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Archivum Mathematicum, Vol. 19 (1983), No. 3, 125--131

Persistent URL: http://dml.cz/dmlcz/107165

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TOLERANCES AND ORDERINGS ON SEMILATTICES

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(Received April 4, 1982)

Let $\mathcal{A} = (A, F)$ be an algebra. A binary relation $R$ on $\mathcal{A}$ has the *Substitution Property*, briefly $SP$, if $R$ is a subalgebra of the direct product $\mathcal{A} \times \mathcal{A}$. We shall denote by $\Delta$ the so called diagonal $\{\langle x, x \rangle; x \in A\}$ of $\mathcal{A}$. Clearly $\Delta$ has $SP$. By a *tolerance* we shall mean a reflexive and symmetric relation on $\mathcal{A}$ having $SP$. Denote by $LT(\mathcal{A})$ the lattice of all tolerances on $\mathcal{A}$ ordered by set inclusion. Clearly $LT(\mathcal{A})$ is an algebraic lattice, where $\Delta$ is its least and $A \times A$ its greatest element; see [3] and [5]. Hence, there exists the least tolerance $T(a, b)$ containing $\langle a, b \rangle$ for every two elements $a, b$ of $\mathcal{A}$.

By an *ordering* on $\mathcal{A}$ we shall mean a reflexive, transitive and antisymmetric binary relation on $\mathcal{A}$ having $SP$. Let $\leq$ be a (fix) ordering on $\mathcal{A}$. Following [1] and [7],

$$LD(\mathcal{A}) = \{R; \Delta \subseteq R \subseteq \leq \text{ and } R \text{ has } SP \text{ on } \mathcal{A}\}$$

is the lattice of all reflexive (i.e. diagonal) binary relations having $SP$ on $\mathcal{A}$ and contained in $\leq$. Clearly $LD(\mathcal{A})$ is an algebraic lattice with respect to set inclusion. When $a$ and $b$ are two elements of $\mathcal{A}$ such that $a \leq b$, we denote by $D(a, b)$ the least element of $LD(\mathcal{A})$ containing $\langle a, b \rangle$.

Let $\mathfrak{U}$ be a lattice and $\leq$ its ordering. D. Schweigert [7] and H.-J. Bandelt [1] proved that the lattices $LT(\mathfrak{U})$ and $LD(\mathfrak{U})$ are isomorphic. We proceed to show that the situation is different for semilattices.

**Theorem 1.** Let $\mathfrak{S} = (S, \lor)$ be a semilattice and $\leq$ its induced ordering, i.e. $a \leq b$ if and only if $a \lor b = b$. Then

(i) there exists a subset $L$ of $LT(\mathfrak{S})$ which is a lattice with respect to the order on $LT(\mathfrak{S})$, and $LD(\mathfrak{S})$ is isomorphic to $L$;

(ii) the isomorphism of (i) is a mapping $\psi : LD(\mathfrak{S}) \to L$, where $\psi(R) = \{\langle x, y \rangle; \langle x, x \lor y \rangle \in R \land \langle y, x \lor y \rangle \in R\}$;

*) The present paper was written during the scientific activity of the first of authors at the Computing Center of J. E. Purkyně University of Brno, 1982.
L and LT(\(\mathfrak{S}\)) have a common least and a common greatest element i.e. \(\psi(\leq) = S \times S\) and \(\psi(\Delta) = \Delta\).

**Proof.** Let \(\zeta : LT(\mathfrak{S}) \to LD(\mathfrak{S})\) be a mapping given by \(\zeta(T) = T \cap \leq\). It is clear that \(\zeta\) and \(\psi\) are order-preserving and
\[
\zeta(\psi(R)) = \zeta(\{(\langle x, y \rangle; \langle x, x \lor y \rangle \in R\text{ and } \langle y, x \lor y \rangle \in R\}) = \{\langle x, y \rangle; \langle x, x \lor y \rangle \in \text{ and } \langle y, x \lor y \rangle \in R\} \cap \leq = R.
\]
Hence, \(\psi\) is an order-preserving one-to-one mapping of \(LD(\mathfrak{S})\) into \(LT(\mathfrak{S})\), i.e. \(LD(\mathfrak{S})\) is mapped by \(\psi\) isomorphically to a lattice \(L\) which is a subset of \(LT(\mathfrak{S})\).

