Archivum Mathematicum

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Archivum Mathematicum, Vol. 20 (1984), No. 3, 125--132

Persistent URL: http://dml.cz/dmlcz/107195

ARCH. MATH. 3, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XX: 125–132, 1984

FIXED POINT CHARACTERIZATION OF COMPLETENESS ON LATTICES FOR RELATIVELY ISOTONE MAPPINGS

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(Received April 25, 1983)

0. Introduction

In this paper we present theorems about fixed points of mappings of partially ordered sets (posets) and derive a new characterization of completeness for lattices.

This paper is organized as follows. In the first section we derive our basic tool the fixed point theorem for relatively isotone mappings of a complete lattice into itself. Without appealing to the Axiom of choice or its equivalent we also prove the fixed point theorem for chain complete posets (partially ordered sets in which every chain has a supremum). In the second section we discuss the relationship between the completeness in lattices and the chain completeness in semiuniform posets, respectively, and the existence of fixed points of selfmappings. We prove, using the fixed point theory, that lattices, having a fixed point for a selfmapping of a certain kind, are complete lattices. We also present a related characterization of a chain completeness in semiuniform posets in terms of the fixed points.

We begin with some definitions and notation from the theory of partially ordered sets and from the theory of sets, which will be used throughout the paper. The poset denotes a partially ordered set (i.e. a set with a reflexive, antisymmetric and transitive relation $\leqq$), 0 and 1 being its least and greatest elements (if they exist), respectively. Let S be a subset of a poset P. An element x of P is an upper (lower) bound of S if $s \leqq x$ ($s \geqq x$) for all s in S. The terms the least upper bound and the greatest lower bound will be abbreviated to sup and inf, respectively. Let S be a subset of a poset P and let S^* denote the set of all upper bounds of S and S^+ the set of all lower bounds of S. In particular $\emptyset^* = P$. A subset S of a poset P is updirected provided every pair of elements in S has an upper bound in S. Dually, we call S downdirected if every pair of elements in S has a lower bound in S.

A poset P is said to be semiuniform (see [5]) if for each chain C in P, the set C^* is downdirected. A chain is (dually) well ordered if every nonempty subset of it has a minimum (maximum) element. We will say that a poset P is chain complete if every chain (including the empty set) of P has a sup in P. A mapping $f: P \to Q$ is called isotone if $x \leqq y$ implies $f(x) \leqq f(y)$ for all x, y from P. For any mapping $f: P \to P$ an element $x \in P$ is called a fixed point of f if $x = f(x)$ and we write $\mathrm{Fix}(f)$ for the fixed point set.

A poset P is said to have the fixed point property if every isotone mapping of P into itself has a fixed point. While the general problem of characterizing the partially ordered sets with the fixed point property remains unsolved a number of sufficient conditions as well as some necessary conditions, for this property to hold, are known. For lattices the fixed point property is equivalent to lattice completeness (Tarski [7] and Davis [4]).

1. Some new fixed point theorems

Tarski's classical result (see [7]) states, that the fixed point theorem holds for every complete lattice. We can obtain a slightly stronger theorem if isotonicity is weakened.

Definition 1.1. *A mapping f of a poset P into itself is called relatively isotone if $x, y \in P$, $x \leqq y$, $x \leqq f(y)$, $f(x) \leqq y$ implies $f(x) \leqq f(y)$.*

Theorem 1.2. *Let L be a complete lattice and let f be a relatively isotone mapping of L into itself. Then f has a fixed point.*

Proof. A subset S of L will be called closed if it has the following three properties

(1) $0 \in S$,

(2) $x \in S$ implies $f(x) \in S$,

(3) if $X \subseteq S$, then $\sup X \in S$.

There exists a closed subset of L, for example, L itself. Let T be the intersection of all closed subsets of L. Certainly, T is closed. By assumption, T has a supremum, say $u = \sup T$. Since T is closed, (3) implies $u \in T$. Applying (2), we have $f(u) \leqq u$. We prove that u is a fixed point of f.

Let us suppose the contrary, that is $f(u) < u$. Let R be a subset of T defined as $R = \{x \in T \mid x \leqq f(u)\}$. It is evident that the set R satisfies the conditions (1) and (3). Let x be now an arbitrary element in R, i.e. we have $x \leqq u$ and $x \leqq f(u)$. As x is also in T, it holds $f(x) \leqq u$, since T satisfies (3). Hence $f(x) \leqq f(u)$, as f is relatively isotone. Thus $f(x) \in R$ and (2) is also satisfied. We conclude that R is closed and R is properly contained in T, in contradiction to the minimality of T. This makes u a fixed point of f.

