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HAMILTON EXTREMALS IN HIGHER ORDER MECHANICS

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1. INTRODUCTION

The geometrical and physical significance of the problem of a generalization of the Hamilton equations of classical mechanics to higher order mechanics and field theory ranks it still in the field of interest of mathematicians and theoretical physicists. One possible approach to this problem consists in assigning to the Euler—Lagrange equations of a given lagrangian a new system of *extended* equations, having more solutions than the initial ones, in such a way that (1) to each extremal (= a solution of the Euler—Lagrange equations) there corresponds a solution of the extended equations, and (2) in case that the lagrangian satisfies a regularity condition, the correspondence between the extremals and the solutions of the extended equations is bijective (see [2]). The possibility of constructing the extended equations is based on the existence of a Lepagean equivalent of a lagrangian proved independently in [3] and [6]. An important feature of the extended equations is that for regular lagrangians of order one they may be interpreted as the Hamilton equations of the given lagrangian.

The purpose of this paper is to discuss the Hamilton equations of higher order integral variational problems of one independent variable, and to give a geometric interpretation of these equations in terms of a certain distribution. To this purpose we use the theory of Lepagean differential forms developed in [5], [6], and [7], and we closely follow the general ideas of the paper [2] such as the extension idea by means of a Lepagean form, and the regularity condition. For further results in this direction, we refer to [1] and [4].

As the basic underlying structure we choose a fibered manifold over a one-dimensional base manifold. Within this framework, we construct, under a regularity hypothesis, the Hamilton (one-dimensional) distribution, which corresponds to the so called Hamilton vector field from the variational theory of curves, and whose integral sections are precisely the solutions of the extended equations. We also express this distribution in terms of linear differential forms, which allows us to take into account general (not necessarily regular) lagrangians; the distribution corresponding to a general lagrangian is then shown to be completely integrable if and only if the lagrangian is regular.

2. LEPAGEAN FORMS IN HIGHER ORDER MECHANICS

In this section we recall main definitions and facts of the theory of Lepagean differential forms. Our exposition is adapted to the case of variational problems whose underlying fibered manifolds have one-dimensional bases.

Throughout this paper, X is a one-dimensional manifold, $\pi : Y \to X$ is a fibered manifold, and we denote $m = \dim Y - 1$. $\Gamma(\pi)$ denotes the set of local sections of π, $\pi_r : j^r Y \to X$ is the r-jet prolongation of π, and $\pi_{r,s} : j^r Y \to j^s Y$, $0 \leqq s < r$, is the natural projection of jet spaces. If (V, ψ), $\psi = (t, q^\sigma)$, $1 \leqq \sigma \leqq m$, is a fiber chart on Y, then the fiber chart on $j^r Y$, associated with (V, ψ), is denoted by (V_r, ψ_r), $\psi_r = (t, q_i^\sigma)$, where $1 \leqq \sigma \leqq m$, $0 \leqq i \leqq r$. The r-jet prolongation of the fibered manifold $\pi_s : j^s Y \to X$ is denoted by $(\pi_s)_r : j^r(j^s Y) \to X$. In accordance with the above notation, the chart on $j^r(j^s Y)$, associated with (V, ψ), is denoted by $((V_s)_r, (\psi_s)_r)$, $(\psi_s)_r = (t, q^\sigma_{i,j})$, where $1 \leqq \sigma \leqq m, 0 \leqq i \leqq s, 0 \leqq j \leqq r$, and $q_{i,0} = q_i$.