Finally,
\[
\psi(\leq) = \{\langle x, y \rangle; x \leq x \lor y \text{ and } y \leq x \lor y\} = S \times S
\]
and
\[
\psi(\Delta) = \{\langle x, y \rangle; x = x \lor y \text{ and } y = x \lor y\} = \Delta.
\]

**Remark.** The lattice of Theorem 1 need not be a sublattice of \(LT(\mathfrak{S})\). Indeed, if \(\mathfrak{S}\) is a \(\lor\)-semilattice of three elements \(a, b\) and \(c\) such that \(a \lor b = c\), with \(R_1 = \{(a, c)\} \cup \Delta\) and \(R_2 = \{(b, c)\} \cup \Delta\), then clearly \(R_1, R_2 \in LD(\mathfrak{S})\) and \(\psi(R_1 \lor R_2) = \psi(\leq) = S \times S \neq \{(a, c), (c, a), (b, c), (c, b)\} \cup \Delta = \psi(R_1) \lor \psi(R_2)\), where the join on the left is formed in \(LD(\mathfrak{S})\) and the join on the right is formed in \(LT(\mathfrak{S})\).

The next theorem characterizes semilattices \(\mathfrak{S}\) for which \(LD(\mathfrak{S})\) and \(LT(\mathfrak{S})\) are isomorphic.

**Theorem 2.** Let \(\mathfrak{S} = (S, \lor)\) be a semilattice and \(\leq\) its induced ordering. If \(\mathfrak{S}\) is a chain, then \(LD(\mathfrak{S})\) and \(LT(\mathfrak{S})\) are isomorphic. If \(\mathfrak{S}\) is not a chain, then \(LD(\mathfrak{S})\) is isomorphic to a proper sublattice of \(LT(\mathfrak{S})\).

**Proof.** Let \(\zeta\) and \(\psi\) be the mappings of the proof of Theorem 1. When \(\mathfrak{S}\) is a chain, then \(\psi(\zeta(T)) = T\) for every \(T \in LT(\mathfrak{S})\), because \(\psi(R)\) is the symmetric envelop of \(R\). Applying now Theorem 1, we have \(LD(\mathfrak{S}) \cong LT(\mathfrak{S})\).

On the contrary, suppose \(\mathfrak{S}\) is not a chain, i.e. there exist elements \(x, y\) of \(\mathfrak{S}\) such that \(\{x, y, x \lor y\}\) constitutes a three-element subsemilattice \(\mathfrak{C}\) of \(\mathfrak{S}\). Now we can define two different tolerances \(T_1 \in LT(\mathfrak{S}), T_2 \in LT(\mathfrak{S})\) such that \(\mathfrak{C}\) is contained in a single block of \(T_2\), but in \(T_1\) it is divided into two blocks one containing \(\{x, x \lor y\}\) and the other \(\{y, x \lor y\}\); elsewhere \(T_1 = T_2\) (see [3]). Then \(\zeta(T_1) = \zeta(T_2)\). Suppose that there exist relations \(R_1 \in LD(\mathfrak{S}), R_2 \in LD(\mathfrak{S})\) such that \(T_1 = \psi(R_1), T_2 = \psi(R_2)\). As \(\zeta(\psi(R)) = R\) for each \(R \in LD(\mathfrak{S})\), we have \(R_1 = \zeta(\psi(R_1)) = \zeta(T_1) = \zeta(T_2) = \zeta(\psi(R_2)) = R_2\). But this implies also \(\psi(R_1) = T_1 = \psi(R_2)\), which is a contradiction. Thus at least one of the relations \(T_1, T_2\) is not an image of a relation from \(LD(\mathfrak{S})\) in the mapping \(\psi\), and \(\psi\) maps \(LD(\mathfrak{S})\) onto a proper subset of \(LT(\mathfrak{S})\), not onto whole \(LT(\mathfrak{S})\).
would be isomorphic to its proper subset and evidently it would be infinite. We have:

**Corollary 1.** Let \( \mathcal{S} = (S, \vee) \) be a finite semilattice and \( \leq \) its induced ordering. The lattices \( LD(\mathcal{S}) \) and \( LT(\mathcal{S}) \) are isomorphic if and only if \( \mathcal{S} \) is a chain.