Since each isotone mapping of a poset P into itself is relatively isotone, we obtain, as an immediate consequence of the previous Theorem, the well known fixed point theorem by A. Tarski

Corollary 1.3. Let L be a complete lattice and let f be an isotone mapping of L into itself. Then a fixed point of f exists.

Remark 1.4. Unlike the isotone mapping the set Fix (f) of fixed points of a relative isotone mapping f of a given complete lattice L into itself need not form, with respect to the inductive partial order, a complete lattice. Further, for a commuting family F of relatively isotone mappings of a complete lattice L into itself a common fixed point of all mappings from F need not exist, which is obvious from the following:

Example 1.5. The set of all positive integers will be denoted by N. Let $P = \{0, 1, x_1, x_2, ...\}$ be the poset, where 0 (1) is the smallest (greatest) element of P and $\{x_1, x_2, ...\}$ is an infinite antichain. For every $n \in N$ let us define the mappings f_n from P into itself in the following way:

$$f_n(0) = f_n(1) = f_n(x_i) = x_n \qquad \text{for } i \leqq n,$$
$$f_n(x_i) = x_i \qquad \text{for } i > n.$$

It is easy to see that $F = \{f_n \mid n \in N\}$ is the commuting family of relatively isotone mappings of P into itself. From the definition of f_n it follows at once that for each $n \in N$ the set Fix (f_n) of all fixed points of the mapping f_n is formed by incomparable elements $\{x_n, x_{n+1}, ...\}$. But for the set Fix (F) of common fixed points of all mappings f_n, $n \in N$, we have

$$\text{Fix}(F) = \bigcap_{n \in N} \text{Fix}(f_n) = \emptyset.$$

The next theorem is the strengthening and generalization of the corresponding result for isotone mappings according to Abian and Brown [1] and we will suggest a proof without using any of the equivalent statements of the Axiom of Choice.

Theorem 1.6. *Let P be a chain complete poset and f be a relatively isotone mapping of P into itself. Then f has a fixed point.*

Proof. A subset S of P will be called chain closed if it has the following three properties

(1) $0 \in S$,

(2) if $x \in S$, then $f(x) \in S$,

(3) if C is a chain in S, then $\sup C \in S$.

Since $f: P \to P$ and since P is chain complete, the poset P satisfies the above conditions. Therefore, there exists a chain closed subset. Let T be the intersection of all chain closed subsets. It is easy to see that T is a chain closed subset of P. Since T is the smallest chain closed set, any set satysfying the above conditions and contained in T must be T. We will make heavy use of this fact in proving that T is a chain. An element c in T will be called chainable if

(i) c is comparable with every element in T,

(ii) if $y \in T$ and $y < c$, then $f(y) \leqq c$,

(iii) if $y \in T$ and $c < y$, then $c \leqq f(y)$.

There exists a chainable element in P, for example 0 is chainable.

For an arbitrary chainable element $c \in T$ we have by (2) $f(c) \in T$ and hence by (i) either $c \leqq f(c)$ or $f(c) \leqq c$. We want to show that for each chainable element it holds $c \leqq f(c)$. Let us suppose, on the contrary, that there exists a chainable element $d \in T$ such that $f(d) < d$. We will prove that the set $R = \{x \in T \mid x \leqq f(d)\}$ is chain closed. It follows from the definition of the least element and from the definition of suprema, respectively, that R satisfies the conditions (1) and (3). Let x be in R. Since d is chainable and $x \leqq f(d) < d$, (ii) implies $f(x) \leqq d$. Applying relative isotonicity, we have $f(x) \leqq f(d)$ and hence $f(x) \in R$. This proves that R satisfies (2). Therefore R is a chain closed subset properly contained in T, and this contradicts the minimality of T.

