HAMILTON EXTREMALS 
IN HIGHER ORDER MECHANICS

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1. INTRODUCTION

The geometrical and physical significance of the problem of a generalization of the Hamilton equations of classical mechanics to higher order mechanics and field theory ranks it still in the field of interest of mathematicians and theoretical physicists. One possible approach to this problem consists in assigning to the Euler–Lagrange equations of a given lagrangian a new system of extended equations, having more solutions than the initial ones, in such a way that (1) to each extremal (= a solution of the Euler–Lagrange equations) there corresponds a solution of the extended equations, and (2) in case that the lagrangian satisfies a regularity condition, the correspondence between the extremals and the solutions of the extended equations is bijective (see [2]). The possibility of constructing the extended equations is based on the existence of a Lepagean equivalent of a lagrangian proved independently in [3] and [6]. An important feature of the extended equations is that for regular lagrangians of order one they may be interpreted as the Hamilton equations of the given lagrangian.

The purpose of this paper is to discuss the Hamilton equations of higher order integral variational problems of one independent variable, and to give a geometric interpretation of these equations in terms of a certain distribution. To this purpose we use the theory of Lepagean differential forms developed in [5], [6], and [7], and we closely follow the general ideas of the paper [2] such as the extension idea by means of a Lepagean form, and the regularity condition. For further results in this direction, we refer to [1] and [4].

As the basic underlying structure we choose a fibered manifold over a one-dimensional base manifold. Within this framework, we construct, under a regularity hypothesis, the Hamilton (one-dimensional) distribution, which corresponds to the so called Hamilton vector field from the variational theory of curves, and whose integral sections are precisely the solutions of the extended equations. We also express this distribution in terms of linear differential forms, which allows us to
take into account general (not necessarily regular) lagrangians; the distribution corresponding to a general lagrangian is then shown to be completely integrable if and only if the lagrangian is regular.

2. LEPAGEAN FORMS IN HIGHER ORDER MECHANICS

In this section we recall main definitions and facts of the theory of Lepagean differential forms. Our exposition is adapted to the case of variational problems whose underlying fibered manifolds have one-dimensional bases.

Throughout this paper, $X$ is a one-dimensional manifold, $\pi : Y \to X$ is a fibered manifold, and we denote $m = \dim Y - 1$. $\Gamma(\pi)$ denotes the set of local sections of $\pi$, $\pi_s : j^s Y \to X$ is the $s$-jet prolongation of $\pi$, and $\pi_{s+1} : j^{s+1} Y \to X$, $0 \leq s < r$, is the natural projection of jet spaces. If $(V, \psi)$, $\psi = (t, q^\alpha)$, $1 \leq \sigma \leq m$, is a fiber chart on $Y$, then the fiber chart on $j^s Y$, associated with $(V, \psi)$, is denoted by $(V_s, \psi_s) = (t, q^\alpha_s)$, where $1 \leq \sigma \leq m$, $0 \leq i \leq r$. The $r$-jet prolongation of the fibered manifold $\pi_s : j^s Y \to X$ is denoted by $j^r(\pi_s) : j^r(j^s Y) \to X$. In accordance with the above notation, the chart on $j^r(j^s Y)$, associated with $(V, \psi)$, is denoted by $((V_{s+r}, \psi_{s+r})), (\psi_{s+r}) = (t, q^\alpha_{s+r})$, where $1 \leq \sigma \leq m$, $0 \leq i \leq s, 0 \leq j \leq r$, and $q_{s+1} = q_1$.

