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ON SOME MINIMAL PROBLEM

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It is known that not any cyclic order has a linear extension. The corresponding counterexample in [2] is constructed on a 13-elemented set. This paper deals with the problem of the minimal positive integer n with the property: There exists a cyclic order on an n-elemented set which has no linear extension.

1. TERNARY RELATIONS

1.1. Definition. *Let G be a set. A ternary relation T on the set G is any subset of the* 3^{rd} *cartesian power of G, i.e.* $T \subseteq G^3$.

1.2. Definition. *Let G be a set, T a ternary relation on G. This relation is called:*
asymmetric, iff $(x_1, x_2, x_3) \in T \Rightarrow (x_3, x_2, x_1) \notin T$
strongly asymmetric, iff $(x_1, x_2, x_3) \in T \Rightarrow (x_{i_1}, x_{i_2}, x_{i_3}) \notin T$ *for any odd permutation* (i_1, i_2, i_3) *of* $(1, 2, 3)$
cyclic, iff $(x_1, x_2, x_3) \in T \Rightarrow (x_2, x_3, x_1) \in T$
transitive, iff $(x_1, x_2, x_3) \in T$, $(x_1, x_3, x_4) \in T \Rightarrow (x_1, x_2, x_4) \in T$
complete, iff $x_1, x_2, x_3 \in G$, $x_1 \neq x_2 \neq x_3 \neq x_1 \Rightarrow (x_{i_1}, x_{i_2}, x_{i_3}) \in T$ *for some permutation* (i_1, i_2, i_3) *of* $(1, 2, 3)$.

1.3. Notation. Let T be a ternary relation on a set G. We denote the *cyclic hull* of T with T^c, i.e. $T^c = \{(x_1, x_2, x_3) \in G^3;$ there exists an even permutation (i_1, i_2, i_3) of $(1, 2, 3)$ with $(x_{i_1}, x_{i_2}, x_{i_3}) \in T\}$.
Evidently, T^c is the least cyclic ternary relation on G containing T.

1.4. Lemma. *Let T be a ternary relation on a set G. Then it holds:*
(1) *If T is cyclic, then T is strongly asymmetric iff it is asymmetric*
(2) *If T is strongly asymmetric, then* T^c *is asymmetric.*

Proof. (1) Let T be cyclic. If T is strongly asymmetric, then it is asymmetric. If T is asymmetric and $(x_1, x_2, x_3) \in T$, then $(x_2, x_3, x_1) \in T$, $(x_3, x_1, x_2) \in T$ so that $(x_3, x_2, x_1) \bar{\in} T$, $(x_1, x_3, x_2) \bar{\in} T$, $(x_2, x_1, x_3) \bar{\in} T$ and T is strongly asymmetric.

(2) Let T be strongly asymmetric and $(x_1, x_2, x_3) \in T^c$. Suppose $(x_3, x_2, x_1) \in$ $\in T^c$. Then there exists an even permutation (i_1, i_2, i_3) of $(1, 2, 3)$ with $(x_{i_1}, x_{i_2}, x_{i_3}) \in$

$\in T$ and an even permutation (j_1, j_2, j_3) of (3, 2, 1) with $(x_{j_1}, x_{j_2}, x_{j_3}) \in T$. But then (j_1, j_2, j_3) is and odd permutation of (i_1, i_2, i_3) which contradicts the strong asymmetry of T.

1.5. Lemma. *Let T be a strongly asymmetric ternary relation on a set G, let $(x_1, x_2, x_3) \in T$. Then $x_1 \neq x_2 \neq x_3 \neq x_1$.*

Proof. Suppose $(x_1, x_2, x_3) \in T$ and card $\{x_1, x_2, x_3\} \leqq 2$. If $x_1 = x_2$, then $(x_2, x_1, x_3) \in T$, if $x_1 = x_3$, then $(x_3, x_2, x_1) \in T$, if $x_2 = x_3$, then $(x_1, x_3, x_2) \in T$. This contradicts in all cases the strong asymmetry of T.

1.6. Theorem. *Let T be a cyclic ternary relation on a set G. T is transitive iff one of the following equivalent conditionS holds:*

(1) $\quad (x_1, x_2, x_3) \in T, \quad (x_1, x_3, x_4) \in T \Rightarrow (x_1, x_2, x_4) \in T$

(2) $\quad (x_1, x_2, x_3) \in T, \quad (x_1, x_3, x_4) \in T \Rightarrow (x_2, x_3, x_4) \in T$

(3) $\quad (x_1, x_2, x_4) \in T, \quad (x_2, x_3, x_4) \in T \Rightarrow (x_1, x_2, x_3) \in T$

(4) $\quad (x_1, x_2, x_4) \in T, \quad (x_2, x_3, x_4) \in T \Rightarrow (x_1, x_3, x_4) \in T$

Proof. [3], Theorem 1.6.

