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A CHARACTERIZATION OF INDUCTIVE POSETS

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Abstract. In this paper some aspects of the fixedpoint theory of posets are studied. A new type of selfmapping on posets so called ascending mappings is defined. This new concept enables to prove a fixedpoint theorem which is a generalization of the Bourbaki's fixedpoint theorem. This result is applied to two characterizations of inductiveness for posets. The first one shows that inductiveness in semilattices is equivalent to the existence of fixedpoints of extensive mappings The second characterization proves the inductiveness property for general posets.

Key words: partially ordered sets, fixedpoints, isotone mappings.

The aim of this short paper is to give some characterizations of inductiveness for posets in a way similar to that of A. C. Davis [4]. First we derive our basic tool, the fixed point theorem for ascending mappings of an inductive poset into itself. Further we discuss the relationship between the inductiveness in posets and the existence of fixed points. We also present a related characterization of an inductiveness for semilattices in terms of the fixed points.

Throughout this paper a poset is a partially ordered set and a chain will mean a totally ordered set (it may be empty). A lower directed set will mean an ordered set having a lower bound for each finite subset. Lower directed subsets must be nonempty, since they must contain a lower bound for the empty set. For elements x and y of a poset P, $x < y$ means $x \leqq y$ and $x \neq y$. The set of all lower (upper) bounds of a subset X in a poset P is denoted $X^+(X^*)$; in particular, it is $\emptyset^* = P = \emptyset^+$ and denote $(X^*)^+ = X^{*+}$. A mapping f of a poset P into itself is called extensive if and only if $x \leqq f(x)$ for every $x \in P$. When we have a family F of mappings from a poset P into itself, then we will denote the set of all common fixed points of F by $\mathrm{Fix}(F) = \{x \in P \mid x = f(x) \text{ for all } f \in F\}$. Speaking about commuting family of mappings we mean that compositions are commutative.

Definition 1. A poset P is called *inductive* (see [5]) if each nonempty chain of P has an upper bound in P.

A poset P is called *chain complete* if every chain in P has the least upper bound in P.

Bourbaki's classical result (see [2]) states, that the fixed point theorem for extensive mappings holds in inductive posets. We can obtain a slightly stronger theorem if extensivity is weakened.

Definition 2. A mapping f of a poset P into itself is called *ascending* if $x, y \in P$, $f(x) \leqq y$ implies $f(x) \leqq f(y)$.

Theorem 3. *Let P be an inductive poset and let f be an ascending mapping on P. Then f has fixed point.*

Proof. Let S be a collection of all chains C of P with the following property: if $x \in C$, then $f(x) \in C$ and $x \leqq f(x)$. Since $f(x) \leqq f(x)$ and f is ascending, we have $f(x) \leqq f^2(x)$ for all $x \in P$. Therefore, S is nonempty, because a chain $\{f^n(x) \mid n = 1, 2, \ldots\}$ belongs to S for all $x \in P$. Let M be a maximal chain in S and let u be any upper bound of M. As $x \leqq f(x) \in M$ for all $x \in M$, it follows $f(x) \leqq u$ and hence $x \leqq f(x) \leqq f(u)$ for all $x \in M$, since f is ascending. Thus $f(u)$ is an upper bound of M. Further it is $f(u) \leqq ff(u)$, since f is ascending. If there were $f(u) \neq f^2(u)$, then $M \cup \{f^n(u) \mid n = 1, 2, \ldots\}$ would be a chain in S properly containing M, and this contradicts the maximality of M. Therefore $f(u)$ i the fixed point of f.

Corollary 4. (Zermelo [7]) *Let P be a chain complete poset and f an extensive mapping on P. Then f has a fixed point.*

Proof. Since a chain complete poset is inductive and as every extensive mapping is ascending, the proof follows from the theorem 3.

Corollary 5. (Bourbaki's fixed point theorem [2]) *Let P be an inductive poset and f an extensive mapping on P. Then f has a fixed point.*

Theorem 6. *Let S be an upper semilattice. Then the following conditions are equivalent:*

(A) *S is an inductive poset,*

(B) *each extensive mapping on S has a fixed point.*

Proof. (A) implies (B) by the Corollary 5. To verify that (B) implies (A), assume that an upper semilattice S is not inductive. Then S contains a chain C, that does not have an upper bound in S. Let U be a well ordered chain cofinal with C. For each $x \in S$ set $U(x) = \{u \in U \mid u \nleqq x\}$, As $C^* = \emptyset$, for every $x \in S$ the set $U(x)$ is nonempty. We are now ready to define an extensive mapping that does not have a fixed point in S. Let us define a mapping f from S into itself according to the following prescription:

$$f(x) = x \vee \min U(x).$$

A mapping f is well defined, since S is an upper semilattice and U is a well ordered chain. As $\min U(x) \nleqq x$, we have $x < x \vee \min U(x)$ for all $x \in S$, so f is an extensive mapping of S into itself, which does not have a fixed point.

