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THE GROUP OF DIVISIBILITY OF \( \mathbb{Z} \)

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1. In [2] we investigated a notion of a topological group of divisibility of a GCD-domain in the following way: Let \((K, T)\) be a topological field and let \(A\) be a GCD-domain in \(K\) with \(K\) as a quotient field. Suppose that the group \(U(A)\) of units of \(A\) is closed in a multiplicative group \((K^*, \cdot, T|K^*)\). Then the factor topological group \(G(A) = K^*/U(A)\) is called a topological group of divisibility of \(A\), in symbol \(G(A) = (K, T, A)\), if \(G(A)\) is a topological lattice. More generally, for a topological lattice-ordered group \(G\) we set \(G = (K, T, A)\) if \(K\) is a topological field with a topology \(T\), \(A\) is a Bezout domain with the quotient field \(K\), the group of units \(U(A)\) of \(A\) is closed in \(K^*\), and the topological factor group \(K^*/U(A)\) is a tl-group which is tl-isomorphic to \(G\). In this case we say that \(G\) has a representation. Let us recall that a tl-group is a triple \((G, \leq, F)\) where \(G\) is a group, \(\leq\) is a partial order, and \(F\) is a topology on the underlying set \((G)\) of \(G\) such that \((G, \leq)\) is an l-group, \((G, F)\) is a topological group, and \((|G|, \leq, F)\) is a topological lattice. Moreover, we say that two tl-groups are tl-isomorphic if there is a homeomorphism between them which is both a lattice and group isomorphism.

In [2] we have observed that there are tl-groups which have no representation. On the other hand, it is possible to construct examples of tl-groups with a representation. The example of a tl-group \((G, F)\) we consider here is not a complete space and hence we may construct the completion \((\hat{G}, \hat{F})\) of \((G, F)\). It is well known that \((\hat{G}, \hat{F})\) is a tl-group and the natural question arising here is whether \((\hat{G}, \hat{F})\) admits a representation. To tell the truth we cannot solve this question as stated here. On the other hand if we somewhat modify the notion of a representation we are able to answer affirmatively this question. To do it, we say that a tl-group \((G, F)\) admits a general representation \((K, T, A)\) (in symbol \((G, F) \sim = (K, T, A)\)), if \(K\) is a ring (commutative) with possible zero divisors, \(A\) is a subring in \(K\) such that \(K\) is a total quotient ring of \(A\), \(T\) is a ring topology on \(K\) such that \((U(K), T|U(K))\) is a topological group with \(U(A)\) as a closed subgroup and the factor topological group \(G(A) = U(K)/U(A)\) is a tl-group (with ordering defined by \((U(K)/U(A))_+ = A^*/U(A)\), where \(A^*\) is the set of regular elements
of $A$, therefore, a po-group $G(A)$ is a value group of $A$ in the sense of [4]) which is tl-isomorphic to $G$. In a sequel we use a method of non-standard analysis introduced by A. ROBINSON [5] and, especially, we employ a variant of nonstandard analysis introduced by E. ZAKON [7] since it requires only rudiments of first order logic.

2. The groups of divisibility we are dealing with are of the form $\mathbb{Z}^{(I)}$, where $I$ is a subset of the set $N$ of integers. Clearly, every such a group is a group of divisibility of a domain $A_I = \bigcap_{i \in I} R_{w_i} \subset Q$, where $Q$ is the field of rationals and $w_i$ is the $p_i$-adic valuation on $Q$. Let $F$ be the topology on $\mathbb{Z}^{(I)}$ with a subbase of neighbourhoods of zero consisting of prime $I$-ideals

$$H_i = \{x \in \mathbb{Z}^{(I)} : \alpha_i = 0\}, \quad i \in I.$$ 

Then clearly $(\mathbb{Z}^{(I)}, F)$ is a tl-group (see [6]) and if $\text{card } I = \aleph_0$, then $F$ is a non-discrete topology. If we denote by $T_{w_i}$ the field topology on $Q$ defined by $w_i$ with a subbase of neighbourhoods of zero consisting of the sets $U_{w_i, a} = \{x \in Q : w_i(x) > a\}, a \in \mathbb{N}$, we obtain the following proposition.