The module of $p$-forms (resp. $\pi_s$-horizontal $p$-forms, resp. $\pi_s$-horizontal $p$-forms) on $j^s Y$ is denoted by $\Omega^p(j^s Y)$ (resp. $\Omega^p_\pi(j^s Y)$, resp. $\Omega^p_\pi(j^s Y)$). Let $\eta \in \Omega^p(j^s Y)$ be a form. There exist unique forms $h(\eta) \in \Omega^{s+1}_\pi(j^{s+1} Y)$ and $p(\eta) \in \Omega^{s+1}_\pi(j^{s+1} Y)$ such that for each $\gamma \in \Gamma(\pi)$, $j^s \gamma \ast \eta = j^{s+1} \gamma \ast h(\eta)$, such that $\pi_{s+1}, \eta = h(\eta) + p(\eta)$; the mapping $\eta \to h(\eta)$ is called the $n$-horizontalization. A form $\eta \in \Omega^p(j^s Y)$ is $\pi_s$-horizontal if and only if $\pi_{s+1}, \eta = h(\eta)$; $\eta$ is called contact if $\pi_{s+1}, \eta = p(\eta)$, i.e., $h(\eta) = 0$. For $p \geq 2\eta$ is obviously contact.

The following definitions slightly differ from the ones introduced in [6]. Let $p \geq 1$. A form $\eta \in \Omega^p(j^s Y)$ is called $1$-contact, if (1) it is contact, and (2) for each $\pi_s$-vertical vector field $\xi$ on $j^s Y$, the form $i_\xi \eta$ is $\pi_s$-horizontal (0-contact); if $k \geq 2$, $\eta$ is called $k$-contact, if $i_\xi \eta$ is $(k - 1)$-contact. We denote by $\Omega^{p,k}(j^s Y)$ the module of $k$-contact $p$-forms on $j^s Y$. Each form $\eta \in \Omega^p(j^s Y)$ admits a unique decomposition

$$\eta = \eta_0 + \eta_1 + \ldots + \eta_p,$$

where $\eta_k \in \Omega^{p,k}(j^{s+1} Y)$.

Let $\varphi \in \Omega^1(j^s Y)$ be a form. There exist uniquely defined forms $E \in \Omega^{2,1}(j^{s+1} Y)$, $F \in \Omega^{2,2}(j^{s+1} Y)$ such that

$$\pi_{s+1}, \varphi = E + F.$$

We say that $\varphi$ is a Lepagean form if $E \in \Omega^{2,1}(j^{s+1} Y)$, i.e. if $E$ is $\pi_{s+1,0}$-horizontal.

Let $\varphi \in \Omega^1(j^s Y)$. In a fiber chart $(V, \psi)$, $\psi = (t, q^\alpha)$, $\pi_{s+1,0} \varphi$ has an expression

$$\pi_{s+1,0} \varphi = f_0 dt + \sum_{l=0}^s f_{s+1} \omega_l^\alpha,$$
where

\[(2.4) \quad \omega_i^\sigma = dq_i^\sigma - q_{i+1}^\sigma dt.\]

It can be easily verified that \( \varphi \) is Lepagean if and only if

\[(2.5) \quad f^i_\sigma = \sum_{k=0}^{s+1-i} (-1)^k \frac{d^k}{dt^k} \frac{\partial f_0}{\partial q^\sigma_{i+k}}, \quad 1 \leq i \leq s.\]

This implies, in particular, that if a form \( \varphi \in \Omega^1(j^rY) \) is Lepagean, then the form \( \lambda = h(\varphi) \) is \( \pi_{s+1, k} \)-projectable for some \( k \leq (s + 1)/2 \).

A lagrangian of order \( r \) for \( \pi \) is by definition an element \( \lambda \in \Omega^1_X(j^rY) \). A Lepagean equivalent of a lagrangian \( \lambda \in \Omega^1_X(j^rY) \) is a Lepagean form \( \varphi \in \Omega^1(j^rY) \), where \( s \) is an integer, such that \( h(\varphi) = \lambda \). Writing in a fiber chart \((V, \psi)\), \( \psi = (t, q^\sigma) \),

\[(2.6) \quad \lambda = L dt\]