As known, the compact elements of \( LT(\mathcal{U}) \) are finite joins of tolerances \( T(a, b) \) for elements \( a, b \) of \( \mathcal{U} \), see [3]. Clearly the compact elements of \( LD(\mathcal{U}) \) are the finite joins of \( D(a, b) \) for \( a \leq b \), where \( \leq \) is the fixordering of \( \mathcal{U} \). A semilattice \( \mathcal{S} \) is called a tree-semilattice if the interval \([a, b]\) is a chain for every pair \( a \leq b \) of elements \( a, b \) in \( \mathcal{S} \). If \( \mathcal{S} \) is a finite tree-semilattice, its Hasse diagram is a tree in the graph theoretical sense.

**Theorem 3.** Let \( \mathcal{S} \) be a semilattice and \( \leq \) its induced ordering, let \( a \leq b \) in \( \mathcal{S} \). Then:

1. \( \psi(D(a, b)) \supseteq T(a, b) \);
2. \( \psi(D(a, b)) = T(a, b) \) for every pair \( a \leq b \) of \( \mathcal{S} \) if and only if \( \mathcal{S} \) is a tree-semilattice.

**Proof.** If \( a \leq b \) in \( \mathcal{S} \), then \( D(a, b) = \{\langle x, y \rangle; x = a \lor c, y = b \lor c \text{ for } c \in \} \cup A \). Hence,

\[
\psi(D(a, b)) = \{\langle x, y \rangle; \langle x, x \lor y \rangle \in D(a, b) \text{ and } \langle y, x \lor y \rangle \in D(a, b)\} = \\
= \{\langle x, y \rangle; x = a \lor c, y = a \lor d, x \lor y = b \lor c = b \lor d \text{ for } c, d \in \mathcal{S}\} \cup A.
\]

Choosing \( c = a \) and \( d = b \) we obtain, \( \langle a, b \rangle \in \psi(D(a, b)) \subseteq LT(\mathcal{S}) \), and thus \( T(a, b) \subseteq \psi(D(a, b)) \).

Now, let \( \mathcal{S} \) be a tree-semilattice. Then \( a \leq a \lor d \leq b \lor d \) and \( a \leq a \lor c \leq b \lor c = b \lor d \), whence both \( a \lor c \) and \( a \lor d \) lie in the interval \([a, b \lor d]\). Since \( \mathcal{S} \) is a tree-semilattice, \([a, b \lor d]\) is a chain, whence \( a \lor c \) and \( a \lor d \) are comparable. Then

\[
\psi(D(a, b)) = \{\langle x, y \rangle; \langle x, y \rangle \in D(a, b)\} \cup \{\langle x, y \rangle; \langle y, x \rangle \in D(a, b)\} \cup A = T(a, b).
\]