**Proposition 1.** $(\mathbb{Z}^{(I)}, F) = (Q, \text{sup } \{T_{w_i} : i \in I\}, A_I)$.

**Proof.** At first we observe that $U(A)$ is closed in $Q$, since $U(R_{w_i})$ is closed for every $i \in I$. Let

$$\varphi : G(A_I) = Q^*/U(A_I) \to \mathbb{Z}^{(I)}$$

be defined such that $\varphi(w(x)) (i) = \varphi(x U(A_I)) (i) = w_i(x), \ i \in I$. Clearly, $\varphi$ is an $o$-isomorphism. By [2], Lemma 1, to prove the proposition it remains to show that $\varphi$ is open and continuous. We have

$$\varphi^{-1}(H_i) = w(U(R_{w_i})) = U(R_{w_i})/U(A_I),$$

and it is an open neighbourhood of zero in $G(A_I)$ since $U(R_{w_i}) = w_i^{-1}(0)$ is open in $(Q, T_I)$ for $T_I = \sup \{T_{w_i} : i \in I\}$. On the other hand

$$\varphi(U_{w_i, a}/U(A_I)) = \{x \in \mathbb{Z}^{(I)} : \alpha_i > a\} \quad (= B)$$

as follows using the approximation theorem for valuations in $Q$. Since for every $\alpha \in B$ we have $\alpha + H_i \subset B$, $B$ is open in $F$ and, therefore, $\varphi$ is a homeomorphism.

Now, let $(\hat{Q}, \hat{T})$ be the completion of $(Q, T_I)$ and let $\hat{A}_I$ be the closure of $A_I$ in $\hat{Q}_I$. It is well known that $\hat{Q}_I$ has zero divisors, so that $(\hat{Q}_I, \hat{T}_I, \hat{A}_I)$ cannot be a representation of any tl-group. On the other hand, it may be a general representation and, in fact, we shall prove the following main result for $G = \mathbb{Z}^{(I)}$.

**Theorem 2.** $(G, \hat{F}) \sim (\hat{Q}_I, \hat{T}_I, \hat{A}_I)$.

The proof of this theorem will be a consequence of several independent propositions which describe structures of $\hat{Q}_I$ and $G$, respectively. As we have mentioned above, for an investigation of algebraic properties of $\hat{Q}_I$ and $G$ we use a method
which is based on a notion of an enlargement from the tools of nonstandard analysis. Included solely for the convenience of the reader, we introduced the basic facts about enlargements.

For any set \( A = A_0 \) of individuals, the superstructure on \( A \) is the set \( \mathcal{A} = \bigcup_{n \in \mathbb{N}} A_n \), where \( A_{n+1} \) is the set of all subsets of \( A_0 \cup A_n \). The first order language \( \mathcal{L} \) we need is a simple modification of a classical one, namely, we assume that all constants of \( \mathcal{L} \) are in 1–1 correspondence with elements of \( \mathcal{A} \) and identify the constants with the corresponding elements. Well-formed formulae (WFF) and sentences (WFS) are defined as usual with the restriction that all quantifiers must have form \( (\forall x \in C) \) or \( (\exists x \in C) \) with \( C \) a constant (i.e. \( C \in \mathcal{A} \)). Now, let \( A, B \) be two sets with superstructures \( \mathcal{A}, \mathcal{B} \), respectively, and let

\[ * : \mathcal{A} \rightarrow \mathcal{B} \]

be a map of \( \mathcal{A} \) into \( \mathcal{B} \). We write \( *C \) for \( *(C) \). Let \( *A = \bigcup_{n \in \mathbb{N}} A_n \) (since \( A_n \in \mathcal{A} \)).