and using (2.3) and (2.5) we easily obtain that there exists exactly one Lepagean equivalent of \( \lambda \) which we denote by \( \Theta_\lambda \) and call the Poincaré–Cartan equivalent of \( \lambda \); we note that for \( \dim X > 1 \) there exist more Lepagean equivalents than one. Substituting \( f_0 = L \) in (2.5) we obtain the coordinate expression of \( \Theta_\lambda \) for the fiber chart \((V, \psi)\). Obviously, \( f^{i+1}_\sigma = f^{i+1}(t, q^0, q^1, \ldots, q^s_{2r-l-1}) \) which implies that in general, the Poincaré–Cartan equivalent \( \Theta_\lambda \) of \( \lambda \) is defined on \( j^{2r-1}Y \).

Consider the decomposition (2.2). We have

\[(2.7) \quad \pi_{2r, 2r-1}^* d\Theta_\lambda = E_\lambda + F_\lambda,\]

where

\[(2.8) \quad E_\lambda = E_\sigma(L) dq^\sigma \wedge dt, \quad E_\sigma(L) = \sum_{k=0}^{r} (-1)^k \frac{d^k}{dt^k} \frac{\partial L}{\partial q^\sigma_k},\]

\[(2.9) \quad F_\lambda = \sum_{i=0}^{r-1} \sum_{k=0}^{2r-1-i} \frac{\partial f^{i+1}_\sigma}{\partial q^\sigma_k} \omega_k^\sigma \wedge \omega_i^\sigma.\]

3. HAMILTON EXTREMALS

Recall that a section \( \delta \in \Gamma(\pi_{2r-1}) \) is called a Hamilton extremal of a lagrangian \( \lambda \in \Omega^1_X(j^rY) \) if for each \( \pi_{2r-1} \)-vertical vector field \( \xi \) on \( j^{2r-1}Y \)

\[(3.1) \quad \delta^*i_\xi \ d\Theta_\lambda = 0.\]

Let \( h : \Omega^p(j^{2r-1}Y) \to \Omega^q_X(j^r(j^{2r-1}Y)) \) be the \( \pi_{2r-1} \)-horizontalization. Then (3.1) is equivalent to the condition

\[(3.2) \quad j^1\delta^*h(i_\xi \ d\Theta_\lambda) = 0,\]

where \( j^1\delta \) is the 1-jet prolongation of \( \delta \).
Consider a lagrangian $\lambda \in \Omega_{\mathcal{X}}^k(j^*Y)$ and its chart expression (2.6) with respect to a fiber chart $(V, \psi)$, $\psi = (t, q^r)$, on $Y$. In this fiber chart, the Poincaré–Cartan equivalent $\Theta_\lambda$ of $\lambda$ has an expression

$$\Theta_\lambda = L\ dt + \sum_{i=0}^{r-1} f^{i+1}_\sigma \omega_i^\sigma,$$

where

$$f^{i+1}_\sigma = \sum_{k=0}^{2r-i} (-1)^k \frac{d}{dt^k} \frac{\partial L}{\partial q^r_{j+k}}, \quad 1 \leq j \leq 2r - 1.$$

The following is a direct consequence of (3.2).

**Theorem 1.** A section $\delta \in \Gamma(\pi_{2r-1})$ is a Hamilton extremal of the lagrangian $\lambda$ if and only if along $j^1 \delta$

$$\sum_{k=0}^{2r-1-i} \frac{\partial f^{k+1}_\sigma}{\partial q^r_i} (q^r_{k,1} - q^r_{k+1}) = 0, \quad r \leq i \leq 2r - 1,$$

$$\sum_{k=0}^{r-1} \left( \frac{\partial f^{k+1}_\sigma}{\partial q^r_i} - \frac{\partial f^{k+1}_\sigma}{\partial q^r_k} \right) (q^r_{k,1} - q^r_{k+1}) - \sum_{k=r}^{2r-1-i} \frac{\partial f^{i+1}_\sigma}{\partial q^r_k} (q^r_{k,1} - q^r_{k+1}) = 0,$$

$$1 \leq i \leq r - 1,$$

$$E_\sigma(L) + \sum_{k=0}^{r-1} \frac{\partial f^{k+1}_\sigma}{\partial q^r_0} (q^r_{k,1} - q^r_{k+1}) - \sum_{k=0}^{2r-1-i} \frac{\partial f^{i+1}_\sigma}{\partial q^r_k} (q^r_{k,1} - q^r_{k+1}) = 0.$$