1.7. Remark. Let T be a transitive ternary relation on a set G. Then T^c need not be transitive.

1.8. Example. Let $G = \{x, y, z, u\}$, $T = \{(x, y, z), (x, z, u), (x, y, u)\}$. Evidently T is transitive. But $(z, u, x) \in T^c$, $(z, x, y) \in T^c$, $(z, u, y) \notin T^c$ so that T^c is not transitive.

In what follows, we shall deal with asymmetric, cyclic and transitive ternary relations. To see that a given ternary relation T (on a finite set) is asymmetric and cyclic is very simple. But it is often not easy to show that T is transitive. The following theorem gives a method which simplifies this problem.

1.9. Theorem. *Let T be a strongly asymmetric ternary relation on a set G. T^c is transitive iff the following condition holds:*

For every four elements $x_1, x_2, x_3, x_4 \in G$:

(T) (1) *either there exists no permutation (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$ with $(x_{i_1}, x_{i_2}, x_{i_3}) \in T^c$, $(x_{i_1}, x_{i_3}, x_{i_4}) \in T^c$,*

(2) *or there exists a permutation (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$ with $(x_{i_1}, x_{i_2}, x_{i_3}) \in T^c$, $(x_{i_1}, x_{i_3}, x_{i_4}) \in T^c$, $(x_{i_1}, x_{i_2}, x_{i_4}) \in T^c$, $(x_{i_2}, x_{i_3}, x_{i_4}) \in T^c$.*

Proof. The necessity of the condition (T) is clear with respect to 1.6. We shall prove its sufficiency. Note that T^c is asymmetric by 1.4. Let $(x_1, x_2, x_3) \in T^c$, $(x_1, x_3, x_4) \in T^c$, $(x_1, x_2, x_4) \bar{\in} T^c$. Then it does not hold (1) and thus a permutation (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$ must exist with $(x_{i_1}, x_{i_2}, x_{i_3}) \in T^c$, $(x_{i_1}, x_{i_3}, x_{i_4}) \in T^c$, $(x_{i_1}, x_{i_2}, x_{i_4}) \in T^c$, $(x_{i_2}, x_{i_3}, x_{i_4}) \in T^c$ and with the property that no even permutation of sequences (i_1, i_2, i_3), (i_1, i_3, i_4), (i_1, i_2, i_4), (i_2, i_3, i_4) is equal to (1, 2, 4). Thus some of these sequences is an odd permutation of (1, 2, 4) and by a simple counting of all possibilities we get a contradiction to the asymmetry of T^c. Let us show the case when (i_1, i_3, i_4) is an odd permutation of (1, 2, 4). We have the possibilities:

$(i_1, i_3, i_4) = (4, 2, 1) \Rightarrow (i_1, i_2, i_4) = (4, 3, 1)$, i.e. $(x_4, x_3, x_1) \in T^c$
$(i_1, i_3, i_4) = (2, 1, 4) \Rightarrow (i_2, i_3, i_4) = (3, 1, 4)$, i.e. $(x_3, x_1, x_4) \in T^c$,
$(x_4, x_3, x_1) \in T^c$
$(i_1, i_3, i_4) = (1, 4, 2) \Rightarrow (i_1, i_2, i_4) = (1, 3, 2)$, i.e. $(x_1, x_3, x_2) \in T^c$,
$(x_3, x_2, x_1) \in T^c$.

1.10. Example. Let $G = \{x, y, z, u, v, w\}$, $T = \{(x, y, z), (x, u, w), (z, y, v), (u, v, w), (x, y, w), (v, x, y), (v, z, x)\}$.
T is strongly asymmetric and it is easy to see that the only quadruplet with the property (2) of (T) is $\{x, y, z, v\}$, namely $(z, x, y) \in T^c$, $(z, y, v) \in T^c$, $(z, x, v) \in T^c$, $(x, y, v) \in T^c$. Thus T^c is transitive.

2. CYCLIC ORDERS AND THEIR EXTENSIONS

2.1. Definition. *Let G be a set, C a ternary relation on G which is asymmetric, cyclic and transitive. Then C is called a cyclic order on G and the pair (G, C) is called a cyclically ordered set. If, moreover, card $G \geqq 3$ and C is complete, then C is called a complete (linear) cyclic order on G and (G, C) is called a linearly cyclically ordered set or a cycle.*

From 1.4 and 1.9 we obtain directly.