Let D be the set of all chainable elements. We have shown that $d \leqq f(d)$ for each $d \in D$. We assert that if $d \in C$ and $x \in T$, then either $x \leqq d$ or $f(d) \leqq x$. To prove this assertion, let $d \in C$ and define $D(d) = \{x \in T \mid x \leqq d \text{ or } f(d) \leqq x\}$. It suffices to show that $D(d)$ is chain closed. Since $0 \in T$ and it holds $0 \leqq d$, (1) is satisfied. Let $x \in D(d)$. Then we have either $x < d$, or $x = d$ or $f(d) \leqq x$. If $x < d$, then $f(x) \leqq d$, because $d \in D$. If $x = d$, then $f(d) \leqq f(x)$. If $f(d) \leqq x$ then applying (iii) we have $d \leqq f(x)$, because $d < x$. Since f is relatively isotone and $d \leqq x$, $d \leqq f(x)$, $f(d) \leqq x$ we have $f(d) \leqq f(x)$. Thus in every case $f(x) \in D(d)$ and (2) is satisfied. Next, let C be a chain in $D(d)$. Then either $c \leqq d$ for each $c \in C$, in which case $\sup C \leqq d$, or there exists $c \in C$ such that $f(d) \leqq c$. Hence $f(d) \leqq c \leqq \sup C$. Thus $\sup C \in D(d)$ and (3) is also satisfied. We conclude that $D(d)$ is chain closed and so $T = D(d)$.

We next assert that each element of T is chainable. We prove this by showing that D is chain closed. Since 0 is the least element there is no element $y \in P$ such that $y < 0$ and therefore 0 satisfies (ii) trivially. It follows immediately from the definition of the least element that D satisfies (i) and (iii). As we observed above, 0 is a chainable element and therefore D satisfies (1). Furthermore, let $d \in D$ and $y \in T$ be such that $y < f(d)$. Since $y \in T = D(d)$, we have $y \leqq d$. (The inequality $f(d) \leqq y$ being impossible.) If $y < d$, the definition of D yields $f(y) \leqq d \leqq f(d)$. If $y = d$ then $f(y) \leqq f(d)$. Hence $f(d)$ satisfies (ii). Let $y \in T$ be now such that $f(d) < y$, then we have $d \leqq f(y)$, since $d \leqq f(d)$ and $d \in D$. From these inequalities and from the relative isotonicity of f it follows $f(d) \leqq f(y)$, so (iii) is satisfied. The expressions $T = D(d)$ and $d \leqq f(d)$ involve the fact that $f(d)$ satisfies (i). Hence $f(d)$ is also chainable.

Further, let C be a chain in D and let $y \in T$ have the property $y < \sup C$. Since $y \in T = D(c)$ for each $c \in C$, we have either $y \leqq c$ for some $c \in C$ or $f(c) \leqq y$ for every $c \in C$. If the latter alternative were true, we would have $\sup C \leqq$

$\leqq \sup\{f(c) \mid c \in C\} \leqq y$ which is impossible. Thus there is some $c \in C$ such that $y \leqq c$. If $y < c$, then, since $c \in D$, we have $f(y) \leqq c \leqq \sup C$. If $y = c$, then $y \in D$ and $\sup C \in T = D(y)$. As $y < \sup C$ we obtain again $f(y) \leqq \sup C$. Thus $\sup C$ satisfies (ii). Let $y \in T$ be such that $\sup C < y$. Then $c < y$ for each $c \in C$. Hence $c \leqq f(y)$ for each $c \in C$, i.e. $\sup C \leqq f(y)$. This proves that $\sup C$ satisfies (iii). To show that $\sup C$ satisfies also (i), let x be an arbitrary element in T. Since $x \in T = D(c)$ for every $c \in C$, we have either $x \leqq c$ for some $c \in C$ or $f(c) \leqq x$ for every $c \in C$. In the first case we have $x \leqq \sup C$ and in the latter one we obtain $\sup C \leqq x$, since $c \leqq f(c)$ holds for every $c \in C$. We conclude that $\sup C$ satisfies (i) and hence it is chainable. This proves that D satisfies (3). Therefore D is chain closed and we have $T = D$.

We conclude from the above arguments that if $x \in T$ and $y \in T = D(x)$, (i) implies either $y \leqq x$ or $x \leqq y$. Accordingly, T is a chain. Let $u = \sup T$. Since T is chain closed, (3) implies $u \in T$. Applying (2), we have $f(u) \leqq u$. On the other hand we have $u \leqq f(u)$, since u is chainable. Therefore u is the fixed point of f.

As an immediate consequence of this Theorem we have the following

Corollary 1.7. (Abian and Brown [1]) Let P be a chain complete poset and let f be an isotone mapping of P into itself. Then f has a fixed point.

A brief proof of this Corollary based on Zorn's lemma may be found in Wong [8, Theorem 1]. The preceeding Corollary, however, may be proved independently and without using of Axiom of Choice. A more involved proof that does not use any form of the Axiom of Choice is given by Abian and Brown [1, Theorem 2].

