{col}(x_n(t), y_n(t)) = \int_{t_1}^{t_2} \mathscr{G}(t, \tau) \operatorname{col}(\eta_{10}(\tau), \eta_{20}(\tau) + \xi(\tau))\, d\tau \qquad \text{for } t_1 < t < t_2,$$

where the matrix $\mathscr{G}$ is defined by (51),

(64) $$A(t) = \exp\Big(-\int_{t_1}^{t} [g_{10}(\tau) + g_{20}(\tau)]\, d\tau\Big), \qquad w = -u_2(t_1),$$

(u_i, v_i) are solutions of (58) and $u_i(t_i) = 0$, $v_i(t_i) = 1$ $(i = 1, 2)$.

Taking into account (61), (62) and Lemma 7, we conclude that the terms

$$\frac{(-1)^{i+1}}{w}\, u_{3-i}(t) \int_{t_i}^{t} u_i(\tau)\, \xi(\tau)\, A(\tau)\, d\tau \qquad (i = 1, 2)$$

are nonnegative for $t_1 < t < t_2$ if $x_n(t) < 0$ and nonpositive otherwise. Thus, applying (15), (61) and the definition of the constants A and B, from the first component of the equality (63) we obtain

(65) $$|x_n(\,)| \leqq \frac{1}{w}[u_1(t)\,\varkappa_2(t) - u_2(t)\,\varkappa_1(t)] \leqq$$
$$\leqq \frac{A^2 \lambda \mu_1(b)}{B}\left[\eta_0 + \frac{1}{I_0(a, t)\, I_0(t, b)} \int_a^b \mu_1(\tau)\, v_1(\tau) \eta(\tau)\, d\tau\right],$$

$$(66) \qquad \Big|\int_{t_i}^{t} | u_i(\tau)\,\zeta(\tau) \,|\, A(\tau)\,\mathrm{d}\tau\Big| \leqq \varkappa_i(t) + \left|\frac{u_i(t)}{u_{3-i}(t)}\right| \varkappa_{3-i}(t),$$

where

$$(67) \quad \varkappa_i(t) = \Big|\int_{t_i}^{t} | v_i(\tau)\,\eta_{10}(\tau) + u_i(\tau)\,\eta_{20}(\tau) \,|\, A(\tau)\,\mathrm{d}\tau\Big| \qquad \text{for } t_1 < t < t_2 (i = 1, 2).$$

The last estimate along with the second component of the equality (63) gives

$$(68) \quad | y_n(t) | \leqq \frac{6A^3\lambda(1 + [\mu_1(b)]^2)}{B^2 I_0(a, t)\, I_0(t, b)} \left[\eta_0 + \int_a^b \mu_1(\tau)\,v_1(\tau)\,\eta(\tau)\,\mathrm{d}\tau\right] \qquad \text{for } t_1 < t < t_2.$$

Now let $s \in]a, b[$, $x_n(s) = 0$, and let there exist t_1 and t_2 such that

$$t_1 \in [a, s[, \qquad t_2 \in]s, b], \qquad x_n(t) = 0 \qquad \text{for } t_1 \leqq t \leqq t_2.$$

If $[t_1, t_2]$ is the maximal segment with these properties, then for each natural number m we have one of the following possibilities:

(i) there exist $s_j \in [a_m, b_m]$ $(j = 1, 2, \ldots)$ such that $x_n(s_j) \neq 0$ and either $s_j \to t_1$ or $s_j \to t_2$ when $j \to \infty$;

(ii) $[t_1, t_2] \supset [a_m, b_m]$.

Let (i) occur. Then $| y_n(t_0) | \leqq \varrho^*(t_0)$ where $t_0 \in [a_m, b_m]$ is either t_1 or t_2 and $\varrho^*(t)$ is the right-hand side of the inequality (68).

Now let (ii) take place. Then, since $[a_m, b_m] \supset [a_0, b_0]$, the first of the inequalities (42) implies the existence of $t_0 \in [a_m, b_m]$ such that $| y_n(t_0) | \leqq \eta_0$.