The module of p-forms (resp. π_s-horizontal p-forms, resp. $\pi_{s,0}$-horizontal p-forms) on $j^s Y$ is denoted by $\Omega^p(j^s Y)$ (resp. $\Omega^p_X(j^s Y)$, resp. $\Omega^p_Y(j^s Y)$). Let $\eta \in \Omega^p(j^s Y)$ be a form. There exist unique forms $h(\eta) \in \Omega^p_X(j^{s+1} Y)$ and $p(\eta) \in \Omega^p(j^{s+1} Y)$ such that for each $\gamma \in \Gamma(\pi)$, $j^s\gamma^*\eta = j^{s+1}\gamma^* h(\eta)$, such that $\pi^*_{s+1,s}\eta = h(\eta) + p(\eta)$; the mapping $\eta \to h(\eta)$ is called the *π-horizontalization*. A form $\eta \in \Omega^p(j^s Y)$ is π_s-horizontal if and only if $\pi^*_{s+1,s}\eta = h(\eta)$; η is called *contact* if $\pi^*_{s+1,s}\eta = p(\eta)$, i.e., $h(\eta) = 0$. For $p \geqq 2\eta$ is obviously contact.

The following definitions slightly differ from the ones introduced in [6]. Let $p \geqq 1$. A form $\eta \in \Omega^p(j^s Y)$ is called *1-contact*, if (1) it is contact, and (2) for each π_s-vertical vector field ξ on $j^s Y$, the form $i_\xi\eta$ is π_s-horizontal (0-contact); if $k \geqq 2$, η is called *k-contact*, if $i_\xi\eta$ is $(k-1)$-contact. We denote by $\Omega^{p,k}(j^s Y)$ the module of k-contact p-forms on $j^s Y$. Each form $\eta \in \Omega^p(j^s Y)$ admits a unique decomposition

$$\pi^*_{s+1,s}\eta = \eta_0 + \eta_1 + \ldots + \eta_p, \tag{2.1}$$

where $\eta_k \in \Omega^{p,k}(j^{s+1} Y)$.

Let $\varrho \in \Omega^1(j^s Y)$ be a form. There exist uniquely defined forms $E \in \Omega^{2,1}(j^{s+1} Y)$, $F \in \Omega^{2,2}(j^{s+1} Y)$ such that

$$\pi^*_{s+1,s}\, d\varrho = E + F. \tag{2.2}$$

We say that ϱ is a *Lepagean form* if $E \in \Omega^{2,1}_Y(j^{s+1} Y)$, i.e. if E is $\pi_{s+1,0}$-horizontal.

Let $\varrho \in \Omega^1(j^s Y)$. In a fiber chart (V, ψ), $\psi = (t, q^\sigma)$, $\pi^*_{s+1,s}\varrho$ has an expression

$$\pi^*_{s+1,s}\varrho = f_0\, dt + \sum_{i=0}^{s} f^{i+1}_\sigma \omega^\sigma_i, \tag{2.3}$$

where

(2.4) $$\omega_i^\sigma = \mathrm{d}q_i^\sigma - q_{i+1}^\sigma\,\mathrm{d}t.$$

It can be easily verified that ϱ is Lepagean if and only if

(2.5) $$f_\sigma^i = \sum_{k=0}^{s+1-i} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial f_0}{\partial q_{i+k}^\sigma}, \qquad 1 \leqq i \leqq s.$$

This implies, in particular, that if a form $\varrho \in \Omega^1(\mathrm{j}^s Y)$ is Lepagean, then the form $\lambda = h(\varrho)$ is $\pi_{s+1,k}$-projectable for some $k \leqq (s+1)/2$.

A *lagrangian of order r* for π is by definition an element $\lambda \in \Omega_X^1(\mathrm{j}^r Y)$. A *Lepagean equivalent* of a lagrangian $\lambda \in \Omega_X^1(\mathrm{j}^r Y)$ is a Lepagean form $\varrho \in \Omega^1(\mathrm{j}^s Y)$, where s is an integer, such that $h(\varrho) = \lambda$. Writing in a fiber chart (V, ψ), $\psi = (t, q^\sigma)$,

