Archivum Mathematicum

Jacek Michalski

On s-skew elements in polyadic groups

Archivum Mathematicum, Vol. 19 (1983), No. 4, 215--218

Persistent URL: http://dml.cz/dmlcz/107176

ARCH. MATH. 4, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XIX: 215—218, 1983

ON s-SKEW ELEMENTS IN POLYADIC GROUPS

JACEK MICHALSKI, Wrocław

(Received October 15, 1981)

1. This note is a supplement to [2]. We introduce the notion of an s-skew element in a polyadic group (i.e. an n-group for some n) which is a generalization of that of a skew element from [1]. A 1-skew element ($s = 1$) is simply a skew element. That notion enables a simplification of notation and clears up to some extent the structure of creating $(k + 1)$-groups of a given $(n + 1)$ group (see [2]).

We use the same notation as in [2] and we assume also $n = sk$.

2. Post in [3] stated a necessary and sufficient condition for an $(n + 1)$-group to be derived from a $(k + 1)$-group. That condition was expressed in terms of polyads. To put those terms into the language used in [2] suggests the following.

Definition. Let d and c be elements of an $(n + 1)$-group $\mathfrak{G} = (G, f)$. The element d is called an s-skew element to the element c if the following conditions are fulfilled:

$$f(\overset{s}{d}, \overset{(k-1)s}{c}, x) = x \qquad \text{for each } x \in G; \tag{1}$$

$$f(x_1, \ldots, x_i, d, \overset{k-1}{c}, x_{i+1}, \ldots, x_{n+1-k}) = \tag{2}$$

$$= f(x_1, \ldots, x_i, \overset{k-1}{c}, d, x_{i+1}, \ldots, x_{n+1-k}) = f(d, \overset{k-1}{c}, x_1, \ldots, x_{n+1-k})$$

for each $x_1, \ldots, x_{n+1-k} \in G$ and arbitrary $i = 1, \ldots, n + 1 - k$.

The formerly mentioned condition of Post was given in a modified form (adopted to the given in [2] construction of a free covering group) in [2] as Theorem 5. Using the notion of an s-skew element this condition can be reformulated as follows:

Proposition 1. An $(n + 1)$-group $\mathfrak{G} = (G, f)$ is derived from a $(k + 1)$-group if and only if for some element $c \in G$ there exists an element $d \in G$ which is s-skew to c in the $(n + 1)$-group $\mathfrak{G}$. In that case the $(k + 1)$-ary operation g in the $(k + 1)$-

group $\mathfrak{G}_{(s-1)} = (G, g)$ can be given by the formula $g(x_1, \ldots, x_{k+1}) = f(x_1, \ldots, x_{k+1}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c})$

Examining the proof of Theorem 5 from [2] we get a little more, namely

Corollary 1. If an $(n+1)$-group $\mathfrak{G} = (G, f)$ is derived from a $(k+1)$-group $\mathfrak{G}_{(s-1)} = (G, g)$, then for every element $c \in G$ there exists an s-skew element to c in the $(n+1)$-group $\mathfrak{G}$.

Corollary 2. If an $(n+1)$-group $\mathfrak{G} = (G, f)$ is derived from a $(k+1)$-group $\mathfrak{G}_{(s-1)} = (G, g)$, then the following conditions are equivalent:

(a) the element $d \in G$ is skew to the element $c \in G$ in $\mathfrak{G}_{(s-1)}$;

(b) the element $d \in G$ is s-skew to the element $c \in G$ in $\mathfrak{G}$ and $g(x_1, \ldots, x_{k+1}) = f(x_1, \ldots, x_{k+1}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c})$

From this corollary we infer that if we know the skew element to some element from $\mathfrak{G}_{(s-1)} = (G, g)$, then the $(k+1)$-ary operation g is already uniquely determined. There exists a one-to-one correspondence between the set of the creating $(k+1)$-groups of the $(n+1)$-group $\mathfrak{G}$ and the set of all s-skew elements to any element from the $(n+1)$-group $\mathfrak{G}$ (see [3], p. 232).