Given a WFF \( \alpha \), we denote by \( *\alpha \) the formulae obtained from \( \alpha \) by replacing each constant \( C \in \mathcal{A} \) by \( *C \). Elements of the form \( *C \) \( (C \in \mathcal{A}) \) are called standard, their elements are called internal. A 1–1 map \( * : \mathcal{A} \rightarrow \mathcal{B} \) is then called a strict monomorphism if

1. \( *\emptyset = \emptyset \),
2. for every \( y \in \mathcal{A} \), \( y \subseteq *\mathcal{A} \) holds,
3. for every WFS \( \alpha, \mathcal{A} \models \alpha \) iff \( \mathcal{B} \models *\alpha \). A binary relation \( R \) in \( \mathcal{A} \) is said to be concurrent if, for any finite number of elements \( a_1, ..., a_m \in D_1(R) = \{ x : (\exists y) (x, y) \in R \} \), there exists \( b \) such that \( (a_k, b) \in R \) for \( k = 1, ..., m \). Then a strict monomorphism \( * : \mathcal{A} \rightarrow \mathcal{B} \) is called enlarging and \( *\mathcal{A} \) an enlargement of \( \mathcal{A} \), if, for each concurrent relation \( R \) in \( \mathcal{A} \) there is some \( b \in *\mathcal{A} \) such that \( (*a, b) \in *R \) for all \( a \in D_1(R) \), simultaneously.

If \( *\mathcal{A} \) is an enlargement of \( \mathcal{A} \), where \( \mathcal{A} \) is a superstructure on \( A \), we say frequently that \( *A \) \( (\in *\mathcal{A}) \) is an enlargement of \( A \) \( (\in \mathcal{A}) \). For any \( X \subseteq A \) we may consider \( X \) as a subset of \( *X \) and, furthermore, for any binary relation \( R \subseteq X \times Y \), \( X, Y \subseteq A \), we have \( R \subseteq *R \).

Now, let \( K \) be a field with a topology \( T = \sup (T_w : w \in \Omega) \) and let \( \mathcal{K} \) be the superstructure on \( K_0 = K \cup \Omega \cup \bigcup_{w \in \Omega} G_w \), \( \mathcal{K} = \bigcup_{n \in \mathbb{N}} K_n \), and let \( *\mathcal{K} \) be an enlargement of \( \mathcal{K} \). Using the property (3), it can be proved that \( *K \) \( (\in *\mathcal{K}) \) is a field, \( K \subseteq *K \) is a subfield and \( *w \) \( (w \in \Omega \in \mathcal{K}) \) is a valuation on \( *K \) with a value group \( *G_w \) such that the diagram

\[
\begin{array}{ccc}
*K & \xrightarrow{*w} & *G_w \cup \{\infty\} \\
\downarrow & & \downarrow \\
K & \xrightarrow{w} & G_w \cup \{\infty\}
\end{array}
\]
comutates. Let $cG_w$ be the convex closure of $G_w$ in $*G_w$ and let $w$ be a valuation on $*K$ completing the diagram

$$
\begin{array}{ccc}
*K & \xrightarrow{w} & *G_w \cup \{\infty\} \\
& \downarrow{w} & \searrow{\text{nat}} \\
G_w/cG_w & & 
\end{array}
$$

Let $M_w$ be the maximal ideal of $R_w$ and let

$$M = \bigcap_{w \in \Omega} M_w.$$

Then $M$ is a subgroup of $(*K, +)$ and on the factor group $*K/M$ we may define a topology such that a subbase $B$ of neighbourhoods of $0$ consists of the sets

$$U_{w,\alpha}/M = \{x + M \in *K/M : *w(x) > \alpha\},$$

where $w \in \Omega$, $\alpha \in G_w^+$. Clearly, $*K/M$ is then a topological group.

Now, under the injection $x \mapsto x + M$, $x \in K$, we may identify $K$ with a subgroup in $*K/M$. Let $K$ be the closure of $K$ in $*K/M$. In [3] we have proved that $K$ is homeomorphic with the completion $\hat{K}$ of $K$.