(2.6) $$\lambda = L\,\mathrm{d}t$$

and using (2.3) and (2.5) we easily obtain that there exists exactly one Lepagean equivalent of λ which we denote by Θ_λ and call the *Poincaré – Cartan equivalent* of λ; we note that for dim $X > 1$ there exist more Lepagean equivalents than one. Substituting $f_0 = L$ in (2.5) we obtain the coordinate expression of Θ_λ for the fiber chart (V, ψ). Obviously, $f_\sigma^{i+1} = f_\sigma^{i+1}(t, q_0^\sigma, q_1^\sigma, \ldots, q_{2r-i-1}^\sigma)$ which implies that in general, the Poincaré – Cartan equivalent Θ_λ of λ is defined on $j^{2r-1}Y$. Consider the decomposition (2.2). We have

(2.7) $$\overset{*}{\pi}_{2r,2r-1}\,\mathrm{d}\Theta_\lambda = E_\lambda + F_\lambda,$$

where

(2.8) $$E_\lambda = E_\sigma(L)\,\mathrm{d}q^\sigma \wedge \mathrm{d}t, \qquad E_\sigma(L) = \sum_{k=0}^{r} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial L}{\partial q_k^\sigma},$$

(2.9) $$F_\lambda = \sum_{i=0}^{r-1} \sum_{k=0}^{2r-1-i} \frac{\partial f_\sigma^{i+1}}{\partial q_k^\nu}\, \omega_k^\nu \wedge \omega_i^\sigma.$$

3. HAMILTON EXTREMALS

Recall that a section $\delta \in \Gamma(\pi_{2r-1})$ is called a *Hamilton extremal* of a lagrangian $\lambda \in \Omega_X^1(j^r Y)$ if for each π_{2r-1}-vertical vector field ξ on $j^{2r-1}Y$

(3.1) $$\delta^* i_\xi\,\mathrm{d}\Theta_\lambda = 0.$$

Let $\tilde{h} : \Omega^p(j^{2r-1}Y) \to \Omega_X^p(\mathrm{j}^1(j^{2r-1}Y))$ be the π_{2r-1}-horizontalization. Then (3.1) is equivalent to the condition

(3.2) $$j^1\delta^* \tilde{h}(i_\xi\,\mathrm{d}\Theta_\lambda) = 0,$$

where $j^1\delta$ is the 1-jet prolongation of δ.

Consider a lagrangian $\lambda \in \Omega_X^1(j^r Y)$ and its chart expression (2.6) with respect to a fiber chart (V, ψ), $\psi = (t, q^\sigma)$, on Y. In this fiber chart, the Poincaré–Cartan equivalent Θ_λ of λ has an expression

$$\Theta_\lambda = L\, dt + \sum_{i=0}^{r-1} f_\sigma^{i+1} \omega_i^\sigma, \tag{3.3}$$

where

$$f_\sigma^j = \sum_{k=0}^{2r-j} (-1)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial q_{j+k}^\sigma}, \qquad 1 \leqq j \leqq 2r - 1. \tag{3.4}$$

The following is a direct consequence of (3.2).

Theorem 1. A section $\delta \in \Gamma(\pi_{2r-1})$ is a Hamilton extremal of the lagrangian λ if and only if along $j^1\delta$

$$\sum_{k=0}^{2r-1-i} \frac{\partial f_\nu^{k+1}}{\partial q_i^\sigma} (q_{k,1}^\nu - q_{k+1}^\nu) = 0, \qquad r \leqq i \leqq 2r - 1, \tag{3.5}$$

$$\sum_{k=0}^{r-1} \left(\frac{\partial f_\nu^{k+1}}{\partial q_i^\sigma} - \frac{\partial f_\sigma^{i+1}}{\partial q_k^\nu} \right) (q_{k,1}^\nu - q_{k+1}^\nu) - \sum_{k=r}^{2r-1-i} \frac{\partial f_\sigma^{i+1}}{\partial q_k^\nu} (q_{k,1}^\nu - q_{k+1}^\nu) = 0,$$
$$1 \leqq i \leqq r - 1,$$

$$E_\sigma(L) + \sum_{k=0}^{r-1} \frac{\partial f_\nu^{k+1}}{\partial q_0^\sigma} (q_{k,1}^\nu - q_{k+1}^\nu) - \sum_{k=0}^{2r-1} \frac{\partial f_\sigma^1}{\partial q_k^\nu} (q_{k,1}^\nu - q_{k+1}^\nu) = 0.$$