In both the cases from the inequality

$$| y_n'(t) | \leqq g_2(t)\, | y_n(t) | + \eta(t) \qquad \text{for } t_1 < t < t_2,$$

which is due to the second of the conditions (42), we obtain

$$| y_n(t) | \leqq \left[| y_n(t_0)| + \int_{a_m}^{b_m} \eta(\tau)\,\mathrm{d}\tau\right] \exp\left(\int_a^b g_2(s)\,\mathrm{d}s\right)$$
$$\text{for } t \in [t_1, t_2] \cap [a_m, b_m] \; (m = 1, 2, \ldots).$$

Thus considering (59), (65) and (68) we conclude that (53) is fulfilled for all $s \in I_m$ where I_m is a certain set dense in $[a_m, b_m]$. Therefore (53) is valid for all $s \in [a_m, b_m]$, and without loss of generality we may assume that the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are uniformly convergent on each segment of $]a, b[$.

Suppose that $a^* \in]a, b[$, $3I_0(a, a^*) < I_0(a, b)$ and $s \in]a, a^*[$. If $x_n(s) \neq 0$ for a certain natural number n, choose $t_1 \in [a, s[$ and $t_2 \in]s, b]$ satisfying (62). Then from (15), (23), (38) and (63) we obtain

$$| x_n(s) | \leqq \frac{A^2\lambda}{B} \int_a^{a^*} [h_0(\tau)\,\eta_0 + \mu_1(\tau)\,\eta(\tau)]\,\mathrm{d}\tau,$$

when $t_2 \leqq a^*$, and

$$|x_n(s)| \leqq \frac{A^2\lambda}{B} \int_a^{a^*} [h_0(\tau)\eta_0 + \mu_1(\tau)\eta(\tau)]\, d\tau +$$

$$+ \frac{A^2\lambda}{B}\mu_1(t)\left[1 + \frac{\nu_1(a^*)}{\mu_1(a^*)} + \frac{3}{I_0(a, b)}\right][\eta_0 + \int_{a^*}^{b} \nu_1(\tau)\eta(\tau)\, d\tau],$$

when $t_2 > a^*$. These inequalities give (54). Moreover, by the similar ar we can show that (55) also holds. Thus the sequence $(x_n)_{n=1}^{\infty}$ uniformly co on the segment $[a, b]$.

Using (53) and the definition of f_{1n}, f_{2n}, we establish that (x, y) with given by (56) is a solution of the problem (1), (2). This completes the prc

Proof of Theorem 4. By virtue of Lemmas 2, 7 and 8 it is easy to ve there exist $A \in [1, +\infty[$ and $B \in]0, 1]$ such that for any points $t_1 \in [a,$ $\in [c, b]$ and any measurable functions $h_{i0}, g_{i0} :]a, b[\to R$ $(i = 1, 2)$ sa the conditions

$$\begin{aligned}
&h_0(t) \leqq h_{10}(t) \leqq h_1(t), \quad h_{20}(t) = h_2(t) \quad \text{for } a < t < c,\\
(69) \qquad &h_{10}(t) = h_1(t), \quad h_0(t) \leqq h_{20}(t) \leqq h_2(t) \quad \text{for } c < t < b,\\
&|g_{i0}(t)| \leqq g_i(t) \quad \text{for } a < t < b
\end{aligned}$$

we have in $]a, b[$

$$B_1 |I_1(t_1, t)| \leqq |u_1(t)| \leqq A |I_1(t_1, t)|, \quad u_2(t) \geqq B, \quad |v_2(t)| \leqq A[\mu_1$$
$$\text{for } t \leqq c,$$

$$|u_1(t)| \leqq A[v_2(t)]^{-1}, \quad v_1(t) \geqq B, \quad B|I_2(t_2, t)| \leqq |v_2(t)| \leqq A |I_2($$
$$\text{for } t \geqq c,$$

$$|v_1(t)| \leqq A \max\left\{1, \frac{I_1(t, t_1)}{\mu_1(t)}\right\} \quad \text{for } t < t_1, \quad |v_1(t)| \leqq A \quad \text{for } t \geqq$$

$$|u_2(t)| \leqq A \quad \text{for } t \leqq t_2, \quad |u_2(t)| \leqq A \max\left\{1, \frac{I_2(t_2, t)}{v_2(t)}\right\} \quad \text{for } t$$

where

$$I_i(s, t) = \int_s^t h_{i0}(\tau)\, d\tau \qquad (i = 1, 2),$$

(u_1, v_1), (u_2, v_2) are solutions of the system (24) and

$$(70) \qquad u_1(t_1) = 0, \ v_1(t_1) = 1; \qquad u_2(t_2) = 1, \ v_2(t_2) = 0.$$