3. From Proposition 1 and the Corollaries resulting from it one can obtain some statements concerning homomorphisms and sub-$(k+1)$-groups of creating $(k+1)$-groups.

Corollary 3. Let $(n+1)$-groups $\mathfrak{A} = (A, f)$ and $\mathfrak{B} = (B, f)$ be derived from $(k+1)$-groups $\mathfrak{A}_{(s-1)} = (A, g)$ and $\mathfrak{B}_{(s-1)} = (B, g)$. If $h : \mathfrak{A} \to \mathfrak{B}$ and $h(\bar{c}^{(g)}) = \overline{h(c)}^{(g)}$ for some $c \in A$, then $h : \mathfrak{A}_{(s-1)} \to \mathfrak{B}_{(s-1)}$.

Proof. Using Corollary 2 the element $d = \bar{c}^{(g)}$ is s-skew to c in the $(n+1)$-group $\mathfrak{A}$. Hence $h(g(x_1, \ldots, x_{k+1})) = h(f(x_1, \ldots, x_{k+1}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c})) = f(h(x_1), \ldots, h(x_{k+1}), \overset{s-1}{h(d)}, \overset{(k-1)(s-1)}{h(c)}) = g(h(x_1), \ldots, h(x_{k+1}))$. □

Proposition 2. Let $(n+1)$-groups $\mathfrak{A} = (A, f)$ and $\mathfrak{B} = (B, f)$ be derived from $(k+1)$-groups $\mathfrak{A}_{(s-1)} = (A, g)$, $\mathfrak{B}_{(s-1)} = (B, g)$ and $h : \mathfrak{A}_{(s-1)} \to \mathfrak{B}_{(s-1)}$. If $\mathfrak{D} = (D, f)$ is an $(n+1)$-group, $h = h_2 h_1$ where $h_1 : \mathfrak{A} \to \mathfrak{D}$, $h_2 : \mathfrak{D} \to \mathfrak{B}$ and h_2 is a monomorphism, then $\mathfrak{D}$ is derived from a unique $(k+1)$-group $\mathfrak{D}_{(s-1)} = (D, g)$ such that $h_1 : \mathfrak{A}_{(s-1)} \to \mathfrak{D}_{(s-1)}$, $h_2 : \mathfrak{D}_{(s-1)} \to \mathfrak{B}_{(s-1)}$.

Proof. Take an element $c_1 \in A$ and an element $d_1 \in A$ to be skew to c_1 in the $(k+1)$-group $\mathfrak{A}_{(s-1)}$. The element d_1 is s-skew to c_1 in the $(n+1)$-group $\mathfrak{A}$. Let $c = h_1(c_1)$ and $d = h_1(d_1)$. We show that the element d is s-skew to c in $\mathfrak{D}$. Using the assumption $h(c_1)$ is s-skew to $h(d_1)$ in the $(n+1)$-group $\mathfrak{B}$ (since

$h : \mathfrak{A}_{(s-1)} \to \mathfrak{B}_{(s-1)}$), whence we get $h_2(f(\overset{s}{d}, \overset{(k-1)s}{c}, x)) = f(\overset{s}{h_2(d)}, \overset{(k-1)s}{h_2(c)}, h_2(x)) = f(\overset{s}{h_2h_1(d_1)}, \overset{(k-1)s}{h_2h_1(c_1)}, h_2(x)) = f(\overset{s}{h(d_1)}, \overset{(k-1)s}{h(c_1)}, h_2(x)) = h_2(x)$. But the homomorphism h_2 is a monomorphism, whence $f(\overset{s}{d}, \overset{(k-1)s}{c}, x) = x$. This equality shows that the elements d and c fulfil condition (1) of Definition. Similarly one can prove that the elements d and c fulfil condition (2). Thus, in view of Proposition 1 and Corollary 2, the $(n+1)$-group $\mathfrak{D}$ is derived from such a $(k+1)$-group $\mathfrak{D}_{(s_1)} = (D, g)$ that the element $h_1(d_1) = d$ is skew to $h_1(c_1) = c$ in $\mathfrak{D}_{(s-1)}$. From Corollary 3 we infer that $h_1 : \mathfrak{A}_{(s-1)} \to \mathfrak{D}_{(s-1)}$. Since $h : \mathfrak{A}_{(s-1)} \to \mathfrak{B}_{(s-1)}$ and d_1 is skew to c_1 in $\mathfrak{A}_{(s-1)}$, the element $h_2(d) = h(d_1)$ is skew to $h_2(c) = h(c_1)$. Hence, by Corollary 3, $h_2\colon \mathfrak{D}_{(s-1)} \to \mathfrak{B}_{(s-1)}$. The operation g in the $(k+1)$-group $\mathfrak{D}_{(s-1)}$ is given by the formula $g(x_1, \ldots, x_{n+1}) = h^{-1}(g(h_2(x_1), \ldots, h_2(x_{k+1})))$.

