Jacek Michalski
On $s$-skew elements in polyadic groups

Archivum Mathematicum, Vol. 19 (1983), No. 4, 215--218

Persistent URL: http://dml.cz/dmlcz/107176

Terms of use:
© Masaryk University, 1983

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
ON S-SKEW ELEMENTS IN POLYADIC GROUPS

JACEK MICHALSKI, Wroclaw
(Received October 15, 1981)

1. This note is a supplement to [2]. We introduce the notion of an s-skew element in a polyadic group (i.e. an n-group for some n) which is a generalization of that of a skew element from [1]. A 1-skew element (s = 1) is simply a skew element. That notion enables a simplification of notation and clears up to some extent the structure of creating (k + 1)-groups of a given (n + 1) group (see [2]).

We use the same notation as in [2] and we assume also n = sk.

2. Post in [3] stated a necessary and sufficient condition for an (n + 1)-group to be derived from a (k + 1)-group. That condition was expressed in terms of polyads. To put those terms into the language used in [2] suggests the following.

Definition. Let d and c be elements of an (n + 1)-group (G, f). The element d is called an s-skew element to the element c if the following conditions are fulfilled:

\[ s \ (k - 1) \ s \]
\[ f(d, c, x) = x \quad \text{for each } x \in G; \]
\[ k - 1 \]
\[ f(x_1, \ldots, x_i, d, c, x_{i+1}, \ldots, x_{n+1-k}) = \]
\[ k - 1 \]
\[ f(x_1, \ldots, x_i, c, d, x_{i+1}, \ldots, x_n, x_{n+1-k}) = f(d, c, x_1, \ldots, x_{n+1-k}) \]

for each \( x_1, \ldots, x_{n+1-k} \in G \) and arbitrary \( i = 1, \ldots, n + 1 - k \).

The formerly mentioned condition of Post was given in a modified form (adopted to the given in [2] construction of a free covering group) in [2] as Theorem 5. Using the notion of an s-skew element this condition can be reformulated as follows:

Proposition 1. An (n + 1)-group (G, f) is derived from a (k + 1)-group if and only if for some element \( c \in G \) there exists an element \( d \in G \) which is s-skew to \( c \) in the (n + 1)-group (G, f). In that case the (k + 1)-ary operation \( g \) in the (k + 1)-
group $G_{(s-1)} = (G, g)$ can be given by the formula $g(x_1, \ldots, x_{k+1}) = s - 1 (k - 1) (s - 1) = f(x_1, \ldots, x_{k+1}, d, c)$. 

Examining the proof of Theorem 5 from [2] we get a little more, namely

**Corollary 1.** If an $(n + 1)$-group $G = (G, f)$ is derived from a $(k + 1)$-group $G_{(s-1)} = (G, g)$, then for every element $c \in G$ there exists an $s$-skew element to $c$ in the $(n + 1)$-group $G$.

**Corollary 2.** If an $(n + 1)$-group $G = (G, f)$ is derived from a $(k + 1)$-group $G_{(s-1)} = (G, g)$, then the following conditions are equivalent:

(a) the element $d \in G$ is skew to the element $c \in G$ in $G_{(s-1)}$;
(b) the element $d \in G$ is $s$-skew to the element $c \in G$ in $G$ and $g(x_1, \ldots, x_{k+1}) = s - 1 (k - 1) (s - 1) = f(x_1, \ldots, x_{k+1}, d, c)$.

From this corollary we infer that if we know the skew element to some element from $G_{(s-1)} = (G, g)$, then the $(k + 1)$-ary operation $g$ is already uniquely determined. There exists a one-to-one correspondence between the set of the creating $(k + 1)$-groups of the $(n + 1)$-group $G$ and the set of all $s$-skew elements to any element from the $(n + 1)$-group $G$ (see [3], p. 232).