Let $\mathcal{K}$ be now the superstructure on the field $K = \mathbb{Q}$. Let $P_I$ ($\Omega_I$) be the set of all $i$-th prime numbers $p_i$ (prime valuations on $\mathbb{Q}$) for $i \in I$ and let $\mu$ be a WFS such that

$$\mu = (\forall p \in N) (\forall x \in N) (\forall y \in N) (p \in P_I \Rightarrow (p \neq 1 \land (p = x \land y = x = 1) \lor y = 1)).$$

Then $\mathcal{K} \models \mu$ states that $P_I$ is a set of prime numbers in $N$. Since $*\mathcal{K} \models *\mu$, the set $*P_I$ is a set of "prime numbers," in $*N$. Analogously, let $\gamma$ be a WFS which states that for every $p \in P_I$ there exists a $p$-adic valuation $w_p$ in $\mathbb{Q}$ with a value group $\mathbb{Z}$. Since $\mathcal{K} \models \gamma$, we have $*\mathcal{K} \models *\gamma$ and it follows that for every $p \in *P_I$ there exists a "$p$-adic" valuation $w_p$ in a field $*\mathbb{Q}$ with a value group $*\mathbb{Z}$. Clearly, for $p \in P_I \subset *P_I$ we have $w_p = *w_p$.

The following proposition describes fully the set $U(\hat{Q}_I)$. The elements of $\hat{Q}_I$ we denote by $x (= x + M)$, where $x \in *\mathbb{Q}$.

**Proposition 3.** Let $x \in *\mathbb{Q}$. Then $x \in U(\hat{Q}_I)$ if and only if $*w_i(x) \in \mathbb{Z}$ for each $i \in I$.

**Proof.** Let $x \in U(\hat{Q}_I)$. If there exists $i \in I$ such that $*w_i(x) = \omega$ for some $\omega \in *N - N$, then for $y \in \hat{Q}_I$ such that $x \cdot y = 1$ we have $*w_i(x \cdot y - 1) \in *N - N$ and it follows $*w_i(y) = -\omega$. Then for any $z \in \mathbb{Q}$ we have

$$-\omega = *w_i(y - z) < n, \quad \forall n \in N,$$

and $y \notin \hat{Q}_I$, a contradiction.

Conversely, without loss of generality we may suppose that $*w_i(x) = -a_i$ for each $i \in I$ and $a_i \in N$. Then according to [3], Prop. 3.5, $\tilde{w}_i(z) = *w_i(z) \geq 0,$
$i \in I$, where $z = x^{-1}$ and $\tilde{w}_i$ is the continuous extension of a valuation $w_i$ onto a (Manis) valuation in a ring $\hat{Q}_I$. By [1], Prop. 6 and Lemma 11, we obtain

\[ \hat{A}_I = \bigcap_{i \in I} R_{w_i}, \text{ and it follows } z \in \hat{A}_I \subseteq \hat{Q}_I. \]

Hence, for every pair $(i, a) \in I \times \mathbb{N}$ there exists $y_{i,a} \in A_I$ such that

\[ *w_i(z - y_{i,a}) > a + 2a_i. \]

Since $*w_i(z) = a_i < a + 2a_i$, we have $y_{i,a} \neq 0$ and $y_{i,a}^{-1} \in Q$,

\[ a_i = *w_i(z) = w_i(y_{i,a}). \]

Then we obtain

\[ *w_i(x - y_{i,a}^{-1}) = *w_i(x(y_{i,a} - z)y_{i,a}^{-1}) > -a_i + a + 2a_i - a_i = a. \]

Therefore, we have proved

\[ \forall (i, a) \in I \times \mathbb{N} \exists z_{i,a} \in Q \text{ such that } *w_i(x - z_{i,a}) > a. \]

Now, let $i_1, \ldots, i_m \in I$, $a_1, \ldots, a_m \in \mathbb{N}$. Using the approximation theorem for valuations in $Q$ we may find an element $y \in Q$ such that

\[ w_{i_1}(y - z_{i_1,a_1}) > a_1, \quad t = 1, \ldots, m. \]

Hence,

\[ *w_{i_t}(x - z) = *w_{i_t}(x - z_{i_t,a_t} + z_{i_t,a_t} - y) > a_t, \quad 1 \leq t \leq m, \]

and it follows $x \in \hat{Q}_I$. Clearly, $x \cdot z = 1$ in $\hat{Q}_I$ and $x \in U(\hat{Q}_I)$.