Theorem 7. *A poset P is inductive if and only if every ascending mapping on P has a fixed point.*

Proof. The necessity of a given condition follows from the Theorem 3. To prove the converse, assume that P is a poset that is not inductive. Then P contains a chain $C \neq \emptyset$ that does not have an upper bound in P. Let U be a well ordered chain cofinal with C. If we set $U(x) = \{u \in U \mid u \nleq x\}$, then for each $x \in P$ the set $U(x)$ is nonempty, since $U^* = C^* = \emptyset$. Let us define a mapping $f\colon P \to P$ as follows:

$$f(x) = \min U(x).$$

Then f is well defined, since U is a well ordered chain. Let x and y be elements in P with the property $f(x) \leqq y$. As $f(x) \in U$, it follows by the definition of $U(y)$, $f(x) \leqq u$ for all $u \in U(y)$. Hence we have $f(x) \leqq \min U(y) = f(y)$. By the definition of f we have $f(x) \nleq x$ for all $x \in P$. So f is an ascending mapping on P, which does not have a fixed point.

Theorem 8. *Let P be an inductive poset and f, g commuting ascending mappings on P. Then* Fix (f, g) = Fix (fg) *is an inductive poset.*

Proof. First we will prove that the mapping $h = fg$ is an ascending one. To prove this assertion, let $x, y \in P$, $fg(x) \leqq y$. As f is ascending, it follows $fg(x) \leqq f(y)$ and hence $gf(x) \leqq gf(y)$, since g is ascending and g, f commute. Thus $h(x) \leqq h(y)$ and h is an ascending mapping on P.

Applying now the theorem 3, we obtain that Fix (h) is nonempty. To prove that Fix (h) is inductive, let C be any chain in Fix (h). As P is inductive, $C^* \neq \emptyset$. If $y \in C^*$, then $h(x) \leqq y$ for all $x \in C$. Hence $x = h(x) \leqq h(y)$ for all $x \in C$. Therefore h maps C^* into itself, so $h \mid C^*$ is an ascending mapping of an inductive poset C^* into itself. Applying now Theorem 3 for this case, we obtain a fixed point of h, which is an upper bound of C in Fix (h). Thus Fix (h) is an inductive poset.

Next, we prove that Fix (h) is exactly the set of all common fixed points of f and g. Let $x \in$ Fix (h), i.e. $fg(x) = x$. Hence $gfg(x) = g(x)$. As f is ascending, it follows $g(x) = fgg(x) \leqq fg(x) = gf(x) = x$. By reflexivity of $\leqq$ we have $gf(x) \leqq x$ and hence $gf(x) = x \leqq g(x)$, since g is ascending. By the antisymmetry of $\leqq$ we obtain $x = g(x)$. Analogously we prove $x = f(x)$. Therefore Fix $(fg) \subsetneqq$ Fix (f, g). On the other hand, for any $x \in$ Fix (f, g) we have $x = g(x)$ and hence $x = f(x) = fg(x)$. Thus Fix (fg) = Fix (f, g) and the proof is complete.

Corollary 9. *Let P be an inductive poset and f be an ascending mapping on P. Then* Fix (f) *is an inductive poset.*

Corollary 10. *Let P be an inductive poset and let* $F = \{f_1, f_2, \dots, f_n\}$ *be a commuting set of ascending mappings on P. Then* Fix $(f_1, \dots, f_n)$ = Fix (F) *is an inductive poset.*

Proof follows from the Theorem 8 by induction.

Example 11. We will show, that for any commuting family F of ascending mappings on a inductive poset P a common fixed point of all mappings from F need not exist.

Let P be a chain $x_1 > x_2 > \dots > x_n > \dots$ Let N be the set of all positive integers and let us define the mappings f_n from P into itself for all $n \in \mathrm{N}$ in the following way:

$$f_n(x_i) = \begin{cases} x_n & \text{if } i \leqq n, \\ x_i & \text{if } i > n. \end{cases}$$

It is easy to see that $F = \{f_n \mid n \in \mathrm{N}\}$ is a commuting family of ascending mappings of an inductive poset P into itself. From the definition of f_n it follows at once that we have:

$$\mathrm{Fix}\,(f_n) = \{x_n, x_{n+1}, \dots\} \qquad \text{for all } n \in \mathrm{N}.$$

But for the set Fix (F) of all common fixed points of all mappings f_n we have:

$$\mathrm{Fix}\,(F) = \bigcap_{n \in \mathrm{N}} \mathrm{Fix}\,(f_n) = \emptyset.$$

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