**Proof.** There exists one and only one form $H_\lambda \in \Omega^2(j^1(j^{2r-1}Y))$ such that for each $\pi_{2r-1}$-vertical vector field $\xi$ on $2^{2r-1}Y$, $i_{\xi} \delta H_\lambda = \delta(i_\xi \ d\Theta_\lambda)$. Condition (3.2) implies that $\delta$ is a Hamilton extremal of $\lambda$ if and only if $H_\lambda$ vanishes along $j^1 \delta$. But in the fiber chart $(V, \psi)$,

$$H_\lambda = \sum_{i=0}^{2r-1} F^{i+1}_\sigma \ dq^r_i \wedge dt,$$

where the functions $F^{i+1}_\sigma$ are precisely the left-hand side expressions (3.5). This proves Theorem 1.

The form $H_\lambda$ (3.6) is called the Hamilton form of the lagrangian $\lambda$ [8].

4. REGULARITY

A Hamilton extremal $\delta \in \Gamma(\pi_{2r-1})$ of a lagrangian $\lambda \in \Omega_{\mathcal{X}}^k(j^*Y)$ is said to be regular, if $\delta = j^{2r-1} \gamma$ for some extremal $\gamma \in \Gamma(\pi)$ of $\lambda$.

A sufficient condition of regularity of a Hamilton extremal can be deduced from equations (3.5). Notice that the first two of these equations form a system of linear, homogeneous equations for the unknowns $q^r_{k,1} - q^r_{k+1}$, $0 \leq v \leq m$,.
0 \leq k \leq 2r - 2; the number of the equations is \( m(2r - 1) \), and is equal to the number of the unknowns. Let us consider the matrix of this system. Labelling rows (resp. columns) by \( i = 1, 2, \ldots, 2r - 1 \) (resp. \( q'_{0,1} - q'_{1}, \ldots, q'_{2r-2,1} - q'_{2r-1} \)) we obtain for this matrix

\[
\begin{pmatrix}
\frac{\partial f^2}{\partial q_{2r-2}^g} \\
\frac{\partial f^r}{\partial q_{2r-2}^r} \\
\vdots \\
\frac{\partial f^r}{\partial q_r^v} \\
\vdots \\
\frac{\partial f^1}{\partial q_{2r-1}^v} \\
\end{pmatrix}
\]

where by (3.4)

\[
\frac{\partial f^k}{\partial q_{2r-k}^g} = \frac{\partial f^k}{\partial q_{2r-k}^r} = (-1)^{r-k} \frac{\partial^2 L}{\partial q_r^v \partial q_r^v}, \quad 1 \leq k \leq r.
\]

Hence a necessary and sufficient condition that (4.1) be a regular matrix (at a point) is that the matrix \((\delta^2 L/\delta q_r^v \delta q_r^v)\) whose elements are labelled by \( \sigma \) and \( v \), is regular (at this point). Since the regularity of (4.1) is a sufficient condition for the first two equations (3.5) to have the trivial solution only, we have the following result.

**Theorem 2.** Let \( \delta : U \to j^{2r-1}Y \) be a Hamilton extremal of \( \lambda \), defined on an open set \( U \subset X \). Assume that to each point \( x \in U \) there exists a fiber chart \((V, \psi)\), \( \psi = (t, q^v) \), on \( Y \) such that \( \delta(x) \in V_{2r-1} \) and

\[
\det \left( \frac{\partial^2 L}{\partial q_r^v \partial q_r^v} \right) \neq 0
\]

at \( \pi_{2r-1,\delta(x)} \), where the function \( L \) is defined by the chart expression \( \lambda = L \, dt \). Then \( \delta \) is regular.

