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ON A VECTOR MULTIPOINT BOUNDARY VALUE PROBLEM

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Abstract. Existence and uniqueness of the solution to a multipoint boundary value problem for nth-order nonlinear differential systems is proved by using the estimates for derivatives of the scalar functions and by introducing admissible system of functions with respect to the Green function. The theory of positive linear operators gives that the obtained result is the best in a certain sense.

Key words. Vector multipoint boundary value problem, generalized Banach space, Green function, admissible system of functions, associated system of constants, Lipschitz condition, ordered Banach space with positive cone, positive eigenvalue.

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In the paper a multipoint boundary value problem for n-th order nonlinear differential systems is studied. It is shown how the facts and methods from the scalar case can be applied to the vector one. Existence theorems for De la Vallée Poussin problem are obtained by means of the estimates for derivatives of scalar functions or by introducing admissible system of functions with respect to the Green function. Here the theory of positive linear operators is applied. The obtained results extend and generalize some theorems proved by R. P. Agarwal and J. Vosmanský in [2].

In the paper the following vector multipoint boundary value problem

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)}), \tag{1}$$

$$x^{(i-1)}(t_j) = a_{i,j}, \qquad i = 1, \dots, r_j, \qquad j = 1, 2, \dots, m \tag{2}$$

will be considered where

$n \geqq 2, 2 \leqq m \leqq n, 1 \leqq r_j$ are natural numbers such that $r_1 + r_2 + \dots + r_m = n$, $-\infty < a = t_1 < \dots < t_m = b < \infty$ are real numbers, $a_{i,j} \in R^d$ are vectors, $d \geqq 1$ and throughout the whole paper we assume that

$$f \in C(D, R^d) \qquad \text{where} \qquad D = [a, b] \underbrace{\times R^d \times R^d \times \dots \times R^d}_{n \text{ times}}.$$

The scalar case ($d = 1$) has been thoroughly studied. See the monograph [7] by I. T. Kiguradze, [4] by S. R. Bernfeld and V. Laksmikantham. As to the vector case ($d > 1$), there are substantially less papers devoted to problem (1), (2). Among them the paper [2] written by R. P. Agarwal, J. Vosmanský attracted the attention. The ideas from that paper are deepened and developed here.

PRELIMINARIES

If $x = (x_1, \ldots, x_d)^T$ is a column vector, then we denote $|x| = (|x_1|, \ldots, |x_d|)^T$. A partial ordering in R^d can be introduced by the relation:

If $x = (x_1, \ldots, x_d)^T$, $y = (y_1, \ldots, y_d)^T$ belong to R^d, then

$$x \leqq y \quad \text{iff} \quad x_j \leqq y_j \quad \text{for } j = 1, \ldots, d.$$

Further we denote $u_d = (1, \ldots, 1)^T \in R^d$.

The set of all real $d \times d$ matrices will be denoted as $M_{d \times d}$. Similarly as in the case of vectors, if $L = (l_{ij})$, then $|L| = (|l_{ij}|)$, $i, j = 1, \ldots, d$. Further $L \leqq \bar{L}$ iff $l_{ij} \leqq \bar{l}_{ij}$ for $i, j = 1, \ldots, d$ and $L = (l_{ij})$, $\bar{L} = (\bar{l}_{ij})$. U_d (0_d) will mean the matrix from $M_{d \times d}$, all elements of which are 1 (0). E_d will denote the unit matrix. As usual, the spectral radius $\varrho(L)$ of the matrix $L \in M_{d \times d}$ $\varrho(L) = \max_i |\lambda_i|$ where λ_i are all eigenvalues of L.

POUSSIN CONSTANTS

The first result is based on the following estimates for scalar functions.