On the contrary, if \( \mathcal{S} \) is not a tree-semilattice, there exist elements \( a, b, c \) of \( \mathcal{S} \) such that \( a \) and \( b \) are non-comparable and \( c \) is a lower bound of \( a \) and \( b \). Thus \( \{c, a, b, a \lor b\} \) constitutes a four-element subsemilattice of \( \mathcal{S} \), where we denote briefly \( d = a \lor b \). Since \( \langle a, d \rangle = \langle a \lor c, a \lor d \rangle \in D(c, d) \) and \( \langle b, d \rangle = \langle b \lor c, b \lor d \rangle \in D(c, d) \), we have \( D(c, d) = \{\langle c, d \rangle, \langle a, d \rangle, \langle b, d \rangle\} \cup A \), and moreover, \( \psi(D(c, d)) = \{\langle c, d \rangle, \langle d, c \rangle, \langle a, d \rangle, \langle d, a \rangle, \langle b, d \rangle, \langle d, b \rangle, \langle a, b \rangle, \langle b, a \rangle\} \cup A \). The other parts but \( \langle a, b \rangle \in \psi(D(c, d)) \) are trivial, and \( \langle a, b \rangle \in \psi(D(c, d)) \) follows from \( \langle a, a \lor b \rangle = \langle a, d \rangle \in D(c, d) \) and \( \langle b, a \lor b \rangle = \langle b, d \rangle \in D(c, d) \). However, \( T(c, d) = \{\langle c, d \rangle, \langle d, c \rangle, \langle a, d \rangle, \langle d, a \rangle, \langle b, d \rangle, \langle d, b \rangle\} \cup A \) as we can easily see [6], [8]. Hence \( T(c, d) \neq \psi(D(c, d)) \), and the assertion follows. \( \square \)
The foregoing theorem gives a characterization of tree-semilattices by means of tolerances $T(a, b)$ and diagonal relations $D(a, b)$. In the next part we proceed to give an explicite description of $D(a, b)$.

Let $\leq$ be a (fix) ordering on an algebra $\mathfrak{A}$. We denote by $LO(\mathfrak{A})$ the set of all orderings on $\mathfrak{A}$ contained in $\leq$. Clearly also $LO(\mathfrak{A})$ is a complete lattice. Hence, if $a \leq b$ in $\mathfrak{A}$, there is a least element in $LO(\mathfrak{A})$ containing $\langle a, b \rangle$, and we shall denote that element by $P(a, b)$.

**Theorem 4.** Let $\mathfrak{S}$ be a semilattice and $\leq$ its induced ordering. If $D \in LD(\mathfrak{S})$, then the transitive closure $C(D)$ of $D$ is an ordering on $\mathfrak{S}$, i.e. $C(D) \in LO(\mathfrak{S})$.

**Proof.** Because $D \subseteq \leq$, also $C(D) \subseteq \leq$. Now, if $C(D)$ has $SP$, then it is an ordering on $\mathfrak{S}$, and thus it remains to prove $SP$ for $C(D)$. Suppose $\langle a, b \rangle, \langle c, d \rangle \in C(D)$. Then there exist elements $x_0, x_1, ..., x_m, y_0, y_1, ..., y_n$ such that $a = x_0 \leq x_1 \leq ... \leq x_m = b$ and $c = y_0 \leq y_1 \leq ... \leq y_n = d$, where $\langle x_i, x_{i+1} \rangle \in D$ for $i = 0, 1, ..., m - 1$ and $\langle y_j, y_{j+1} \rangle \in D$ for $j = 0, 1, ..., n - 1$. Without loosing generality we assume that $m \leq n$, and put $x_i = b$ for $m \leq i \leq n$. Let now $z_i = x_i \lor y_i$ for $i = 0, 1, ..., n$. Then $a \lor c = z_0 \leq z_1 \leq ... \leq z_n = b \lor d$ and $\langle z_i, z_{i+1} \rangle \in D$ for $i = 0, 1, ..., n$ because of $SP$ of $D$. Hence $\langle a \lor c, b \lor d \rangle \in C(D)$ and $C(D)$ has $SP$.

**Theorem 5.** Let $\mathfrak{S}$ be a semilattice with the induced ordering $\leq$, $a, b$ two elements of $\mathfrak{S}$, and $a \leq b$. Then $D(a, b) = P(a, b)$.

