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A TYPE OF CONTINUOUS PROJECTIONS

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1. Introduction

Let $S$ be a nonempty set and $V \subseteq S$. A mapping $E : S \rightarrow V$ satisfying $E(S) = V$ and $E^2 = E$ is said to be projection from $S$ onto $V$. If $S$ is a topological space, $V$ a subspace of $S$ and $E$ a continuous mapping, then $E$ is called continuous projection. Continuous projections in function spaces can be viewed as approximations of given functions in function subspaces. For instance, the orthogonal projection onto a closed subspace $V$ of a Banach space is the best approximation with respect to $V$ (see, e.g., [2]).

In practice we can comparatively easily solve problems of linear approximations. In this paper we show that a type of operators defined by means of linear approximations are continuous projections. This can be used for parameters estimation. We present the following examples in which $f$ denotes a given function (experimental data) to be fitted by a function $g$ using the least squares method (i.e., $\int (f - g)^2 = \min$)

1. $g = \frac{1}{ax^2 + bx + c}$; An approximation of the exact solution can be obtained solving the problem

$$f_1 = \frac{1}{f}, \quad g_1 = ax^2 + bx + c,$$

which is linear with respect to the parameters $a, b, c$.

2. $g = ae^{bx}$;

$$f_1 = \ln f, \quad g_1 = bx + \ln a$$

3. $g = ae^{bx} + c$;

$$f_2 = \frac{df}{dx}, \quad g_2 = by - d.$$
Solving of this problem determines \( b^0 \neq 0, d^0 \). We put \( b^0c^0 = d^0 \) and solve the problem

\[
\begin{align*}
  f_1 &= f, \\
  g_1 &= ae^{b_0x} + c^0.
\end{align*}
\]

Solving of this linear problem determines \( a^0 \). From the main theorem of this paper follows that the mapping

\[
f \mapsto a^0e^{b_0x} + c^0
\]

is a continuous projection in a space of sufficiently smooth functions.

Parameters estimations of such types were used in optimization programs package OPTIPACK [3] which was developed in Institute of Physical Metallurgy Computing Department of Czechoslovak Academy of Sciences.

Let \( R \) be a normed space, \( V \subseteq S \subseteq R \). Then a mapping \( E \) from \( S \) onto \( V \) is a continuous projection from \( S \) onto \( V \) iff for every \( z \in V \) the following condition holds

\[
\lim_{\| y - z \| \to 0} \| E(y) - z \| = 0
\]

i.e., for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for \( y \in S \) satisfying \( \| y - z \| < \delta \) it holds \( \| E(y) - z \| < \varepsilon \).

2. Preliminary Lemmas

Notations. Throughout the following text we shall use the symbol \( R \) for a normed linear space over the field \( T \) of all real numbers. The norm in \( R \) is denoted by \( \| . \| \).

Further we shall consider the norm \( [ . ] \) in \( T^n \) defined by

\[
[(a_1, \ldots, a_n)] = \max \{ |a_1|, \ldots, |a_n| \}.
\]

For \( y_1, \ldots, y_n, y_0 \in R \) and \( \delta > 0 \) we put

\[
\langle y_1, \ldots, y_n, y_0, \delta \rangle = \{(a_1, \ldots, a_n) \in T^n; \| a_1y_1 + \ldots + a_ny_n + y_0 \| < \delta \}.
\]

Lemma 1. \( \langle y_1, \ldots, y_n, 0, \delta \rangle \) is a convex subset of \( T^n \) which is bounded iff \( y_1, \ldots, y_n \) are linearly independent.

Notation. For the sake of simplicity we shall use the following notation:

\[
\sup \langle y_1, \ldots, y_n, y_0, \delta \rangle = \sup \{ [x]; x \in \langle y_1, \ldots, y_n, y_0, \delta \rangle \}.
\]

If \( V \) is a finite-dimensional subspace of \( R \) and \( x \in R \), we denote

\[
\varrho_v(x) = \min_{y \in V} \| y - x \|
\]

Lemma 2. Let \( y_1, \ldots, y_n \) be linearly independent elements in \( R \). Then for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\sup \langle y_1, \ldots, y_n, 0, \delta \rangle < \varepsilon.
\]
Lemma 3. Let $y_1, \ldots, y_n$ be linearly independent elements from $\mathbb{R}$, $A_1, \ldots, A_n \in \mathbb{R}$ and $\delta > 0$. Let us denote

$$A_1 = \langle y_1 + A_1, \ldots, y_n + A_n, A_0, \delta \rangle,$$
$$A_2 = \langle y_1, \ldots, y_n, 0, \delta \rangle.$$

Then for every $\varepsilon > 0$ there exists $\sigma > 0$ such that $\|A_i\| < \sigma$ for every $i$ ($1 \leq i \leq n$) implies

$$\text{sup } A_1 - \text{sup } A_2 < \varepsilon.$$

Proof. Suppose that there exists $\varepsilon_0 > 0$ such that for every $\sigma > 0$ from $\|A_i\| \leq \sigma$ ($1 \leq i \leq n$) it follows

$$\text{sup } A_1 - \text{sup } A_2 \geq \varepsilon_0.$$

Let us denote:

$$e_k = k \frac{\varepsilon_0}{n + 2},$$
$$s_k = \text{sup } A_2 + e_k$$

for $k = 1, \ldots, n + 1$.