**3.** From Proposition 1 and the Corollaries resulting from it one can obtain some statements concerning homomorphisms and sub-$(k + 1)$-groups of creating $(k + 1)$-groups.

**Corollary 3.** Let $(n + 1)$-groups $G = (A, f)$ and $H = (B, f)$ be derived from $(k + 1)$-groups $G_{(s-1)} = (A, g)$ and $H_{(s-1)} = (B, g)$, if $h : G \rightarrow H$ and $h(c) = \bar{c}(\theta)$ for some $c \in A$, then $h : G_{(s-1)} \rightarrow H_{(s-1)}$.

**Proof.** Using Corollary 2 the element $d = \bar{c}(\theta)$ is $s$-skew to $c$ in the $(n + 1)$-group $G_{(s-1)}$.

Hence $h(g(x_1, \ldots, x_{k+1})) = h(f(x_1, \ldots, x_{k+1}, d, c)) = f(h(x_1), \ldots, h(x_{k+1}), h(d), h(c)) = g(h(x_1), \ldots, h(x_{k+1})).$ 

**Proposition 2.** Let $(n + 1)$-groups $G = (A, f)$ and $H = (B, f)$ be derived from $(k + 1)$-groups $G_{(s-1)} = (A, g)$ and $H_{(s-1)} = (B, g)$, if $D = (D, f)$ is an $(n + 1)$-group, $h = h_2 h_1$ where $h_1 : G \rightarrow D$, $h_2 : D \rightarrow H$ and $h_2$ is a monomorphism, then $D$ is derived from a unique $(k + 1)$-group $D_{(s-1)} = (D, g)$ such that $h_1 : G_{(s-1)} \rightarrow D_{(s-1)}$, $h_2 : D_{(s-1)} \rightarrow B_{(s-1)}$.

**Proof.** Take an element $c_1 \in A$ and an element $d_1 \in A$ to be skew to $c_1$ in the $(k + 1)$-group $G_{(s-1)}$. The element $d_1$ is $s$-skew to $c_1$ in the $(n + 1)$-group $G$. Let $c = h_1(c_1)$ and $d = h_1(d_1)$. We show that the element $d$ is $s$-skew to $c$ in $D$. Using the assumption $h(c_1)$ is $s$-skew to $h(d_1)$ in the $(n + 1)$-group $H$ (since
\[ h : \mathfrak{H}_{(s-1)} \to \mathfrak{B}_{(s-1)} \], whence we get
\[ s \begin{pmatrix} k-1 \\ k \end{pmatrix} = f(h_2(d), \quad h_2(c), \quad h_2(x)) =
\begin{pmatrix} s \\ k-1 \end{pmatrix} = f(h_2(d_1), \quad h_2(h_1(c_1), \quad h_2(x)) = f(h(d_1), \quad h(c_1), \quad h_2(x)) = h_2(x). \]

But the homomorphism \( h_2 \) is a monomorphism, whence \( f(d, \quad c, \quad x) = x \). This equality shows that the elements \( d \) and \( c \) fulfill condition (1) of Definition. Similarly one can prove that the elements \( d \) and \( c \) fulfill condition (2). Thus, in view of Proposition 1 and Corollary 2, the \((n + 1)\)-group \( \mathfrak{D} \) is derived from such a \((k + 1)\)-group \( \mathfrak{D}_{(s-1)} \) = \((D, g)\) that the element \( h_1(d_1) = d \) is skew to \( h_1(c_1) = c \) in \( \mathfrak{D}_{(s-1)} \). From Corollary 3 we infer that \( h_1 : \mathfrak{H}_{(s-1)} \to \mathfrak{D}_{(s-1)} \). Since \( h : \mathfrak{H}_{(s-1)} \to \mathfrak{B}_{(s-1)} \) and \( d_1 \) is skew to \( c_1 \) in \( \mathfrak{H}_{(s-1)} \), the element \( h_2(d) = h(d_1) \) is skew to \( h_2(c) = h(c_1) \). Hence, by Corollary 3, \( h_2 : \mathfrak{D}_{(s-1)} \to \mathfrak{B}_{(s-1)} \). The operation \( g \) in the \((k + 1)\)-group \( \mathfrak{D}_{(s-1)} \) is given by the formula \( g(x_1, \ldots, x_{n+1}) = h^{-1}(g(h_2(x_1), \ldots, h_2(x_{n+1}))). \)