To show that $(\hat{Q}_I, \hat{T}_I, \hat{A}_I)$ is a general representation we have to prove that $U(\hat{Q}_I)$ with induced topology is a topological group (and not only a topological semigroup).

**Proposition 4.** $(U(\hat{Q}_I), \cdot, \hat{T}_I \mid U(\hat{Q}_I))$ is a topological group.

**Proof.** We show that a map $x \mapsto x^{-1}$ is continuous. In fact, let $U = (1 + \bigcap_{s=1}^n U_{w_{i_s,a_s}}) \cap U(\hat{Q}_I)$ be an arbitrary neighbourhood of 1. Since $(Q, T_I)$ is a topological field, there exists a neighbourhood $V = (1 + \bigcap_{s=1}^m U_{w_{j_s,b_s}}) \cap Q^*$ of 1 in $Q$ such that

\[ V^{-1} \subseteq 1 + \bigcap_{i=1}^n U_{w_{i_t,a_t}} = U. \]

Let $z \in V = (1 + \bigcap_{s=1}^m U_{w_{j_s,b_s}}) \cap U(\hat{Q}_I)$. Then by Prop. 3, $*w_i(z) \in Z$ for every $i \in I$ and $*w_i(z - 1) > b_s$, $s = 1, \ldots, m$. Without loss of generality we may assume that $\{w_{j_1}, \ldots, w_{j_m}\} \cap \{w_{i_1}, \ldots, w_{i_m}\} = \emptyset$. Since $z \in \hat{Q}_I$, there exists $x \in Q$ such that

\[ *w_{j_s}(z - x) > b_s, \quad s = 1, \ldots, m, \]

\[ *w_{i_t}(z - x) > \max (a_t + 2w_{i_t}(z), w_{i_t}(z)), \quad t = 1, \ldots, n. \]
Since \( w_j(x - 1) = \ast w_j(x - z + z - 1) > b \), we have \( x \in V \) and \( x^{-1} \in U \). Then it is easy to see that
\[
\ast w_t(z^{-1} - 1) = \ast w_t(z^{-1} - x^{-1} + x^{-1} - 1) > a_t, \quad 1 \leq t \leq n,
\]
and \( V^{-1} \subseteq U \).

Now, the same method of enlargement we may use for investigation of properties of the completion \( \hat{G} \) of \((Z^{1}), F)\). As in a case of topological fields we may do it in a more general way.

To do it, let \( G \) be a tl-group with a subbase \( \mathcal{H} \) of zero consisting of prime \( l \)-ideals, \( \mathcal{H} = \{H_i : i \in J\} \). Let \( \mathcal{G} \) be a superstructure on the set \( G_0 = G \cup J \) and let \( \ast \mathcal{G} \) be an enlargement of \( \mathcal{G} \). Let
\[
H = \bigcap_{j \in J} \ast H_j.
\]
Then \( H \) is an o-ideal of \( \ast G \) and in a group \( \ast G \) we may define a topology in such a way that \( \{\ast H_j : j \in J\} \) is a subbase of neighbourhoods of zero. Clearly, \( \ast G \) is a tl-group. Since \( H \) is closed \( l \)-ideal of \( \ast G \), we may consider a factor tl-group \( \ast G/H \).

Then the canonical map \( G \to \ast G/H \) is an injection as follows from the fact \( \ast H_j \cap G = H_j, j \in J \). Then the following proposition holds.

**Proposition 5.** The closure \( cG \) of \( G \) in \( \ast G/H \) is tl-isomorphic with the completion \( \hat{G} \) of \( G \).