Let a_n, b_n $(n = 0, 1, \ldots)$ be the same as in Proof of Theorem 3 and

$$I_0(a_0, c)\, I_0(c, b_0) > 0.$$

Put (15), (47) and (48) where

$$\varrho_n = \frac{9A^4\lambda^2}{B^3}(1+\mu_1(c))(1+\nu_2(c))\left(1+\frac{1}{I_0(a,a_n)}+\frac{1}{I_0(b_n,b)}\right)\times$$
$$\times\left[\eta_0+\int_a^c \mu_1(\tau)\eta(\tau)\,d\tau+\int_c^b \nu_2(\tau)\eta(\tau)\,d\tau\right] \qquad (n=1,2,\ldots).$$

For a natural number n consider the system (50) where the functions f_{1n}, f_{2n} are defined by (60). Just as it was carried out in the proof of the previous theorem, we can show that the problem (50), (3) has a solution (x_n, y_n) which, at the same time, satisfies the system

$$x' = g_{10}(t)\,x + h_{10}(t)\,y + \eta_{10}(t) + \xi_1(t),$$
$$y' = h_{20}(t)\,x + g_{20}(t)\,y + \eta_{20}(t) + \xi_2(t)$$

under the conditions (69) and

$$\xi_1(t)=0,\ \ \xi_2(t)\,x_n(t)\geqq 0,\ \ |\eta_{10}(t)|\leqq h_0(t)\,\eta_0,\ \ |\eta_{20}(t)|\leqq \eta(t) \quad \text{for } a<t\leqq c,$$
$$\xi_1(t)\,y_n(t)\geqq 0,\ \ \xi_2(t)=0,\ \ |\eta_{10}(t)|\leqq \eta(t),\ \ |\eta_{20}(t)|\leqq h_0(t)\,\eta_0 \quad \text{for } c\leqq t<b.$$

Let $s\in\,]a,c[$ and $x_n(s)\neq 0$. Then there exist $t_1\in[a,s[$ and $t_2\in\,]s,b]$ such that either

(i) $t_2\leqq c$ and (62) holds

or

(ii) $t_2\geqq c$, $x_n(t_1)=y_n(t_2)=0$, $x_n(t)\neq 0$ on $]t_1,c]$ and $y_n(t)\neq 0$ on $[c,t_2[$ (if $t_2>c$).

Note that since $(h_1,h_2,g_1+g_2)\in\mathscr{P}_{01}(a,c)$,the case (i) has been studied in Proof of Theorem 3. Thus we obtain

$$\text{(71)}\qquad \begin{aligned} |x_n(t)| &\leqq \frac{A^2\lambda}{B}(1+\mu_1(c))\left[\eta_0+\int_a^c \mu_1(\tau)\eta(\tau)\,d\tau\right] \\ |y_n(t)| &\leqq \frac{6A^3\lambda(1+\mu_1(c))}{B^2 I_0(a,t)}\left[\eta_0+\int_a^c \mu_1(\tau)\eta(\tau)\,d\tau\right] \end{aligned} \qquad \text{for } t_1<t<t_2.$$

Now consider the case (ii). From Lemma 5 it follows that

$$\text{(72)}\qquad \operatorname{col}(x_n(t),y_n(t)) = \int_{t_1}^{t_2} \mathscr{G}(t,\tau)\operatorname{col}(\eta_{10}(\tau)+\xi_1(\tau),\eta_{20}(\tau)+\xi_2(\tau))\,d\tau$$
$$\text{for } t_1<t<t_2$$

where the matrix $\mathscr{G}$ is given by (51), (64) and (u_1, v_1), (u_2, v_2) are the solutions of (24) satisfying (70).