Proof. There exists one and only one form $H_\lambda \in \Omega^2(j^1(j^{2r-1}Y))$ such that for each π_{2r-1}-vertical vector field ξ on $2^{2r-1}Y$, $i_{j^1\xi} H_\lambda = \tilde{h}(i_\xi\, d\Theta_\lambda)$. Condition (3.2) implies that δ is a Hamilton extremal of λ if and only if H_λ vanishes along $j^1\delta$. But in the fiber chart (V, ψ),

$$H_\lambda = \sum_{i=0}^{2r-1} F_\sigma^{i+1}\, dq_i^\sigma \wedge dt, \tag{3.6}$$

where the functions F_σ^{i+1} are precisely the left-hand side expressions (3.5). This proves Theorem 1.

The form H_λ (3.6) is called the *Hamilton form* of the lagrangian λ [8].

4. REGULARITY

A Hamilton extremal $\delta \in \Gamma(\pi_{2r-1})$ of a lagrangian $\lambda \in \Omega_X^1(j^r Y)$ is said to be ***regular***, if $\delta = j^{2r-1}\gamma$ for some extremal $\gamma \in \Gamma(\pi)$ of λ.

A sufficient condition of regularity of a Hamilton extremal can be deduced from equations (3.5). Notice that the first two of these equations form a system of linear, homogeneous equations for the unknowns $q_{k,1}^\nu - q_{k+1}^\nu$, $0 \leqq \nu \leqq m$,

$0 \leqq k \leqq 2r - 2$; the number of the equations is $m(2r - 1)$, and is equal to the number of the unknowns. Let us consider the matrix of this system. Labelling rows (resp. columns) by $i = 1, 2, \ldots, 2r - 1$ (resp. $q^{\nu}_{0,1} - q^{\nu}_{1}, \ldots, q^{\nu}_{2r-2,1} - q^{\nu}_{2r-1}$) we obtain for this matrix

$$\begin{bmatrix} & & & & \frac{\partial f^2_\sigma}{\partial q^\nu_{2r-2}} \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & \frac{\partial f^r_\sigma}{\partial q^\nu_r} & & \\ & \frac{\partial f^r_\nu}{\partial q^\sigma_r} & & \emptyset & \\ \cdot^{\cdot^{\cdot}} & & & & \\ \frac{\partial f^1_\nu}{\partial q^\sigma_{2r-1}} & & & & \end{bmatrix}, \tag{4.1}$$

where by (3.4)

$$\frac{\partial f^k_\nu}{\partial q^\sigma_{2r-k}} = \frac{\partial f^k_\sigma}{\partial q^\nu_{2r-k}} = (-1)^{r-k} \frac{\partial^2 L}{\partial q^\sigma_r \, \partial q^\nu_r}, \qquad 1 \leqq k \leqq r. \tag{4.2}$$

Hence a necessary and sufficient condition that (4.1) be a regular matrix (at a point) is that the matrix $(\delta^2 L / \delta q^\sigma_r \delta q^\nu_r)$ whose elements are labelled by σ and ν, is regular (at this point). Since the regularity of (4.1) is a sufficient condition for the first two equations (3.5) to have the trivial solution only, we have the following result.

Theorem 2. Let $\delta : U \to j^{2r-1} Y$ be a Hamilton extremal of λ, defined on an open set $U \subset X$. Assume that to each point $x \in U$ there exists a fiber chart (V, ψ), $\psi = (t, q^\sigma)$, on Y such that $\delta(x) \in V_{2r-1}$ and

$$\det\left(\frac{\partial^2 L}{\partial q^\sigma_r \, \partial q^\nu_r}\right) \neq 0 \tag{4.3}$$

at $\pi_{2r-1,r}\delta(x)$, where the function L is defined by the chart expression $\lambda = L\,\mathrm{d}t$. Then δ is regular.