**Proof.** Consider the Hamilton extremal \( \delta \) of Theorem 2. Let \( x \in U \) be a point, \((V, \psi)\) a fiber chart on \( Y \) for which the assumptions of Theorem 2 are satisfied. Since \( \delta \) satisfies (3.5) and the matrix (4.1) is regular, we have at \( \delta(x) \)

\[
q_{k,1}^v - q_{k+1}^v = 0, \quad 0 \leq k \leq 2r - 2,
\]

that is,

\[
q_{k+1}^v(\delta(x)) = \left( \frac{d}{dt} (q^v \circ \delta) \right)_x = \ldots = \left( \frac{d^{k+1}}{dt^{k+1}} (q^v \circ \delta) \right)_x.
\]
This implies that \( \delta(x) = j_{x}^{2r-1} \gamma \), where \( \gamma = \pi_{2r-1,0} \circ \delta \), and, since the point \( x \) is arbitrary, \( \delta = j^{2r-1} \gamma \). It remains to show that \( \gamma \) is an extremal of \( \lambda \). This follows, however, from the third equation (3.5) which coincides, along \( j^{2r} \gamma \), with the Euler—Lagrange equation of \( \lambda \).

We note that the condition (4.3) is independent of the choice of the fibered chart \( (V, \psi) \); in this paper we call it the regularity condition. If the regularity condition is satisfied at each point of \( j'Y \), then every Hamilton extremal is regular, and the system (3.5) is equivalent with the Euler—Lagrange equation.

4. THE LEGENDRE TRANSFORMATION AND THE HAMILTON EQUATIONS

In this section we assume that we are given a lagrangian \( \lambda \in \Omega_{k}^{1}(j'Y) \) satisfying at each point of \( j'Y \), the regularity condition (4.3).

Let \( (V, \psi) \), \( \psi = (t, q^\rho) \), be a fiber chart on \( Y \), \( \lambda = L \, dt \) the expression of \( \lambda \) for the fiber chart \( (V_r, \psi_r) \), \( \psi_r = (t, q_k^r), \) \( 0 \leq k \leq r \). Put

\[
p^r_{-1} = f'_r(t, q^0 \rho, \ldots, q^r),
\]

where by (3.4), \( f'_r = \partial L / \partial q^r \). Obviously, the regularity condition (4.3) guarantees that \( (V_r, \Psi_r) \), \( \Psi_r = (t, q_j^r, p_{-1}^r) \), \( 1 \leq j \leq r - 1 \), is a chart on \( j'Y \). Consider the fiber chart \( (V_{2r-1}, \psi_{2r-1}) \), \( \psi_{2r-1} = (t, q_k^{2r-1}), 0 \leq k \leq 2r - 1 \), on \( j^{2r-1} Y \), associated with \( (V, \psi) \), and put

\[
p^r_{-j-1} = f^r_{-j}(t, q^0 \rho, \ldots, q^r), \quad 1 \leq j \leq r - 1.
\]

As before, the regularity condition guarantees that \( (V_{2r-1}, \Psi_{2r-1}) \), \( \Psi_{2r-1} = (t, q_j^r, p_k^r), 1 \leq v \leq m, 0 \leq j, k \leq r - 1 \), is a fiber chart on \( j^{2r-1} Y \) (over \( j'Y \)). We call the coordinates \( (t, q_j^r, p_k^r) \) the Legendre coordinates on \( V_{2r-1} \), and the transformation \( \Psi_{2r-1} \psi_{2r-1}^{-1} \) of coordinates is called the Legendre (coordinate) transformation.