When set $t=c$ in (72) and compare the signs of the functions u_i, v_i, ξ_i with the signs of $x_n(c)$ and $y_n(c)$, we obtain

$$\chi\leqq\frac{u_2(c)}{u_1(c)}\varkappa_1(c)+\varkappa_2(c) \qquad \text{if} \qquad x_n(c)\,y_n(c)\geqq 0,$$

$$\chi \leqq \frac{v_2(c)}{v_1(c)} \varkappa_1(c) + \varkappa_2(c) \qquad \text{if} \qquad x_n(c)\, y_n(c) < 0,$$

where $\varkappa_i$ are defined by (67) and

$$\chi = \int_c^{t_2} | v_2(\tau)\, \xi_1(\tau) |\, A(\tau)\, d\tau.$$

Furthermore, from the first component of (72) by the analogy with (66) we have

$$\chi_i(t) \leqq \varkappa_i(t) + \frac{u_i(t)}{u_{3-i}(t)} \varkappa_{3-i}(t) + \frac{u_1(t)}{u_{3-i}(t)} \chi \qquad \text{for } t_1 < t \leqslant c,$$

where

$$\chi_i(t) = \left| \int_{t_i}^{t} | u_i(\tau)\, \xi_2(\tau) |\, A(\tau)\, d\tau \right| \qquad \text{for } t_1 \leqq t \leqq t_2 \quad (i = 1, 2).$$

Considering the estimates established above and applying the definition of the constants λ, A and B, we conclude from (72) that on $]t_1, c]$

$$| x_n(t) | \leqq \frac{1}{| w |} [u_2(t)\, \varkappa_1(t) + u_1(t) (\varkappa_2(t) + \chi)] \leqq$$

$$\leqq \frac{2A^3 \lambda}{B^2} (1 + \mu_1(c)) (1 + \nu_2(c)) [\eta_0 + \int_a^c \mu_1(\tau)\, \eta(\tau)\, d\tau + \int_c^b \nu_2(\tau)\, \eta(\tau)\, d\tau],$$

$$| y_n(t) | \leqq \frac{1}{| w |} [| v_2(t) | (\varkappa_1(t) + \chi_1(t)) + | v_1(t) | (\varkappa_2(t) + \chi_2(t) + \chi)] \leqq$$

$$\leqq \frac{9A^4 \lambda (1 + \mu_1(c)) (1 + \nu_2(c))}{B^3 I_0(a, t)} [\eta_0 + \int_a^c \mu_1(\tau)\, \eta(\tau)\, d\tau + \int_c^b \nu_2(\tau)\, \eta(\tau)\, d\tau].$$

From these inequalities and (71) we may derive by the method used in Proof of Theorem 3 that $| x_n(t) | + | y_n(t) | \leqq \varrho_m$ on $[a_m, c]$ $(m = 1, 2, \ldots)$ and (54) holds. Moreover, it may be similarly shown that $| x_n(t) | + | y_n(t) | \leqq \varrho_m$ on $[c, b_m]$ $(m = 1, 2, \ldots)$ and

$$\sup \{| y_n(t) | : n = 1, 2, \ldots\} \to 0 \qquad \text{when } t \to b.$$

Thus without loss of generality we may assume that the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are uniformly convergent on each segment contained within $[a, b[$ and $]a, b]$ respectively and so (x, y) defined by (56) is a solution of the problem (1), (3). This completes the proof.

2. Uniqueness theorems.

Theorem 5. *Let the inequalities*

$$-g_1(t) | x_1 - x_2 | + h_{11}(t) | y_1 - y_2 | \leqq$$

$$\leqq [f_1(t, x_1, y_1) - f_1(t, x_2, y_2)] \operatorname{sign} (y_1 - y_2) \leqq g_1(t) | x_1 - x_2 | + h_{12}(t) | y_1 - y_2 |,$$

$$(73) \qquad h_{22}(t)\,|\,x_1 - x_2\,| - g_2(t)\,|\,y_1 - y_2\,| \leqq$$
$$\leqq [f_2(t, x_1, y_1) - f_2(t, x_2, y_2)] \operatorname{sign}(x_1 - x_2) \leqq h_{21}(t)\,|\,x_1 - x_2\,| + g_2(t)\,|\,y_1 - y_2\,|$$

hold in $]a, b[\times R^2$ *and let the conditions* (40) *be fulfilled for a certain integer* k. *Moreover, suppose that* (6) *with* μ_1, ν_1 *and* h_i *defined by* (5) *and* (32) *is valid. Then the problem* (1), (2) *has at most one solution.*

Theorem 6. *Let the inequalities* (73) *hold in* $]a, b[\times R^2$ *and let the conditions* (41) *be fulfilled for a certain integer* k. *Moreover, suppose that* (11) *with* μ_1, ν_2 *and* h_i *defined by* (5) *and* (32) *is valid. Then the problem* (1), (3) *has at most one solution.*