Proposition 3. Let B be a sub-$(n+1)$-group of an $(n+1)$-group $\mathfrak{A} = (A, f)$ derived from a $(k+1)$-group $\mathfrak{A}_{(s-1)} = (A, g)$. If for some element $c \in B$ the element d which is skew to c in the $(k+1)$-group $\mathfrak{A}_{(s-1)}$ belongs also to B, then B is a sub-$(k+1)$-group of $\mathfrak{A}_{(s-1)}$.

Proof. Assume that the element $d \in B$ is skew to some element $c \in B$ in $\mathfrak{A}_{(s-1)}$. It follows from Corollary 2 that d is s-skew to c in the $(n+1)$-group $\mathfrak{A}$ and the $(k+1)$-ary operation g in $\mathfrak{A}_{(s-1)}$ is described as in Corollary 2. Simultaneously, the element d is s-skew to c in the $(n+1)$-group $\mathfrak{B} = (B, f)$. Hence, in view of Proposition 1, the $(n+1)$-group $\mathfrak{B}$ is derived from the $(k+1)$-group $\mathfrak{B}_{(s-1)} = (B, g)$ where the operation g is given by the same formula as the corresponding operation g in $\mathfrak{A}_{(s-1)}$. Then $\mathfrak{B}_{(s-1)}$ is a sub-$(k+1)$-group of the $(k+1)$-group $\mathfrak{A}_{(s-1)}$. □

With the aid of Corollary 3, Lemma 2 from [2] can be given a slightly stronger form:

Corollary 4. If $\mathfrak{A}$ is an $(n+1)$-group derived from a $(k+1)$-group $\mathfrak{A}_{(s-1)}$ and $h\colon \mathfrak{A} \to \mathfrak{B}$ is an epimorphism onto an $(n+1)$-group $\mathfrak{B}$, then $\mathfrak{B}$ is also derived from a certain $(k+1)$-group $\mathfrak{B}_{(s-1)}$ such that $h\colon \mathfrak{A}_{(s-1)} \to \mathfrak{B}_{(s-1)}$.

Finally, Corollary 4 can be used to modify Proposition 2 from [2].

Proposition 4. An $(n+1)$-group $\mathfrak{G}$ is derived from a $(k+1)$-group $\mathfrak{G}_{(s-1)}$ if and only if there exists an epimorphism $\varrho_G\colon \mathfrak{G}^{*s}_{(s)} \to \mathfrak{G}$ such that $\varrho_G \tau_G = \mathrm{id}_G$ (where $\langle \mathfrak{G}^{*s}, \tau_G \rangle$ is the free covering $(k+1)$-group of $\mathfrak{G}$). Moreover, the $(k+1)$-group $\mathfrak{G}_{(s-1)}$ can be chosen in such a way, that $\varrho_G\colon \mathfrak{G}^{*s} \to \mathfrak{G}_{(s-1)}$.

REFERENCES

[1] W. Dörnte, *Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Mathematische Zeitschrift 29 (1929), 1—19.

[2] J. Michalski, *Covering k-groups of n-groups*, Arch. Math. (Brno) 17 (1981), 207—226.

[3] E. Post, *Polyadic groups*, Trans. Amer. Math. Soc. 48 (1940), 208—350.

J. Michalski
Instytut Kształcenia Nauczycieli
ul. Dawida 1a, 50—527 Wrocław,
Poland