**Proof.** Evidently, $D(a, b) = \langle \langle a \lor x, b \lor x \rangle; x \in \mathfrak{S} \rangle \cup \Delta$. We shall prove the transitivity of $D(a, b)$. Let $c, d$ and $e$ be elements of $\mathfrak{S}$ such that $c \leq d \leq e$ and $\langle c, d \rangle, \langle d, e \rangle \in D(a, b)$. If $c = d$ or $d = e$, there is nothing to prove. Suppose that $\langle c, d \rangle = \langle a \lor x, b \lor x \rangle$ and $\langle d, e \rangle = \langle a \lor y, b \lor y \rangle$ for some elements $x, y \in \mathfrak{S}$. Then $d = b \lor x = a \lor y$, and moreover, $d = d \lor d = a \lor b \lor x \lor y = b \lor x \lor y \geq b \lor y = e$. Because $d \leq e$ and $d \geq e$, we have $d = e$, whence also $\langle c, e \rangle = \langle c, d \rangle \in D(a, b)$, and thus $D(a, b)$ is transitive. Then $D(a, b) \in LO(\mathfrak{S})$ and the equality $D(a, b) = P(a, b)$ is evident.

**Theorem 6.** Let $\mathfrak{S}$ be a tree-semilattice and $\leq$ its induced ordering. If $a, b \in \mathfrak{S}$ and $a \leq b$, then $D(a, b) = \langle \langle x, b \rangle; a \leq x \leq b \rangle \cup \Delta$.

**Proof.** By Theorem 5, $D(a, b) = P(a, b)$, and thus $D(a, b)$ is the least ordering on $\mathfrak{S}$ containing the ordered pair $a \leq b$. Let $R = \langle \langle x, b \rangle; a \leq x \leq b \rangle \cup \Delta$. By putting $x = a$, we obtain $\langle a, b \rangle \in R$, and according to the definition of $R$, $R \subseteq D(a, b)$. It remains to prove that $R$ has $SP$. Suppose $\langle y_1, z_1 \rangle, \langle y_2, z_2 \rangle \in R$, and if $\langle y_1, z_1 \rangle, \langle y_2, z_2 \rangle \in D(a, b)$, there is nothing to prove. If $\langle y_1, z_1 \rangle \in D$ and $\langle y_2, z_2 \rangle \in R \setminus \Delta$, then $y_1 = z_1, z_2 = b$ and $a \leq y_2 \leq b$. These facts imply that $\langle y_1 \lor y_2, z_1 \lor z_2 \rangle = \langle y_1 \lor y_2, y_1 \lor b \rangle$. If $y_1 \leq b$, then $a \leq y_1 \lor y_2 \leq b$ and $z_1 \lor z_2 = y_1 \lor b = b$, and thus $\langle y_1 \lor y_2, z_1 \lor z_2 \rangle \in R$. In the opposite case $y_1 \lor b > b$. On the other hand $b$ and $y_1 \lor y_2$ belong to the interval $[y_2, y_1 \lor b]$. 128
and because $\mathfrak{S}$ is a tree-semilattice, $b$ and $y_1 \lor y_2$ are comparable. The inequality $y_1 \lor y_2 \leq b$ implies that $y_1 \leq b$, which is a contradiction. Thus $y_1 \lor y_2 > b$, and moreover, $y_1 \lor y_2 = y_1 \lor b$. Hence, $\langle y_1 \lor y_2, z_1 \lor z_2 \rangle = \langle y_1 \lor y_2, y_1 \lor b \rangle \in \mathcal{D} \subseteq R$.

If $\langle y_1, z_1 \rangle \in R \setminus \Delta$ and $\langle y_2, z_2 \rangle \in R \setminus \Delta$, then according to the proof above we have $\langle y_1 \lor y_2, z_1 \lor y_2 \rangle$, $\langle z_1 \lor y_2, z_2 \lor y_2 \rangle \in R$. Because $R$ is trivially transitive, we obtain $\langle y_1 \lor y_2, z_1 \lor z_2 \rangle \in R$, and thus $R$ has $SP$. □

**Theorem 7.** Let $\mathfrak{S}$ be a tree-semilattice with the induced ordering $\leq$ and $P$ a reflexive binary relation on $\mathfrak{S}$ contained in $\leq$. Then $P \in LD(\mathfrak{S})$ if and only if $\langle a, b \rangle \in P$ implies $\langle x, b \rangle \in P$ for any elements $a, b, x \in \mathfrak{S}$ such that $a \leq x \leq b$.