Theorem 7. *Let the inequalities*

$$-g_1(t)\,|\,x_1 - x_2\,| + h_0(t)\,|\,y_1 - y_2\,| \leqq [f_1(t, x_1, y_1) - f_1(t, x_2, y_2)] \operatorname{sign}(y_1 - y_2) \leqq$$
$$(74) \qquad \leqq g_1(t)\,|\,x_1 - x_2\,| + h_1(t)\,|\,y_1 - y_2\,|,$$
$$[f_2(t, x_1, y_1) - f_2(t, x_2, y_2)] \operatorname{sign}(x_1 - x_2) \geqq h_2(t)\,|\,x_1 - x_2\,| - g_2(t)\,|\,y_1 - y_2\,|$$

hold in $]a, b[\times R^2$ *and let the condition* (44) *be fulfilled where* $h_0, g_i \in L([a, b])$, $h_0(t) \geqq 0$ *for* $a \leqq t \leqq b$ *and* h_0 *differs from zero on a set of positive measure. Then the problem* (1), (2) *has at most one solution.*

Theorem 8. *Let* $c \in]a, b[$ *and let the inequalities* (74) *with* $h_0(t) \equiv 0$ *be valid in* $]a, c[\times R^2$ *and the inequalities*

$$[f_1(t, x_1, y_1) - f_1(t, x_2, y_2)] \operatorname{sign}(y_1 - y_2) \geqq -g_1(t)\,|\,x_1 - x_2\,| + h_1(t)\,|\,y_1 - y_2\,|,$$
$$-g_2(t)\,|\,y_1 - y_2\,| \leqq [f_2(t, x_1, y_1) - f_2(t, x_2, y_2)] \operatorname{sign}(x_1 - x_2) \leqq$$
$$\leqq h_2(t)\,|\,x_1 - x_2\,| + g_2(t)\,|\,y_1 - y_2\,|$$

hold in $]c, b[\times R^2$ *where* $h_i \in L_{loc}(]a, b[)$, $g_i \in L([a, b])$ $(i = 1, 2)$ *and* (45) *is fulfilled. Then the problem* (1), (3) *has at most one solution.*

Proof of Theorem 5. Let (x_i, y_i) $(i = 1, 2)$ be solutions of the problem (1), (2). Set

$$(75) \qquad x(t) = x_1(t) - x_2(t), \qquad y(t) = y_1(t) - y_2(t).$$

It immediately follows from the first inequality (73) that

$$-g_1(t)\,|\,x(t)\,| + h_{11}(t)\,|\,y(t)\,| \leqq x'(t) \operatorname{sign} y(t) \leqq g_1(t)\,|\,x(t)\,| + h_{12}(t)\,|\,y(t)\,|$$

in $]a, b[$ and, since f_1 is continuous in the last variable,

$$-g_1(t)\,|\,x(t)\,| \leqq x'(t) \leqq g_1(t)\,|\,x(t)\,| \qquad \text{when } y(t) = 0.$$

The second inequality (73) implies the analogous relations for y'.

Thus (x, y) is a solution of a certain system (24) with measurable coefficients $h_{i0}, g_{i0} :]a, b[\to R$ satisfying (36). But according to Lemma 3 this system has not nontrivial solutions under the conditions (2). This completes the proof.

The proof of Theorem 6 is quite similar.

Proof of Theorem 7. Let (x_i, y_i) $(i = 1, 2)$ be solutions of the problem (1), (2). Set (75).

Using (74) we easily verify that (x, y) is a solution of the system

$$(76) \qquad x' = g_{10}(t)\, x + h_{10}(t)\, y, \qquad y' = h_2(t)\, x + g_{20}(t)\, y + \xi(t)$$

where measurable functions $h_{10}, g_{i0}, \xi :]a, b[\to R$ $(i = 1, 2)$ satisfy (57) and

$$(77) \qquad \xi(t)\, x(t) \geqq 0 \qquad \text{for } a < t < b.$$

Let $x(s) \neq 0$ for a certain $s \in]a, b[$. Choose $t_1 \in [a, s[$ and $t_2 \in]s, b]$ such that

$$x(t_1) = x(t_2) = 0, \qquad x(t) \neq 0 \qquad \text{in }]t_1, t_2[.$$

If $h_{10}(t) = 0$ almost everywhere on $[t_1, t_2]$, then (76) implies $x(t) \equiv 0$ on $[t_1, t_2]$.