**Proof.** At first, \( \hat{G} \) may be considered to be the factor set of the set of all Cauchy filters in \( G \). Elements of this factor set will be denoted by \( \bar{\alpha} \), their elements (i.e. Cauchy filters) by \( \alpha, \beta \), etc. Then \( \bar{\alpha}, \bar{\beta} \in \bar{\gamma} \) iff \( \alpha \cap \beta \) is a Cauchy filter in \( G \). The base of neighbourhoods of 0 in \( \hat{G} \) consists of the sets
\[
[\bigcap_{i=1}^{n} H_i] = \{\bar{\alpha} ; \bigcap_{i=1}^{n} H_i \in \alpha\}, \quad \bar{H}_i \in \mathcal{H}.
\]
The operations in \( \hat{G} \) are defined as follows:
\[
\bar{\alpha} + \bar{\beta} = \bar{\gamma} \quad \text{iff} \quad \gamma \text{ is a filter with a base } \alpha + \beta, \\
\bar{\alpha} \land \bar{\beta} = \bar{\gamma} \quad \text{iff} \quad \gamma \text{ is a filter with a base } \alpha \land \beta.
\]

Let \( \bar{\alpha} \in \hat{G} \). We define a binary relation \( R \) in \( \mathcal{G} \) as follows:
\[
(X, Y) \in R \quad \text{iff} \quad X, Y \in \bar{\alpha}, \ X \subseteq Y.
\]
Then \( R \) is a concurrent relation and there exists \( X \in \ast \alpha \) such that \( X \subseteq \ast Y \) for all \( Y \in \bar{\alpha} \). An element \( X \) with this property will be called an infintesimal element of \( \ast \alpha \). Let \( \alpha \in X \). Then we define a map \( \varphi : \hat{G} \to cG \),
\[
\varphi(\bar{\alpha}) = \alpha + H.
\]
This definition is correct. In fact, let \( \beta \in X \) and let \( i \in J \). Since \( \alpha \) is a Cauchy filter,
there exists $Y \in \mathfrak{a}$ such that $Y - Y \subseteq H_i$. Then $X \subseteq {}^*Y$ and $\beta - \alpha \in X - X \subseteq {}^*Y - {}^*Y \subseteq {}^*H_i$. Thus, $\alpha + H = \beta + H$. Let $Z$ be any other infinitesimal element of $\mathfrak{a}$ and let $\gamma \in Z$. Then $Z \cap X \in {}^*\mathfrak{a}$ is infinitesimal and for $\omega \in Z \cap X$ we have $\omega - \alpha, \omega - \gamma \in H$, hence, $\alpha - \gamma \in H$. Finally, let $\beta \in \mathfrak{a}$ and let $T$ be infinitesimal in ${}^*\beta$, $\beta \in T$. Since $\alpha \cap \beta$ is a Cauchy filter, for any $i \in J$ there exists $Y \in \mathfrak{a} \cap \beta$ such that $Y - Y \subseteq H_i$, and $\alpha - \beta \in X - T \subseteq {}^*Y - {}^*Y \subseteq {}^*H_i$. Thus, $\alpha + H = \beta + H$.

Further, $\varphi(\bar{a}) \in cG$. In fact, let $\varphi(\bar{a}) = \alpha + H$ where $\alpha$ is an element of an infinitesimal element $X$ of $\mathfrak{a}$. Then $\{\beta + H : \beta \in Y : Y \in \mathfrak{a}\}$ is a base (in $G$) of a filter $F$ in ${}^*G/H$ and it is easy to see that $\lim F = \alpha + H$. It follows $\alpha + H \in cG$.

$\varphi$ is injective. Indeed, let $\alpha + H = \varphi(\bar{a}) = p(\bar{b}) = \beta + H$, where $\alpha(\beta)$ is an element of an infinitesimal element $X(\beta)$ of $\mathfrak{a}$ ($\mathfrak{b}$). Then there exist $A \in \mathfrak{a}$ and $B \in \mathfrak{b}$ such that $A - A \subseteq \bigcap H_i$, $B - B \subseteq \bigcap H_i$, $A \cup B \in \mathfrak{a} \cup \mathfrak{b}$. Then it is easy to see that $A \cup B - A \cup B \subseteq \bigcap H_i$ and it follows $\bar{a} = \bar{b}$.

Analogously it may be proved that $\varphi$ is surjective and if we consider $cG$ to be a subgroup of a factor group $*G/H$, then from the fact

$$(\alpha - \beta) + H = \varphi(\varphi^{-1}(\alpha + H) - \varphi^{-1}(\beta + H)) \in cG$$

for $\alpha + H, \beta + H \in cG$ it follows that $\varphi$ is a group isomorphism. Similarly it may be done that $\varphi$ is an $\omega$-isomorphism and homeomorphism.