Let us consider the condition for Hamilton extremals (3.2) in the Legendre coordinates. In these coordinates,

\[
\Theta_\lambda = -H \, dt + \sum_{i=0}^{r-1} p^i \, dq^i,
\]

where

\[
-H = L + \sum_{i=0}^{r-1} p^i q^i_{i+1},
\]

and \( H \) is considered as a function of the Legendre coordinates. Computing now the exterior derivative \( d\Theta_\lambda \), the form \( i_\xi d\Theta_\lambda \), where \( \xi \) is any \( \pi_{2r-1} \)-vertical vector field
on $j^{2r-1}Y$, expressed in the Legendre coordinates, and the Hamilton form $H_\nu$ (see the proof of Theorem 1) one obtains

$$H_\lambda = \left[ \sum_{i=0}^{r-1} \left( -\frac{\partial H}{\partial q_i^\sigma} - p_i^{i,1} \right) dq_i^\sigma + \left( -\frac{\partial H}{\partial p_i^\sigma} + q_i^{i,1} \right) dp_i^\sigma \right] \wedge dt,$$

where the functions $p_i^{i,1}$, $q_i^{i,1}$ are the coordinates of the coordinate system $((V_{2r-1})_1, (\Psi_{2r-1})_1, (\Psi_{2r-1})_1 = (t, q^\sigma, p^\sigma, q_i^{i,1}, p_i^{i,1})$, on $j^1(j^{2r-1}Y)$. Consequently, a section $\delta$ of $j^{2r-1}Y$ is a Hamilton extremal if and only if it satisfies the Hamilton equations

$$\frac{\partial H}{\partial q_i^\sigma} + \frac{d}{dt} (p_i^l \circ \delta) = 0, \quad \frac{\partial H}{\partial p_i^\sigma} - \frac{d}{dt} (q_i^\sigma \circ \delta) = 0.$$

To interpret geometrically these equations in a canonical manner, as in the first order variational theory, we introduce the following definition. We shall say that a vector field $\zeta$ on $j^{2r-1}Y$ is a Hamiltonian vector field, if

$$i_\zeta d\Theta_\lambda = 0.$$

Let us express this condition in the Legendre coordinates. Let $\zeta$ be a vector field on $j^{2r-1}Y$,

$$\zeta = \zeta^0 \frac{\partial}{\partial t} + \sum_{i=0}^{r-1} \left( \zeta_i^\sigma \frac{\partial}{\partial q_i^\sigma} + \overline{\zeta_i^l} \frac{\partial}{\partial p_i^l} \right).$$

By (5.3),

$$i_\zeta d\Theta_\lambda = \left( -i_\zeta dH + \frac{\partial H}{\partial t} \zeta^0 \right) dt +$$

$$+ \sum_{i=0}^{r-1} \left[ \left( \frac{\partial H}{\partial q_i^\sigma} \zeta^0 + \overline{\zeta_i^l} \right) dq_i^\sigma + \left( \frac{\partial H}{\partial p_i^l} \zeta^0 - \zeta_i^l \right) dp_i^l \right].$$

Hence $\zeta$ is a Hamiltonian vector field if and only if

$$\zeta_i^l = \frac{\partial H}{\partial p_i^l} \zeta^0, \quad \overline{\zeta_i^l} = -\frac{\partial H}{\partial q_i^\sigma} \zeta^0, \quad 0 \leq i \leq r - 1,$$

or, which is the same, $\zeta = \zeta^0 \cdot \zeta_0$, where

$$\zeta_0 = \frac{\partial}{\partial t} + \sum_{i=0}^{r-1} \left( \frac{\partial H}{\partial p_i^l} \frac{\partial}{\partial q_i^\sigma} - \frac{\partial H}{\partial q_i^\sigma} \frac{\partial}{\partial p_i^l} \right).$$

Let $\Lambda_\lambda$ denote the one-dimensional (regular) distribution on $j^{2r-1}Y$, locally generated by the vector fields $\zeta_0$ (5.11). We have the following result.