Proof. Consider the Hamilton extremal δ of Theorem 2. Let $x \in U$ be a point, (V, ψ) a fiber chart on Y for which the assumptions of Theorem 2 are satisfied. Since δ satisfies (3.5) and the matrix (4.1) is regular, we have at $\delta(x)$

$$q^\nu_{k,1} - q^\nu_{k+1} = 0, \qquad 0 \leqq k \leqq 2r - 2, \tag{4.4}$$

that is,

$$q^\nu_{k+1}(\delta(x)) = \left(\frac{\mathrm{d}}{\mathrm{d}t}(q^\nu_k \circ \delta)\right)_x = \ldots = \left(\frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}(q^\nu_0 \circ \delta)\right)_x. \tag{4.5}$$

This implies that $\delta(x) = j_x^{2r-1}\gamma$, where $\gamma = \pi_{2r-1,0} \circ \delta$, and, since the point x is arbitrary, $\delta = j^{2r-1}\gamma$. It remains to show that γ is an extremal of λ. This follows, however, from the third equation (3.5) which coincides, along $j^{2r}\gamma$, with the Euler – Lagrange equation of λ.

We note that the condition (4.3) is independent of the choice of the fibered chart (V, ψ); in this paper we call it the *regularity condition.* If the regularity condition is satisfied at each point of $j^r Y$, then every Hamilton extremal is regular, and the system (3.5) is equivalent with the Euler – Lagrange equation.

4. THE LEGENDRE TRANSFORMATION AND THE HAMILTON EQUATIONS

In this section we assume that we are given a lagrangian $\lambda \in \Omega^1_X(j^r Y)$ satisfying at each point of $j^r Y$, the regularity condition (4.3).

Let (V, ψ), $\psi = (t, q^\sigma)$, be a fiber chart on Y, $\lambda = L\,\mathrm{d}t$ the expression of λ for the fiber chart (V_r, ψ_r), $\psi_r = (t, q^\sigma_k)$, $0 \leqq k \leqq r$. Put

$$p^{r-1}_\nu = f^r_\nu(t, q^\sigma_0, \ldots, q^\sigma_r), \tag{5.1}$$

where by (3.4), $f^r_\nu = \partial L/\partial q^\nu_r$. Obviously, the regularity condition (4.3) guarantees that (V_r, Ψ_r), $\Psi_r = (t, q^\sigma_j, p^{r-1}_\sigma)$, $1 \leqq j \leqq r-1$, is a chart on $j^r Y$. Consider the fiber chart (V_{2r-1}, ψ_{2r-1}), $\psi_{2r-1} = (t, q^\sigma_k)$, $0 \leqq k \leqq 2r-1$, on $j^{2r-1}Y$, associated with (V, ψ), and put

$$p^{r-j-1}_\nu = f^{r-j}_\nu(t, q^\sigma_0, \ldots, q^\sigma_{r+j}), \qquad 1 \leqq j \leqq r-1. \tag{5.2}$$

As before, the regularity condition guarantees that (V_{2r-1}, Ψ_{2r-1}), $\Psi_{2r-1} = (t, q^\nu_j, p^k_\nu)$, $1 \leqq \nu \leqq m$, $0 \leqq j, k \leqq r-1$, is a fiber chart on $j^{2r-1}Y$ (over $j^r Y$). We call the coordinates (t, q^ν_j, p^k_ν) the *Legendre coordinates* on V_{2r-1}, and the transformation $\Psi_{2r-1}\psi^{-1}_{2r-1}$ of coordinates is called the *Legendre* (coordinate) *transformation.*