2.2. Theorem. *Let T be a ternary relation on a set G which is strongly asymmetric and has the property (T). Then T^c is a cyclic order on G.*

2.3. Definition. *Let C_1, C_2 be cyclic orders on a set G. If $C_1 \subseteq C_2$, then C_2 is called an extension of C_1 and the cyclically ordered set (G, C_2) is called an extension of (G, C_1). An extension C_2 of a cyclic order C_1 on a set G is called a linear extension of C_1 if C_2 is a linear cyclic order on G.*

Of course, a cyclically ordered set (G, C) can have a linear extension only when card $G \geqq 3$. In [2] there is constructed a cyclically ordered set (G, C) with card $G =$ $= 13$ which has no linear extension. Let us denote N_l the set of all positive integers $n \geqq 3$ with the property: Any cyclically ordered set (G, C) with card $G = n$ has a linear extension, and $N_i = \{3, 4, \ldots\} - N_l$.

2.4. Theorem. *If $n \in N_i$, then $n + 1 \in N_i$.*

Proof. Suppose $n \in N_i$, $n + 1 \in N_l$. Then there exists a cyclically ordered set (G, C), where $G = \{x_1, \ldots, x_n\}$, with no linear extension. Choose an element

$x_{n+1} \notin G$ and put $G' = \{x_1, \ldots, x_n, x_{n+1}\}$. As $n + 1 \in N_l$, the cyclically ordered set (G', C) has a linear extension (G', D). Then $D \cap G^3$ is a linear extension of C on G which contradicts our assumption.

2.5. Corollary. *N_l is an initial segment of* $\{3, 4, \ldots\}$.

Denote i_0 the minimal element of N_i; from [2] it follows $i_0 \leqq 13$. But we shall show:

2.6. Theorem. $i_0 \leqq 10$.

Proof. Put $G = \{x, y, z, a, b, c, d, e, f, g\}$, $T = \{(x, z, a), (y, a, b), (z, b, c), (a, c, d), (b, d, z), (c, z, y), (d, y, x), (z, x, e), (y, e, f), (x, f, g), (e, g, b), (f, b, x), (g, x, y), (b, y, z), (x, e, a), (e, z, a), (x, d, g), (d, y, g), (y, c, b)\}$. Evidently T is strongly asymmetric and by a simple counting we find that T satisfies the condition (T) of 1.9. Thus, T^c is a cyclic order on the set G. Let C be any extension of T^c on G. Suppose $(x, y, z) \in C$. Then $(x, z, a) \in T^c \subseteq C$ implies $(y, z, a) \in C$ and by transitivity of C we get successively $(z, a, b) \in C$, $(a, b, c) \in C$, $(b, c, d) \in C$, $(c, d, z) \in C$, $(d, z, y) \in C$, $(z, y, x) \in C$ which contradicts the asymmetry of C. If we suppose $(z, y, x) \in C$, then we obtain analogously $(y, x, e) \in C$, $(x, e, f) \in C$, $(e, f, g) \in C$, $(f, g, b) \in C$, $(g, b, x) \in C$, $(b, x, y) \in C$, $(x, y, z) \in C$, a contradiction. Thus, $(x, y, z) \in C$, $(z, y, x) \in C$ can hold in no extension C of T^c and T^c has no linear extension on G. As card $G = 10$, it is $10 \in N_i$ and $i_0 \leqq 10$.

We can formulate also another minimal problem: Denote j_0 the minimal positive integer $n \geqq 3$ with the property: There exists a cyclically ordered set (G, C) with card $G = n$ and an ordered triplet $(x_1, x_2, x_3) \in G^3$ such that $x_1 \neq x_2 \neq x_3 \neq x_1$, $(x_3, x_2, x_1) \bar{\in} C$ and $(x_1, x_2, x_3) \bar{\in} C'$ for any linear extension C' of C on G.

2.7. Theorem. $j_0 \leqq 7$.

Proof. Put $G = \{x, y, z, a, b, c, d\}$, $T = \{(x, z, a), (y, a, b), (z, b, c), (a, c, d), (b, d, z), (c, z, y), (d, y, x)\}$. Then T^c is a cyclic order on G and by the same argumentation as in the proof of 2.6 we see that $(x, y, z) \in C$ can hold for no extension C of T^c. As card $G = 7$, we have $j_0 \leqq 7$.

2.8. Remark. The cyclically ordered set (G, T^c) from the proof of 2.7 has a linear extension, namely the cycle (a, b, y, c, x, d, z).

2.9. Problem. Find the explicit value of i_0, j_0.

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