**Proof.** If $P \in LD(\mathfrak{S})$ and $\langle a, b \rangle \in P$, then for any $x, a \leq x \leq b$, $\langle x, b \rangle = \langle a \lor x, b \lor x \rangle \in P$, and the first part of the proof follows.

Conversely, suppose that $P$ has the property $\langle a, b \rangle \in P$ implies $\langle x, b \rangle \in P$ for any $a, b, x \in \mathfrak{S}$ with $a \leq x \leq b$. We shall prove $SP$ of $P$. Let $\langle a, b \rangle$, $\langle c, d \rangle \in P$. If $b$ and $d$ are incomparable, then $a \lor c = b \lor d$, because $\mathfrak{S}$ is a tree-semilattice, and thus $\langle a \lor c, b \lor d \rangle \in \Delta \subseteq P$. If e.g. $b \leq d$, then $b \lor d = d$ and $c \leq a \lor c \leq b \lor d = d$. But then $\langle c, d \rangle \in P$ implies $\langle a \lor c, b \lor d \rangle = \langle a \lor c, d \rangle \in P$ according to the property of $P$. The case $d \leq b$ is analogous. □

**Corollary 2.** Let $\mathfrak{S}$ be a tree-semilattice with the induced ordering $\leq$ and $P$ a reflexive, antisymmetric and transitive binary relation on $\mathfrak{S}$ with $P \subseteq \leq$. Then $P \in LO(\mathfrak{S})$ if and only if $\langle a, b \rangle \in P$ implies $\langle x, b \rangle \in P$ for any elements $a, b, x$ of $\mathfrak{S}$ with $a \leq x \leq b$.

**Remark.** Theorem 7 and its Corollary give a possibility to describe the join operation in $LD(\mathfrak{S})$ and in $LO(\mathfrak{S})$, respectively, when $\mathfrak{S}$ is a tree-semilattice.

The join $\lor$ in $LD(\mathfrak{S})$: $P, Q \in LD(\mathfrak{S}) \Rightarrow P \lor Q = P \cup Q$.

The join $\lor$ in $LO(\mathfrak{S})$: $R, U \in LO(\mathfrak{S}) \Rightarrow R \lor U$ is the transitive closure of $R \cup U$.

The remaining part of the paper is devoted to the extension properties of relations of $LD(\mathfrak{S})$ and $LO(\mathfrak{S})$. The first attempt to study the extension property of other relations than congruences was done by Chajda in [2] for relations of $LT(\mathfrak{S})$.

We recall first briefly the necessary concepts:

A class $\mathcal{C}$ of algebras satisfies the **Tolerance Extension Property** (briefly $TEP$) if for every $\mathfrak{A} \in \mathcal{C}$ and every subalgebra $\mathfrak{L}$ of $\mathfrak{A}$, each tolerance $T$ on $\mathfrak{L}$ is the restriction of some tolerance $T^*$ on $\mathfrak{A}$, i.e. $T = T^* \cap (\mathfrak{L} \times \mathfrak{L})$.

**Proposition.** (Theorem 2 and the Example in [2]) Every class of tree-semilattices satisfies $TEP$. The variety of all semilattices does not satisfy $TEP$.

We can define the extension property analogously for relations of $LD(\mathfrak{A})$ and $LO(\mathfrak{A})$:

**Definition.** Let $\mathcal{C}$ be a class of ordered algebras such that every $\mathfrak{A} \in \mathcal{C}$ is ordered by a fixordering $\leq$. $\mathcal{C}$ satisfies the Extension Property of Orderings if for every
\( \mathcal{U} \in \mathcal{C} \) and for every subalgebra \( \mathcal{L} \) of \( \mathcal{U} \), each \( P \in \text{LO}(\mathcal{L}) \) is the restriction of some \( P^* \in \text{LO}(\mathcal{U}) \). \( \mathcal{C} \) satisfies the D-Extension Property if for every \( \mathcal{U} \in \mathcal{C} \) and for every subalgebra \( \mathcal{L} \) of \( \mathcal{U} \), each \( D \in \text{LD}(\mathcal{L}) \) is the restriction of some \( D^* \in \text{LD}(\mathcal{U}) \).
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