Now assume that $h_{10}(t) \neq 0$ on some set of positive measure from the segment $[t_1, t_2]$. Then according to Lemma 4 we have[1])

$$\operatorname{col}(x(t), y(t)) = \frac{x(s_2)}{u_1(s_2)} \operatorname{col}(u_1(t), v_1(t)) + \frac{x(s_1)}{u_2(s_1)} \operatorname{col}(u_2(t), v_2(t)) + \int_{s_1}^{s_2} \mathscr{G}(t, \tau) \operatorname{col}(0, \xi(\tau))\, d\tau$$

in $]s_1, s_2[$ for all $s_1 \in]t_1, t_2[$, $s_2 \in]s_1, t_2[$ sufficiently close to t_1, t_2 respectively where the matrix $\mathscr{G}$ is defined by (51),

$$(78) \qquad A(t) = \exp\left(-\int_{s_1}^{t} [g_{10}(\tau) + g_{20}(\tau)]\, d\tau\right), \qquad w = -u_2(s_1)$$

and (u_i, v_i) are solutions of (58) under the conditions

$$u_i(s_i) = 0, \qquad v_i(s_i) = 1 \qquad (i = 1, 2).$$

Hence, as (77) holds and $u_1(t) \geqq 0$, $u_2(t) \leqq 0$ on the segment $[s_1, s_2]$, we obtain on this segment

$$|x(t)| \leqq \frac{|x(s_2)|}{u_1(s_2)} u_1(t) + \frac{|x(s_1)|}{u_2(s_1)} u_2(t).$$

By Lemmas 2 and 7 there exist independent on the choice of s_1 and s_2 positive constants A and B such that

$$|x(t)| \leqq \frac{A}{B}(|x(s_1)| + |x(s_2)|) \qquad \text{for } s_1 \leqq t \leqq s_2.$$

---

[1]) Note that in general we may not use the Green formula on the whole $[t_1, t_2]$.

Taking into account the unrestricted closeness of s_i to t_i ($i = 1, 2$), we conclude that $x(t) \equiv 0$ on $[t_1, t_2]$ and thus on $[a, b]$.

(76) gives $h_0(t)\, y(t) = 0$ for $a \leqq t \leqq b$, but since h_0 is not equivalent to zero, y necessarily vanishes in some points of $]a, b[$. On the other hand, according to (74)

$$|\, y'(t) \,| \leqq g_2(t) \,|\, y(t) \,| \tag{79}$$

on $[a, b]$. Therefore $y(t) \equiv 0$. This completes the proof.

Proof of Theorem 8. Let (x_i, y_i) ($i = 1, 2$) be solutions of the problem (1), (3). Set (75).

Suppose that $x(c) \neq 0$ and that t_1 is the largest zero of x on $[a, c]$. Furthermore, denote by t_2 the smallest zero of y on $[c, b]$.

From the conditions of the theorem it follows that (x, y) is a solution of the system

$$x' = g_{10}(t)\, x + h_{10}(t)\, y + \xi_1(t), \qquad y' = h_{20}(t)\, x + g_{20}(t)\, y + \xi_2(t)$$

where $g_{i0}, h_{i0}, \xi_i :]a, b[\to R$ ($i = 1, 2$) are certain measurable functions satisfying (69) with $h_0(t) \equiv 0$ and

$$\begin{array}{llll} \xi_1(t) = 0, & \xi_2(t)\, x(t) \geqq 0 & \text{for } a < t \leqq c, \\ \xi_1(t)\, y(t) \geqq 0, & \xi_2(t) = 0 & \text{for } c \leqq t < b. \end{array} \tag{80}$$