Let us consider the condition for Hamilton extremals (3.2) in the Legendre coordinates. In these coordinates,

$$\Theta_\lambda = -H\,\mathrm{d}t + \sum_{i=0}^{r-1} p^i_\sigma\,\mathrm{d}q^\sigma_i, \tag{5.3}$$

where

$$-H = L + \sum_{i=0}^{r-1} p^i_\sigma q^\sigma_{i+1}, \tag{5.4}$$

and H is considered as a function of the Legendre coordinates. Computing now the exterior derivative $\mathrm{d}\Theta_\lambda$, the form $i_\xi\,\mathrm{d}\Theta_\lambda$, where ξ is any π_{2r-1}-vertical vector field on $j^{2r-1}Y$, expressed in the Legendre coordinates, and the Hamilton form H_ν (see the proof of Theorem 1) one obtains

$$(5.5) \qquad H_\lambda = \left[\sum_{i=0}^{r-1}\left(-\frac{\partial H}{\partial q_i^\sigma} - p_\sigma^{i,1}\right)dq_i^\sigma + \left(-\frac{\partial H}{\partial p_\sigma^i} + q_{i,1}^\sigma\right)dp_\sigma^i\right]\wedge dt,$$

where the functions $p_\sigma^{i,1}$, $q_{i,1}^\sigma$ are the coordinates of the coordinate system $((V_{2r-1})_1, (\Psi_{2r-1})_1), (\Psi_{2r-1})_1 = (t, q_i^\sigma, p_\nu^k, q_{i,1}^\sigma, p_\nu^{k,1})$, on $j^1(j^{2r-1}Y)$. Consequently, a section δ of $j^{2r-1}Y$ is a Hamilton extremal if and only if it satisfies the *Hamilton equations*

$$(5.6) \qquad \frac{\partial H}{\partial q_i^\sigma} + \frac{d}{dt}(p_\sigma^i \circ \delta) = 0, \qquad \frac{\partial H}{\partial p_\sigma^i} - \frac{d}{dt}(q_i^\sigma \circ \delta) = 0.$$

To interprete geometrically these equations in a canonical manner, as in the first order variational theory, we introduce the following definition. We shall say that a vector field ζ on $j^{2r-1}Y$ is a *Hamiltonian vector field*, if

$$(5.7) \qquad i_\zeta d\Theta_\lambda = 0.$$

Let us express this condition in the Legendre coordinates. Let ζ be a vector field on $j^{2r-1}Y$,

$$(5.8) \qquad \zeta = \zeta^0 \frac{\partial}{\partial t} + \sum_{i=0}^{r-1}\left(\zeta_i^\sigma \frac{\partial}{\partial q_i^\sigma} + \bar{\zeta}_\sigma^i \frac{\partial}{\partial p_\sigma^i}\right).$$

By (5.3),

$$(5.9) \qquad i_\zeta d\Theta_\lambda = \left(-i_\zeta dH + \frac{\partial H}{\partial t}\zeta^0\right)dt + $$
$$+ \sum_{i=0}^{r-1}\left[\left(\frac{\partial H}{\partial q_i^\sigma}\zeta^0 + \bar{\zeta}_\sigma^i\right)dq_i^\sigma + \left(\frac{\partial H}{\partial p_\sigma^i}\zeta^0 - \zeta_i^\sigma\right)dp_\sigma^i\right].$$

Hence ζ is a Hamiltonian vector field if and only if

$$(5.10) \qquad \zeta_i^\sigma = \frac{\partial H}{\partial p_\sigma^i}\zeta^0, \qquad \bar{\zeta}_\sigma^i = -\frac{\partial H}{\partial q_i^\sigma}\zeta^0, \qquad 0 \leq i \leq r-1,$$

or, which is the same, $\zeta = \zeta^0 . \zeta_0$, where

$$(5.11) \qquad \zeta_0 = \frac{\partial}{\partial t} + \sum_{i=0}^{r-1}\left(\frac{\partial H}{\partial p_\sigma^i}\frac{\partial}{\partial q_i^\sigma} - \frac{\partial H}{\partial q_i^\sigma}\frac{\partial}{\partial p_\sigma^i}\right).$$

Let Δ_λ denote the one-dimensional (regular) distribution on $j^{2r-1}Y$, locally generated by the vector fields ζ_0 (5.11). We have the following result.