In Theorem 1.2 we do not need the Axiom of Choice when proving that $\{x \in L \mid x = f(x)\}$ is nonempty. At the same time we do not know whether the fixed point property for the relatively isotone mappings in chain complete posets can be proved with using the Zorn's lemma.

Example 1.8. We will show, that the set $\operatorname{Fix}(f)$ of fixed points of a relatively isotone mapping f from a chain complete poset P into itself need not form a chain complete poset. Further, a common fixed point for a commuting family of relatively isotone mappings need not exist.

Let $P = \{0, 1, x_1, x_2, \ldots\}$ be a poset, where 0 (1) is the smallest (greatest) element of P and $x_1 \leqq x_2 \leqq \ldots$ is a countable chain of type ω. Let N be the set of all positive integers and let us define the mapping f_n from P into itself for all $n \in N$ in the following way

$$\begin{aligned} f_n(0) = f_n(1) = f_n(x_i) &= x_n \qquad \text{for } i \leqq n, \\ f_n(x_i) &= x_i \qquad \text{for } i > n. \end{aligned}$$

It can be seen from the definition of the mappings f_n that $F = \{f_n \mid n \in N\}$ is the commuting family of relatively isotone mappings from P into itself and $\operatorname{Fix}(f_n) = \{x_n, x_{n+1}, \ldots\}$ is not chain complete for any $n \in N$. For the set $\operatorname{Fix}(F)$ of common fixed points of all mappings f_n, $n \in N$, it holds

$$\operatorname{Fix}(F) = \bigcap_{n \in N} \operatorname{Fix}(f_n) = \emptyset.$$

2. A new characterization of completeness for lattices

In this section we shall be concerned with the question of characterization of completeness for posets by fixed points. Davis [4] studied this problem for posets which are lattices and Markowski [6] characterized chain complete posets in terms of the least fixed points of isotone selfmappings. We will be especially interested in the characterization of completeness in lattices for relatively isotone self-mappings.

We believe that the Davis' result gives also the sufficient condition for completeness of lattices in the case of relatively isotone selfmappings. Let each relatively isotone mapping of a lattice L into itself have a fixed point. Then L is complete. The proof of this fact is closely related to the Davis's construction of the fixed point free selfmapping of a given lattice into itself. Hence we obtain the following characterization of completeness for lattices, which is related to characterization by fixed points used by Tarski and Davis A lattice L is complete if and only if each relatively isotone mapping of L into itself has a fixed point.

In general this characterization follows directly from the Theorem 1.2 and from the Theorem of Davis cited above. However, Davis' proof method does not require lattice operations by means of constructing a fixed point free selfmapping. Now, we present a fixed point characterization of completeness which shows that we may restrict our attention to very special relatively isotone mappings in proving that a lattice is complete. In the proof of the next Theorem we present a technique for finding the fixed point free mapping of a given lattice into itself by the use of lattice operations. We need the following definition.

Definition 2.1. *A mapping f of a poset P into itself is called comparable if for each x in P x comparable with $f(x)$.*

Theorem 2.2. *A lattice L is complete if and only if every relatively isotone mapping of L into itself which is comparable has a fixed point.*

Proof. The necessity follows from the Theorem 1.2. To prove the converse, assume that L is a lattice that is not complete. Then L contains a chain C that does not have a supremum. Let U be a well ordered chain cofinal with C. By Hausdorff Maximality Principle there is a maximal chain D in the set C^* of all upper bounds of C. Let V be a dually well ordered chain coinitial with D.

Observe that if there were an element $x \in L$ such that $u \leqq x \leqq v$ for all $u \in U$ and all $v \in V$, then x would be an upper bound of C, and since C does not have a supremum, an element $y \in C^*$ would have to exist with $x \nleqq y$, and as $x \wedge y \in C^*$ and $x \wedge y < x$, the set $D \cup \{x \wedge y\}$ would be a chain in C^* properly containing D, and this contradicts the maximality of D. Therefore, there is no element $x \in L$ such that $u \leqq x \leqq v$ for all $u \in U$ and all $v \in V$.

We are now ready to define a relatively isotone comparable mapping that does not have a fixed point in L. For each $x \in L$ set

$$U_x = \{u \in U \mid u \nleqq x\}, \qquad V_x = \{v \in V \mid x \nleqq v\}.$$

As we observed above, for a given $x \in L$ either U_x or V_x is nonempty. If U_x is nonempty, define u_x to be the least element in U_x. If U_x is empty define v_x to be the greatest element in V_x. Now for any $x \in L$ either $u_x \nleqq x$ or $x \nleqq v_x$.