By our assumptions for every $\sigma > 0$ there exists $(a_1^*, \ldots, a_n^*) \in A_1$ such that

$$[(a_1^*, \ldots, a_n^*)] - \text{sup } A_2 > \varepsilon_{n+1}$$

Let us denote $V_i$ the linear subspace generated by the set $\{y_1, \ldots, y_n\} - \{y_i\}$. Then it holds $\varrho_{V_i}(s_k y_i) \geq \delta$. Clearly, there exists $s = s_m$ satisfying

$$\varrho_{V_i}(s_m y_i) > \delta$$

for every $i$ ($1 \leq i \leq n$).

Then from (1) it follows

$$[(a_1^*, \ldots, a_n^*)] - \text{sup } A_2 > \varepsilon_m \quad \forall \sigma > 0$$

and hence

$$s/[(a_1^*, \ldots, a_n^*)] = (\text{sup } A_2 + \varepsilon_m)/[(a_1^*, \ldots, a_n^*)] > 1.$$

We put

$$\varrho = \min \{\varrho_{V_1}(s y_1), \ldots, \varrho_{V_n}(s y_n)\}.$$ 

In view of (2) we have $\varrho > \delta$. Let us choose $\kappa$ such that

$$0 < \kappa < \varrho - \delta.$$

Now we put $\sigma = \min (\kappa/3, \kappa/3n)$. Let $\|A_i\| < \sigma$ ($1 \leq i \leq n$) and let $(a_1^*, \ldots, a_n^*) \in A_1$ satisfying (1). Further we put

$$K = s/[(a_1^*, \ldots, a_n^*)].$$

Then it holds

$$Ka_i^* \leq s$$

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for every $i$ ($1 \leq i \leq n$) and in view of (3)

\begin{equation}
K < 1.
\end{equation}

Because of

\[ [Ka_i^*, ..., Ka_n^*] = K[(a_i^*, ..., a_n^*)] = s, \]

we have

\[ \| \sum_i Ka_i y_i \| \geq q v_j (s y_j) \geq q > \delta + \kappa, \]

wherein $a_j = [(a_1^*, ..., a_n^*)]$.

Hence

\begin{equation}
\| \sum_i a_i^* y_i \| > \frac{1}{K} (\sigma + \kappa).
\end{equation}

Further we obtain

\begin{equation}
\| \sum_i a_i^* A_i \| \leq \frac{1}{K} \sum_i Ka_i^* \| A_i \| \leq \frac{1}{K} \sum s \| A_i \| \leq \frac{1}{K} \frac{\kappa}{3}.
\end{equation}

Because of $\| A_0 \| \leq \frac{\kappa}{3}$ and using (5) we obtain

\[ \| \sum_i a_i^* y_i + \sum_i a_i^* A_i + A_0 \| \geq (\delta + \kappa) \frac{1}{K} - \frac{1}{K} \frac{\kappa}{3} - \frac{\kappa}{3} > \delta + \frac{\kappa}{3} > \delta \]

contradicting the assumption $(a_1^*, ..., a_n^*) \in A_1$.

3. Main Theorem

**Theorem.** Let $T_1$ be an open subset of $T^n$, $G_0$, $G_1$, ..., $G_m$ continuous mappings from $T^n$ into $R$, $m$, $n$ natural numbers satisfying $m + n \geq 1$ and

\[ V = \{ \sum_{i=1}^m b_i G(a_1, ..., a_n) + G_0(a_1, ..., a_n); (b_1, ..., b_m) \in T^m, (a_1, ..., a_n) \in T_1 \}. \]

Suppose that there exist continuous operators $F_0$, $F_1$, ..., $F_n$ ($F_i : R \to R$) satisfying

\[ x = \sum_{i=1}^m b_i G_i(a_1, ..., a_n) + G_0(a_1, ..., a_n) \Rightarrow F_0(x) + \sum_{i=1}^n a_i F_i(x) = 0 \]

for every $(a_1, ..., a_n) \in T_1$ and for every $(b_1, ..., b_m) \in T^m$. Further suppose

1. $\{F_i(y)\}_{i=1}^n$ is linearly independent set for every $y \in V$.
2. $\{G_j(a_1, ..., a_n)\}_{j=1}^m$ is linearly independent set for every $(a_1, ..., a_n) \in T_1$.