**Theorem 3.** Let $\delta \in \Gamma(\pi_{2r-1})$. The following three conditions are equivalent:

(1) $\delta$ is a Hamilton extremal of $\lambda$. 

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(2) For each fiber chart \((V, \psi), \psi = (t, q^*)\), \(\delta\) satisfies the Hamilton equations (5.6).

(3) \(\delta\) is an integral submanifold of the distribution \(A_\lambda\).

Proof. Since \(\delta\) is a Hamilton extremal of \(A_\lambda\) if and only if the Hamilton form \(H_\lambda\) vanishes along the first jet prolongation \(J^1 \delta\) of \(\delta\), conditions (1) and (2) are obviously equivalent.

Assume that (2) holds. Then, in fiber coordinates, for each point \(x \in X\) and each tangent vector \(\xi \in T_x X\), \(\xi = \xi^0 (d/dt)\_x\)

\[
T_x \delta \cdot \xi = \xi^0 \left[ \frac{\partial}{\partial t} + \sum_{i=0}^{r-1} \left( \frac{d(q^*_i \circ \delta)}{dt} \frac{\partial}{\partial q^*_i} + \frac{d(p^*_i \circ \delta)}{dt} \frac{\partial}{\partial p^*_i} \right) \right].
\]

Since \(\delta\) satisfies the Hamilton equations (5.6), the vector \(T_x \delta \cdot \xi \in T_{\delta(x)} j^{2r-1} Y\) obviously belongs to the subspace \(A_\lambda(\delta(x)) = T_{\delta(x)} j^{2r-1} Y\). The converse is obtained by reversing the arguments.

6. THE EULER-LAGRANGE DISTRIBUTION

Let \(\lambda \in \Omega^1_\lambda (j^r Y)\) be any lagrangian. By definition, a section \(\delta \in \Gamma (\pi_{2r-1})\) is a Hamilton extremal of \(\lambda\) if and only if it is an integral submanifold of the ideal of forms on \(j^{2r-1} Y\), generated by the linear differential forms \(i_\xi d\Theta_\lambda\) (3.1), where \(\xi\) runs over all \(\pi_{2r-1}\)-vertical vector fields on \(j^{2r-1} Y\). We shall now study the problem as to under what conditions these linear differential forms define a distribution on \(j^{2r-1} Y\).

Let \((V, \psi), \psi = (t, q^*)\), be a fiber chart on \(Y\), and consider the Poincaré–Cartan equivalent \(\Theta_\lambda\) in this chart (3.3). Let \(\xi\) be a \(\pi_{2r-1}\)-vertical vector field on \(j^{2r-1} Y\)

\[
(6.1) \quad \xi = \sum_{k=0}^{2r-1} \xi^k \frac{\partial}{\partial q^*_k}. 
\]

By a direct computation

\[
(6.2) \quad i_\xi d\Theta_\lambda = \sum_{k=0}^{2r-1} \xi^k \eta^0_k, 
\]

where

\[
(6.3) \quad \eta^0 = \left( \frac{\partial L}{\partial q^*_0} - \frac{\partial f^1_0}{\partial t} \right) dt + \sum_{k=0}^{r-1} \frac{\partial f^{k+1}_0}{\partial q^*_k} \omega^v_k - \sum_{k=0}^{2r-1} \frac{\partial f^1_0}{\partial q^*_k} dq^*_k, 
\]

\[
\eta^i = \sum_{k=0}^{r-1} \left( \frac{\partial f^{i+1}_k}{\partial q^*_i} - \frac{\partial f^{i+1}_k}{\partial q^*_k} \right) \omega^v_k - \sum_{k=r}^{2r-1} \frac{\partial f^i_k}{\partial q^*_k} \omega^v_k, \quad 1 \leq i \leq r - 1, 
\]