We define a mapping f from L into itself according to the following prescription

$$f(x) = x \vee u_x \quad \text{if } x \notin C^*,$$
$$f(x) = x \wedge v_x \quad \text{if } x \in C^*.$$

Then f is well defined and for any $x \in L$ either $x < f(x)$ or $f(x) < x$, so f is a comparable mapping, which does not have a fixed point. It remains only to show that f is relatively isotone.

Let x and y be elements in L with $x \leqq y$, $x \leqq f(y)$, $f(x) \leqq y$. If U_x is empty then U_y is also empty and inasmuch as $V_y \supseteqq V_x$, it follows that $v_x = v_y$ and hence $f(x) = x \wedge v_x \leqq y \wedge v_y = f(y)$. If both U_x and U_y are nonempty, then again $f(x) = x \vee u_x \leqq y \vee u_y = f(y)$, because $U_y \subseteqq U_x$ and hence $u_x \leqq u_y$. Finally, if U_x is nonempty but U_y is empty, then $u_x \in U$, while $v_y \in V$, so that $u_x \leqq u_y$. As $y \in U^*$, $v_y \in U^*$ we have $y \wedge v_y \in U^*$. Also $u_x \leqq y \wedge v_y$. From our assumption $x \leqq f(y)$, it follows that $x \vee u_x \leqq y \wedge v_y$. We conclude that $f(x) \leqq f(y)$. Let us remark that the case U_x empty and U_y nonempty is, with respect to our assumption $x \leqq y$, impossible. Thus f is relatively isotone and the proof is complete.

At the end of this paper we show, that the existence of fixed point for each relatively isotone mapping from a semiuniform poset into itself is a necessary and sufficient condition for this poset to be chain complete.

Theorem 2.3. *A semiuniform poset P is chain complete if and only if each relatively isotone mapping of P into itself has a fixed point.*

Proof. The necessity follows from the Theorem 1.6. To prove the converse, assume that P is a semiuniform poset that is not chain complete. Then there exists a chain C in P that does not have a supremum. Let U be a well ordered chain cofinal with C. In C^* there exists a maximal chain D. Let V be a dually well ordered chain coinitial with D.

If there were an element $x \in P$ such that $u \leqq x \leqq v$ for all $u \in U$ and all $v \in V$, then x would be an upper bound of C, and since C does not have a supremum, an element $y \in C^*$ would have to exist with $x \nleqq y$. As C^* is down directed, a lower bound $z \in C^*$ of the set $\{x, y\}$ would have to exist such that $z < x$. Then the set $D \cup \{z\}$ would be a chain in C^* greater than D, which yields a contradiction. Hence there is no element $x \in P$ such that $u \leqq x \leqq v$ for all $u \in U$ and all $v \in V$.

For each $x \in P$ set $U_x = \{u \in U \mid u \nleqq x\}$, $V_x = \{v \in V \mid x \nleqq v\}$. As we observed above, for a given $x \in P$ either U_x or V_x is nonempty. If U_x is nonempty, define

$$f(x) = \min U_x,$$

if V_x is nonempty, define

$$f(x) = \max V_x.$$

Now, for any $x \in P$ either $f(x) \nleqq x$ or $x \nleqq f(x)$, so f does not have a fixed point. We must show that f is relatively isotone. Let x and y be elements in P with $x \leqq y$, $x \leqq f(y)$ and $f(x) \leqq y$. We have two possibilities, either $f(x) \in U$ or $f(x) \in V$. If $f(x) \in U$, then $f(y) \in U_y \cup V$, since $f(x) \leqq y$. Hence $f(x) = \min U_x \leqq f(y)$. If $f(x) \in V$, then we have $f(y) \in V$, because $x \leqq y$ implies $V_x \subseteq V_y$. Hence it follows again $f(x) \leqq f(y)$. Then f is a fixed point free relatively isotone selfmapping, which completes the proof of the theorem.

Remark 2.4. It can be easy seen, that the selfmapping f constructed in a proof of the Theorem 2.3, is even isotone. Hence we have the following:

Corollary 2.5. A semiuniform poset P is chain complete if and only if each isotone mapping of P into itself has a fixed point.

Proof. The necessity follows from the Theorem 1.6 and the sufficiency from the Remark 2.4 and the Theorem 2.3.

Acknowledgement. My thanks go to Professor V. Novák for many discussions and helpful suggestions during the preparation of this paper.

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