Then there exists an open subset $S \subseteq R$ satisfying $S \supseteq V$ such that each operator $E : S \to V$ of the form

\[ E(y) = \sum_{i=1}^m b_i^* G_i(a_1^*, ..., a_n^*) + G_0(a_1^*, ..., a_n^*) \]

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with the following properties

\begin{enumerate}
  \item \( \| F_0(y) + \sum_i a_i^* F_i(y) \| = \min_{a_i^*} \| F_0(y) + \sum_i a_i^* F_i(y) \| \)
  \item \( \| G_0(a_1^*, ..., a_n^*) + \sum_i b_i^* G_i(a_1^*, ..., a_n^*) - y \| = \)
    \[ = \min_{b_i^*} \| G_0(a_1^*, ..., a_n^*) + \sum_i b_i^* G_i(a_1^*, ..., a_n^*) - y \| \]
\end{enumerate}

is a continuous projection from \( S \) onto \( V \).

Now we shall prove Lemmas 4, 5, 6, from which the assertion of Theorem follows easily.

**Notation.** In what follows we shall use the following notation. For an arbitrarily chosen \( y \in \mathcal{R} \) we put

\[ y^* = E(y) = \sum_i b_i^* G_i(a_1^*, ..., a_n^*) + G_0(a_1^*, ..., a_n^*) \in V \]

\[ F_i(y) = F_i(y^*) + \Delta F_i(y). \]

**Lemma 4.** For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \| \Delta F_i(y) \| < \delta \) \((1 \leq i \leq n)\) implies

\[ \min_{a_i^*} \| F_0(y) + \sum_i a_i^* F_i(y) \| < \varepsilon. \]

**Proof.** The assertion follows easily from the following relation

\[ \| F_0(y^*) + \Delta F_0(y) + \sum_i a_i^* F_i(y) + \sum_i a_i^* F_i(y) \| = \| \Delta F_0(y) + \sum_i a_i^* F_i(y) \|. \]

**Lemma 5.** For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[ \| y - y^* \| < \delta \Rightarrow [(a_1^*, ..., a_n^*) - (a_1^*, ..., a_n^*)] < \varepsilon. \]

**Proof.** The assertion follows easily from Lemmas 2, 3, 4, and from the continuity of operators \( F_i \).

**Lemma 6.** For every \( \varepsilon > 0 \) there exist \( \delta_1 > 0, \delta_2 > 0 \) such that \( \| y^* - y \| < \delta_2 \), and \( [(a_1^*, ..., a_n^*) - (a_1^*, ..., a_n^*)] < \delta \) implies

\[ \| G_0(a_1^*, ..., a_n^*) + \sum_i b_i^* G_i(a_1^*, ..., a_n^*) - y \| < \varepsilon. \]

**Proof.** Let us choose \( \delta_1 \) so that \( \{G_j(a_1, ..., a_n)\}_{j=1}^m \) is linearly independent set and every \( n \)-tuple \( (a_1^*, ..., a_n^*) - (a_1^*, ..., a_n^*) \) satisfying \( [(a_1^*, ..., a_n^*) - (a_1^*, ..., a_n^*)] < \delta \) belongs to \( T_1 \). Now the assertion follows easily from the relation

\[ \min_{b_j} \| G_0(a_1^*, ..., a_n^*) + \sum_j b_j G_j(a_1^*, ..., a_n^*) - y \| \leq \]

\[ \leq \| G_0(a_1^*, ..., a_n^*) + \sum_j b_j G_j(a_1^*, ..., a_n^*) - y \| = \]
= \| G_0(a_1^*, ..., a_n^*) + G_0(a_{1'}^*, ..., a_{n'}^*) + \sum_j b_j^* G_j(a_1^*, ..., a_n^*) +
\sum_j b_j^* G_j(a_{1'}^*, ..., a_{n'}^*) - y^* + y^* - y \| \leq
\leq \| G_0(a_1^*, ..., a_n^*) + \sum_j b_j G_j(a_1^*, ..., a_n^*) \| + \delta_2.

Proof of Theorem. Let \( \varepsilon > 0 \) be arbitrarily chosen. We chose \( \delta_1 > 0 \) and \( \delta_2 > 0 \) so that the condition (***) is satisfied. Further we choose \( \delta_3 > 0 \) in such a way that
\[ \| y^* - y \| < \delta_3 \Rightarrow [(a_1^*, ..., a_n^*) - (a_1^*, ..., a_n^*)] < \delta_1 \]
(assuming Lemma 5) and put \( \delta = \min(\delta_2, \delta_3) \). Then in view of Lemma 6
\[ \| y^* - y \| < \delta \Rightarrow \| E(y) - y^* \| < \varepsilon. \]

Now we put \( S = \bigcup_{y^* \in V} o_{y^*}, \) where \( o_{y^*} \) is point \( y^* \) \( \delta \)-neighbourhood constructed as above. Then \( E \) is a continuous projection from \( S \) onto \( V \), and \( S \) is an open set satisfying \( S \supseteq V \).

REFERENCES


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