\[
\eta^r = \sum_{k=0}^{2r-1-i} \frac{\partial f^{i+1}_k}{\partial q^*_i} \omega^v_k, \quad r \leq i \leq 2r - 1. 
\]
Since the expression (6.2) is independent of the choice of fiber coordinates and the transformation formulas for the components of the vector field $\xi$ are linear, the transformation formulas for the forms (6.3) are also linear. Consequently, these forms define, in a well-known sense, a distribution (i.e., a vector subbundle of the tangent bundle $Tj^{2r-1}Y$); the distribution is singular in the sense that the dimension of the vector subspace may vary from point to point. We call this distribution the Euler-Lagrange distribution, associated with the lagrangian $\lambda$, and denote it by $A_\lambda$.

If the assumptions of Theorem 2 are satisfied, then the Euler-Lagrange distribution is spanned by the forms

$$\left(\frac{\partial L}{\partial q_0^\sigma} - \frac{\partial f_1^1}{\partial t}\right) dt - \sum_{k=0}^{2r-1} \frac{\partial f_1^1}{\partial q_k^v} dq_k^v, \omega_i^\sigma, \quad 0 \leq i \leq 2r - 2.$$  

This follows from the fact that the second and the third equations (6.3) have the matrix equal to (4.1) which is by assumptions a regular matrix. Since the coefficient at $dq_{2r-1}^v$ in (6.4) is equal to $\partial f_1^1/\partial q_{2r-1}^v = (-1)^{-1} \cdot \partial^2 L/\partial q_{2r-1}^v \partial q_0^v$ (4.2), the Euler-Lagrange distribution $A_\lambda$ is in this case regular, and its dimension is equal to one.

In fact, there are no other lagrangians on $j^rY$, whose Euler-Lagrange distributions are regular, and of dimension one.

**Theorem 4.** Let $\lambda \in \Omega^1_X(j^rY)$ be a lagrangian. The following two conditions are equivalent:

1. The Euler-Lagrange distribution $A_\lambda$ is regular, and its dimension is one.
2. To each point $j^r_\gamma \epsilon j^rY$ there exists a fiber chart $(V, \psi)$, $\psi = (t, q^\sigma)$, on $Y$ such that $j^r_\gamma \epsilon V$, and

$$\det\left(\frac{\partial^2 L}{\partial q_0^\sigma \partial q_r^\sigma}\right) = 0$$

at $j^r_\gamma$, where $L$ is defined by the chart expression $\lambda = L \ dt$.

**Proof.** Assume that $A_\lambda$ is regular and its dimension is one. Since $A_\lambda$ is generated by the forms (6.3), all of these forms are linearly independent, that is, the rank of the matrix of the system (6.3) is $(r + 1) m = \dim j^rY - 1$. Writing out this matrix explicitly, as in (4.1), one obtains that (6.5) must hold at each point, which implies (2).

Conversely, if (2) holds, then the matrix (4.1) is regular, and the rank of the matrix of the system (6.3) is maximal, i.e., equal to $\dim j^rY - 1$. Hence (1) holds.

Obviously, if either of the two equivalent conditions of Theorem 4 is satisfied, then $A_\lambda$ is an integrable distribution, for its dimension is one. It is readily verified that in this case $A_\lambda$ coincides with the distribution, generated by the vector fields $\xi_0$ (5.11). To check it, one has to find, in the Legendre coordinates, linear
differential forms \( \omega \) on \( j^{2r-1}Y \) for which

\[
(6.6) \quad i_\xi \omega = 0,
\]

and then to express these forms in the canonical coordinates. It immediately results that the forms \( \omega \) define the same distribution as the forms (6.4).

REFERENCES


Added in proof: The concept of regularity used in this paper, was discussed by P. Dedecker in C. R. Acad. Sc. Paris, 288 (1979), pp. 827—830, where further references may be found.

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