**Proposition 3.** Let \( B \) be a sub-\((n + 1)\)-group of an \((n + 1)\)-group \( \mathfrak{A} = (A, f) \) derived from a \((k + 1)\)-group \( \mathfrak{A}_{(s-1)} = (A, g) \). If for some element \( c \in B \) the element \( d \) which is skew to \( c \) in the \((k + 1)\)-group \( \mathfrak{A}_{(s-1)} \) belongs also to \( B \), then \( B \) is a sub-\((k + 1)\)-group of \( \mathfrak{A}_{(s-1)} \).

**Proof.** Assume that the element \( d \in B \) is skew to some element \( c \in B \) in \( \mathfrak{A}_{(s-1)} \).

It follows from Corollary 2 that \( d \) is \( s \)-skew to \( c \) in the \((n + 1)\)-group \( \mathfrak{A} \) and the \((k + 1)\)-ary operation \( g \) in \( \mathfrak{A}_{(s-1)} \) is described as in Corollary 2. Simultaneously, the element \( d \) is \( s \)-skew to \( c \) in the \((n + 1)\)-group \( \mathfrak{B} = (B, f) \). Hence, in view of Proposition 1, the \((n + 1)\)-group \( \mathfrak{B} \) is derived from the \((k + 1)\)-group \( \mathfrak{B}_{(s-1)} \) = \((B, g)\) where the operation \( g \) is given by the same formula as the corresponding operation \( g \) in \( \mathfrak{A}_{(s-1)} \). Then \( \mathfrak{B}_{(s-1)} \) is a sub-\((k + 1)\)-group of the \((k + 1)\)-group \( \mathfrak{A}_{(s-1)} \).

With the aid of Corollary 3, Lemma 2 from [2] can be given a slightly stronger form:

**Corollary 4.** If \( \mathfrak{A} \) is an \((n + 1)\)-group derived from a \((k + 1)\)-group \( \mathfrak{A}_{(s-1)} \) and \( h : \mathfrak{A} \to \mathfrak{B} \) is an epimorphism onto an \((n + 1)\)-group \( \mathfrak{B} \), then \( \mathfrak{B} \) is also derived from a certain \((k + 1)\)-group \( \mathfrak{B}_{(s-1)} \) such that \( h : \mathfrak{A}_{(s-1)} \to \mathfrak{B}_{(s-1)} \).

Finally, Corollary 4 can be used to modify Proposition 2 from [2].

**Proposition 4.** An \((n + 1)\)-group \( \mathfrak{G} \) is derived from a \((k + 1)\)-group \( \mathfrak{G}_{(s-1)} \) if and only if there exists an epimorphism \( \varphi_\mathfrak{G} : \mathfrak{G}^{**} \to \mathfrak{G} \) such that \( \varphi_\mathfrak{G} \varphi_\mathfrak{G} = \text{id}_\mathfrak{G} \) (where \( <\mathfrak{G}^{**}, \tau_\mathfrak{G}> \) is the free covering \((k + 1)\)-group of \( \mathfrak{G} \)). Moreover, the \((k + 1)\)-group \( \mathfrak{G}_{(s-1)} \) can be chosen in such a way, that \( \varphi_\mathfrak{G} : \mathfrak{G}^{**} \to \mathfrak{G}_{(s-1)} \).
REFERENCES


J. Michalski
Instytut Kształcenia Nauczycieli
ul. Dawida 1a, 50—527 Wrocław,
Poland