Theorem 3. Let $\delta \in \Gamma(\pi_{2r-1})$. The following three conditions are equivalent:
(1) δ is a Hamilton extremal of λ.

(2) For each fiber chart (V, ψ), $\psi = (t, q^\sigma)$, δ satisfies the Hamilton equations (5.6).

(3) δ is an integral submanifold of the distribution Δ_λ.

Proof. Since δ is a Hamilton extremal of λ if and only if the Hamilton form H_λ vanishes along the first jet prolongation $j^1\delta$ of δ, conditions (1) and (2) are obviously equivalent.

Assume that (2) holds. Then, in fiber coordinates, for each point $x \in X$ and each tangent vector $\xi \in T_xX$, $\xi = \xi^0(\mathrm{d}/\mathrm{d}t)_x$

$$T_x\delta \,.\, \xi = \xi^0\left[\frac{\partial}{\partial t} + \sum_{i=0}^{r-1}\left(\frac{\mathrm{d}(q_i^\sigma \circ \delta)}{\mathrm{d}t}\frac{\partial}{\partial q_i^\sigma} + \frac{\mathrm{d}(p_\sigma^i \circ \delta)}{\mathrm{d}t}\frac{\partial}{\partial p_\sigma^i}\right)\right]. \tag{5.12}$$

Since δ satisfies the Hamilton equations (5.6), the vector $T_x\delta \,.\, \xi \in T_{\delta(x)}j^{2r-1}Y$ obviously belongs to the subspace $\Delta_\lambda(\delta(x)) \subset T_{\delta(x)}j^{2r-1}Y$. The converse is obtained by reversing the arguments.

6. THE EULER-LAGRANGE DISTRIBUTION

Let $\lambda \in \Omega_X^1(j^rY)$ be any lagrangian. By definition, a section $\delta \in \Gamma(\pi_{2r-1})$ is a Hamilton extremal of λ if and only if it is an integral submanifold of the ideal of forms on $j^{2r-1}Y$, generated by the linear differential forms $i_\xi\,\mathrm{d}\Theta_\lambda$ (3.1), where ξ runs over all π_{2r-1}-vertical vector fields on $j^{2r-1}Y$. We shall now study the problem as to under what conditions these linear differential forms define a distribution on $j^{2r-1}Y$.

Let (V, ψ), $\psi = (t, q^\sigma)$, be a fiber chart on Y, and consider the Poincaré–Cartan equivalent Θ_λ in this chart (3.3). Let ξ be a π_{2r-1}-vertical vector field on $j^{2r-1}Y$

$$\xi = \sum_{k=0}^{2r-1}\xi_k^\sigma\frac{\partial}{\partial q_k^\sigma}. \tag{6.1}$$

By a direct computation

$$i_\xi\,\mathrm{d}\Theta_\lambda = \sum_{k=0}^{2r-1}\xi_i^\sigma\eta_\sigma^i, \tag{6.2}$$

where

$$\eta_\sigma^0 = \left(\frac{\partial L}{\partial q_0^\sigma} - \frac{\partial f_\sigma^1}{\partial t}\right)\mathrm{d}t + \sum_{k=0}^{r-1}\frac{\partial f_\nu^{k+1}}{\partial q_0^\sigma}\omega_k^\nu - \sum_{k=0}^{2r-1}\frac{\partial f_\sigma^1}{\partial q_k^\nu}\mathrm{d}q_k^\nu, \tag{6.3}$$

$$\eta_\sigma^i = \sum_{k=0}^{r-1}\left(\frac{\partial f_\nu^{k+1}}{\partial q_i^\sigma} - \frac{\partial f_\sigma^{i+1}}{\partial q_k^\nu}\right)\omega_k^\nu - \sum_{k=r}^{2r-i-1}\frac{\partial f_\sigma^{i+1}}{\partial q_k^\nu}\omega_k^\nu, \qquad 1 \leqq i \leqq r-1,$$

$$\eta_\sigma^i = \sum_{k=0}^{2r-1-i}\frac{\partial f_\nu^{k+1}}{\partial q_i^\sigma}\omega_k^\nu, \qquad r \leqq i \leqq 2r-1.$$