Thus by Lemma 5

$$\operatorname{col}(x(t), y(t)) = \frac{y(s_2)}{v_1(s_2)} \operatorname{col}(u_1(t), v_1(t)) + \frac{x(s_1)}{u_2(s_1)} \operatorname{col}(u_2(t), v_2(t)) + \\ + \int_{s_1}^{s_2} \mathscr{G}(t, \tau) \operatorname{col}(\xi_1(\tau), \xi_2(\tau))\, d\tau \qquad \text{for } s_1 \leqq t \leqq s_2,$$

where $s_1 \in]t_1, c[$, s_2 is c, if $y(c) = 0$, and is an arbitrary point of $]c, t_2[$ otherwise, $\mathscr{G}$ is the matrix defined by (51), (78) and (u_i, v_i) are solutions of (24) under the conditions

$$u_1(s_1) = 0, \qquad v_1(s_1) = 1; \qquad u_2(s_2) = 1, \qquad v_2(s_2) = 0.$$

This equality and (80) imply

$$|\, x(c) \,| \leqq \frac{|\, y(s_2) \,|}{v_1(s_2)} u_1(c) + \frac{|\, x(s_1) \,|}{u_2(s_1)} u_2(c) \qquad \text{when } x(c)\, y(c) \geqq 0,$$

$$|\, y(c) \,| \leqq \frac{|\, y(s_2) \,|}{v_1(s_2)} v_1(c) + \frac{|\, x(s_1) \,|}{u_2(s_1)} |\, v_2(c) \,| \qquad \text{when } x(c)\, y(c) < 0.$$

Now taking into account Lemmas 2 and 8 as well as the unrestricted closeness of s_i to t_i ($i = 1, 2$), we conclude that the first inequality gives $x(c) = 0$ and the second one yields $y(c) = 0$. These contradictions show the falsity of the assumption $x(c) \neq 0$. We may analogously verify that $y(c) = 0$.

Because of (45), $(h_1, h_2, g_1 + g_2) \in \mathscr{P}_{01}(a, c)$. If $x(s) \neq 0$ for some $s \in]a, c[$, then repeating word for word the corresponding argument from Proof of Theorem 7, we get $x(t) = 0$ for $a \leqq t \leqq c$. Hence (79) is valid on $[a, c]$ which, since $y(c) = 0$, gives $y(t) \equiv 0$ on this segment. Similarly, $|x(t)| + |y(t)| \equiv 0$ on $[c, b]$. This completes the proof.

In the case when $f_1(t, x, y) \equiv y$, from Theorems 1,3,5,7 we obtain I. T.Kiguradze existence and uniqueness theorems [3] for the singular problem

$$x'' = f(t, x, x'), \qquad x(a) = x(b) = 0.$$

Moreover, in [3] the effective conditions under which $(1, h, g) \in \mathscr{P}_{k1}(a, b)$ are given (see also [4] and [8]).

The necessity of the main conditions of Theorems 1 – 8 is discussed in [4].

REFERENCES

[1] Tonelli, L.: *Sull'equazione differenziale $y'' = f(x, y, y')$*, Ann. R. Scuola norm. super. Pisa, Sci. fis. e mat., 1939, 8, 75—88.

[2] Epheser, H.: *Über die Existenz der Lösungen von Randwertaufgaben mit gewöhnlichen, nichtlinearen Differentialgleichungen zweiter Ordnung*, Math. Z., 1955, 61, N 4, 435—454.

[3] Kiguradze, I. T.: *Some singular boundary value problems for ordinary differential equations* (Russian), Tbilisi, Tbilisi Univ. Press, 1975.

[4] Shekhter, B. L.: *On a boundary value problem for two-dimensional discontinuous differential systems* (Russian), Proc. I. N. Vekua Inst. Appl. Math., 1980, 8, 79—161.

[5] Lezhava, N. R.: *On the solvability of a nonlinear problem for a system of two differential equations* (Russian), Bull. Acad. Sc. Georg. SSR, 1972, 68, N 3, 545—547.

[6] Krasnoselsky, M. A., Perov, A. I., Povolotsky, A. I., Zabreiko, P. P.: *Vector fields on the plane* (Russian), Moscow, Fizmatgiz, 1963.

[7] Hartman, P.: *Ordinary differential equations* (Russian), Moscow, „Mir", 1970.

[8] Gogiberidze, N. V., Kiguradze, I. T.: *Concerning the disconjugacy of second order singular differential equations* (Russian), Differentsial'nye Uravneniya, 1974, 10, N 11, 2064—2067.

B. L. Skekhter
I. N. Vekua Institute of Applied Mathematics
of Tbilisi State University
University str. 2
380043, Tbilisi, USSR