**Proposition 6.** For every $i \in J$, $\hat{H}_i = *H_i/H \cap \hat{G}$ is the closure of $H_i$ in $\hat{G}$. The set $\tilde{\mathcal{H}} = \{\hat{H}_i : i \in J\}$ is a realizer of $G$ and $\tilde{\mathcal{H}}$ is a subbase of neighbourhoods of $0$ in $\hat{G}$.

**Proof.** The first part of the proposition follows immediately from the fact that $H_i$ is a dense subset in $*H_i/H \cap G$. It may be easily seen that $\hat{H}_i$ is a prime $\omega$-ideal in $\hat{G}$ and

$$\bigcap_{i \in J} \hat{H}_i = \bigcap_{i \in J} ({}^*H_i/H \cap \hat{G}) = \bigcap_{i \in J} ({}^*H_i/H) \cap \hat{G} = \{0\}.$$

Moreover, the topology in $\hat{G}$ is induced from the one in $*G/H$, i.e. the subbase of neighbourhoods of zero consists of the sets $*H_i/H \cap \hat{G} = \hat{H}_i$.

Now, we are able to prove the theorem 2. At first, using the nonstandard construction of $\hat{G}$ we may fully describe elements of $\hat{G}$. So, let $G = \mathbb{Z}^{(\infty)}$ for $I \subseteq N$, card $I = \aleph_0$, and let $*G$ be an enlargement of $G$, $\alpha \in *G$. Then $\alpha + H \in \hat{G}$ if and only if $\alpha_i \in \mathbb{Z}$ for every $i \in I$. This follows immediately from Prop. 6, where $*H_i = \{\beta \in *G : \beta_i = 0\}, i \in I$. Let $\hat{w}$ be a semi-valuation associated with a ring $\hat{A}_I$, i.e.

$$\hat{w} : U(\hat{Q}_I) \rightarrow U(\hat{Q}_I)/U(\hat{A}_I)$$

is a canonical map and let $x \in U(\hat{Q}_I)$. According to Proposition 3, $*_w(x) \in \mathbb{Z}$ for every $i \in I$. Moreover, interpreting a suitable WFS in $*Q$ we may find an element $i_0 \in *I$ such that $w_i(x) = 0$ for any $i \in *I, i > i_0$. Since
\[*G = \{a \in \ast \mathbb{Z}^I : a \text{ is internal and there exists } i_0 \in \ast I \text{ such that } a_i = 0, \forall i > i_0\}, \]
we may find an element \( a \in \ast G \) such that
\[ a_i = \ast w_i(x) \in \mathbb{Z}, \quad i \in I. \]

We define a map \( \varphi \) in the following way:
\[ \varphi : U(\hat{Q}_I)/U(A_I) \rightarrow \hat{G}, \]
\[ \varphi(w(x)) = a + H. \]

If \( x \) and \( y \) are elements in \( Q \) such that \( \ast w_i(x) = \ast w_i(y) \in \mathbb{Z} \) for every \( i \in I \), then we have \( \hat{w}_i(x) = \hat{w}_i(y) \) by [3] and since \( x, y \in U(\hat{Q}_I) \), we have \( x \cdot y^{-1}, y \cdot x^{-1} \in \cap R_{\hat{w}_i} = A_I \) by [1]. It follows \( \hat{w}(x) = \hat{w}(y) \) and the definition of \( \varphi \) is correct. It is clear that \( \varphi \) is an \( o \)-isomorphism. Since \( \varphi^{-1}(H_i) = U(R_{\hat{w}_i}) \), \( i \in I \), \( \varphi \) is open and continuous, hence, \( \varphi \) is a homeomorphism. Since \( U(\hat{Q}_I)/U(A_I) \) is a topological group, by [2], Lemma 1, it is a \( tl \)-group which is \( tl \)-isomorphic to \( \hat{G} \). Hence, the theorem is proved.

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