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NORMAL SUBGROUPS AS IDEALS

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1. INTRODUCTION

We can define *an ideal system on a grupoid* $(S, .)$ as a system $\{A_x : A \subseteq S\}$ fulfilling conditions:

1. $A \subseteq A_x$,
2. $A \subseteq B_x \Leftarrow A_x \subseteq B_x$,
3. $A_x \,.\, B \subseteq A_x$,
4. $A_x \,.\, B \subseteq (A \,.\, B)_x$.

This definition was introduced by K. Aubert [1] in the case where the grupoid operation is both commutative and associative.

We say that $A = A_x$ is *an ideal* and the grupoid operation is called *an ideal operation.*

In [6] F. Voráč studied the structure of a group G with normal subgroups acting as ideals and the commutator operation acting as the ideal operation. In the following we say that such a group G is a *K-group.* Voráč characterized the structure of a K-group G with the aid of the centralizers of the elements of factor groups of G (see [6], Theorem 2.7). Later on it was shown in [5] that a K-group is necessarily nilpotent. In this paper we show that the class of the nilpotent K-group G is at most three. Furthermore, we show that if the commutator operation is associative in G, then G is a K-group. In most cases also the converse result holds.

In this paper G denotes a multiplicative group, G' denotes the commutator subgroup of G and $Z(G)$ the centre of G. The centralizer of an element g in G is denoted by $C_G(g)$ and $N(A)$ means the normal subgroup generated by a subset A of G. Finally, by $*$ we denote the commutator operation.

[1]) This paper was written while the latter author was visiting the University of Brno.

2. BASIC LEMMAS

Let G be a group. From the properties of normal subgroups and the commutator operation it follows that G is a K-group if and only if $N(A) * B \subseteq \subseteq N(A * B)$ for all $A, B \subseteq G$. We first establish a result of Voráč [6], p. 242.

Lemma 2.1. *If G is a K-group, then $C_G(g)$ is a normal subgroup of G for all $g \in G$.*

We also need [6], p. 241.

Lemma 2.2. *Let N be a normal subgroup of G. If $(a * g) * b \in N$, for all $a \in G$ and for all $g * b \in N$, then G is a K-group.*

Lemma 2.3. *Let G be a K-group, $g \in G$ and let $x \in C_A(g)$, then $g * (x * z) = 1$, for all $z \in G$.*

Proof. By lemma 2.1, $z^{-1}xz \in C_G(g)$, so $z^{-1}xzg = gz^{-1}xz$, hence $g^{-1}z^{-1}xzgz^{-1}x^{-1}z = 1$. It follows that $g^{-1}x(x^{-1}z^{-1}xz)\, g(x^{-1}z^{-1}xz)^{-1}x^{-1} = 1$, so $g^{-1}(x * z)\, g(x * z)^{-1} = 1$ and the proof is complete.

As a direct consequence we have

Lemma 2.4. *Let G be a K-group, then $(x * y) * y = 1$, for all $x, y \in G$.*

Proof. Now $y \in C_G(y)$, so by lemma 2.3, $y * (y * x) = 1$. Then, clearly, $(x * y) * y = 1$ and the proof is complete.

Finally, we give two results of Levi [4].

Lemma 2.5. *The commutator operation is associative in a group G if and only if $G' \subseteq Z(G)$* (this result can also be found in [2], p. 87).

Lemma 2..6 *Let $(x * y) * y = 1$ for all $x, y \in G$. Then G is nilpotent of class at most three. Furthermore, if G has no elements of order three, then G is nilpotent of class two.*

A modern treatment of lemma 2.6 is given in [2], p. 288.

3. MAIN RESULTS

Now we are able to establish

Theorem 3.1. *A group G is a K-group if and only if $(x * y) * y = 1$ for all $x, y \in G$.*

Proof. Let G be a K-group. Now lemma 2.4 applies and an assertion is true.

Then suppose that $(x * y) * y = 1$, for all $x, y \in G$. Furthermore, suppose that N is a normal subgroup of G and let $g * b \in N$. Thus $(g * b) * a \in N$, for all $a \in G$. By steps (2) and (3) in the proof of Theorem 6.5 of [2], p. 288–289, we get $)g * b) * a = [(g * a) * b]^{-1} = [(a * g)^{-1} * b]^{-1} = (a * g) * b$, so we can use lemma 2.2. The proof is complete.

Now, by lemma 2.6, a K-group G is nilpotent of class at most three. Lemma 2.5 provides us with

Theorem 3.2. *Let G be a group such that G has no elements of order three. Then G is a K-group if and only if the commutator operation is associative.*

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