Since the expression (6.2) is independent of the choice of fiber coordinates and the transformation formulas for the components of the vector field ξ are linear, the transformation formulas for the forms (6.3) are also linear. Consequently, these forms define, in a well-known sense, a distribution (i.e. a vector subbundle of the tangent bundle $Tj^{2r-1}Y$); the distribution is singulár in the sense that the the dimension of the vector subspace may vary from point to point. We call this distribution the *Euler–Lagrange distribution*, associated with the lagrangian λ, and denote it by Δ_λ.

If the assumptions of Theorem 2 are satisfied, then the Euler–Lagrange distribution is spanned by the forms

$$\left(\frac{\partial L}{\partial q_0^\sigma} - \frac{\partial f_\sigma^1}{\partial t}\right)\mathrm{d}t - \sum_{k=0}^{2r-1} \frac{\partial f_\sigma^1}{\partial q_k^\nu}\mathrm{d}q_k^\nu, \omega_i^\sigma, \qquad 0 \leqq i \leqq 2r-2. \tag{6.4}$$

This follows from the fact that the second and the third equations (6.3) have the matrix equal to (4.1) which is by assumptions a regular matrix. Since the coefficient at $\mathrm{d}q_{2r-1}^\nu$ in (6.4) is equal to $\partial f_\sigma^1/\partial q_{2r-1}^\nu = (-1)^{r-1} \,.\, \partial^2 L/\partial q_r^\sigma\, \partial q_r^\nu$ (4.2), the Euler–Lagrange distribution Δ_λ is in this case regular, and its dimension is equal to one.

In fact, there are no other lagrangians on j^rY, whose Euler–Lagrange distributions are regular, and of dimension one.

Theorem 4. Let $\lambda \in \Omega_X^1(j^rY)$ be a lagrangian. The following two conditions are equivalent:

(1) The Euler–Lagrange distribution Δ_λ is regular, and its dimension is one.

(2) To each point $j_x^r\gamma \in j^rY$ there exists a fiber chart (V, ψ), $\psi = (t, q^\sigma)$, on Y such that $j_x^r\gamma \in V_r$ and

$$\det\left(\frac{\partial^2 L}{\partial q_r^\sigma\, \partial q_r^\nu}\right) \neq 0 \tag{6.5}$$

at $j_x^r\gamma$, where L is defined by the chart expression $\lambda = L\,\mathrm{d}t$.

Proof. Assume that Δ_λ is regular and its dimension is one. Since Δ_λ is generated by the forms (6.3), all of these forms are linearly independent, that is, the rank of the matrix of the system (6.3) is $(r+1)\,m = \dim j^rY - 1$. Writing out this matrix explicitly, as in (4.1), one obtains that (6.5) must hold at each point, which implies (2).

Conversely, if (2) holds, then the matrix (4.1) is regular, and the rank of the matrix of the system (6.3) is maximal, i.e., equal to $\dim j^rY - 1$. Hence (1) holds.

Obviously, if either of the two equivalent conditions of Theorem 4 is satisfied, then Δ_λ is an integrable distribution, for its dimension is one. It is readily verified that in this case Δ_λ coincides with the distribution, generated by the vector fields ζ_0 (5.11). To check it, one has to find, in the Legendre coordinates, linear diffeential rforms ω on $j^{2r-1}Y$ for which

(6.6) $$i_{\xi_0}\omega = 0,$$

and then to express these forms in the canonical coordinates. It immediately results that the forms ω define the same distribution as the forms (6.4).

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Added in proof: The concept of regularity used in this paper, was discussed by P. Dedecker in C. R. Acad. Sc. Paris, 288 (1979), pp. 827—830, where further references may be found.

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