If $x \in C^n([a, b], R)$ satisfies

$$(2'') \qquad x^{(i-1)}(t_j) = 0, \qquad i = 1, \ldots, r_j, \qquad j = 1, 2, \ldots, m,$$

then there exist positive numbers $C_{n,k}$, $k = 0, 1, \ldots, n - 1$, such that

$$(3) \quad |x^{(k)}(t)| \leqq C_{n,k}(b - a)^{n-k} \max_{a \leqq t \leqq b} |x^{(n)}(t)|, \qquad a \leqq t \leqq b, \qquad k = 0, 1, \ldots, n - 1.$$

The constants $C_{n,k}$ have been determined by several authors. E. g. G. A. Bessmertnych, A. Ju. Levin in [6] found that

$$(4) \qquad C_{n,0} = \frac{(n-1)^{n-1}}{n!\, n^n}, \qquad C_{n,k} = \frac{k}{(n-k)!\, n}, \qquad k = 1, \ldots, n - 1.$$

G. A. Bessmertnych in [5] has proved:

Let $l = \min\{r_1, r_m\} > 1$. Then

$$C_{n,k} = \begin{cases} \dfrac{(n-k-1)^{n-k-1}}{(n-k)!\,(n-k)^{n-k}}, & k = 0, 1, \dots, l-1, \\ \dfrac{k-l+1}{(n-k)!\,(n-l+1)}, & k = l, l+1, \dots, n. \end{cases} \tag{5}$$

R. P. Agarwal in [1] derived:
Let $l = \min\{r_1, r_m\}$. Then

$$C_{n,k} = \begin{cases} \dfrac{(n-l)^{n-l}(l-k)^{l-k}}{(n-k)!\,(n-k)^{n-k}}, & k = 0, 1, \dots, l-1, \\ \dfrac{k-l+1}{(n-k)!\,(n-l+1)}, & k = l, l+1, \dots, n-1. \end{cases} \tag{6}$$

For their meaning we shall call $C_{n,k}$ Poussin constants. Thus the following definition will be of use.

Definition 1. Any nonnegative constants $C_{n,k}$, $k = 0, 1, \dots, n-1$, such that for all functions $x \in C^n([a, b], R)$ satisfying (2″) the estimates (3) hold will be called Poussin constants.

Further we shall use the concept of a generalized norm ([4], pp. 225–228). If E is a real vector space, then the generalized norm for E is a mapping $\|\,.\,\|_G : E \to R^d$ denoted by

$$\| x \|_G = (\alpha_1(x), \dots, \alpha_d(x))^T, \tag{7}$$

such that

(a) $\| x \|_G \geqq 0$, that is $\alpha_j(x) \geqq 0$ for $j = 1, \dots, d$, $x \in E$;
(b) $\| x \|_G = 0$ iff $x = 0$;
(c) $\| cx \|_G = | c |\, \| x \|_G$, $c \in R$, $x \in E$;
(d) $\| x + y \|_G \leqq \| x \|_G + \| y \|_G$, $x, y \in E$.

The couple $(E, \|\,.\,\|)_G$ is then called a generalized normed space. The topology in this space is given in the following way. For each $x \in E$, and $\mathscr{E} > 0$, let $B_{\mathscr{E}}(x) = \{y \in E : \| y - x \|_G < \mathscr{E} u_d\}$. Then $\{B_{\mathscr{E}}(x) : x \in E,\ \mathscr{E} > 0\}$ forms a basis for a topology on E. The same topology can be induced by the norm $\|\,.\,\|$ which is defined in this way.

If $\| x \|_G$ is given by (7), then

$$\| x \| = \max(\alpha_1(x), \dots, \alpha_d(x)), \quad x \in E. \tag{8}$$

$\|\,.\,\|$ has all properties of the norm. Since the topology of the normed space $(E, \|\,.\,\|)$ is given by the basis of neighbourhoods $V_{\mathscr{E}}(x) = \{y \in E : \| y - x \| < \mathscr{E}\}$, $x \in E$, $\mathscr{E} > 0$, and $V_{\mathscr{E}}(x) = B_{\mathscr{E}}(x)$, both (7) and (8) define the same topology on E and in this sense they are equivalent. From the topological point of view there is no need for introducing the generalized norm. Yet we have more flexibility when working with generalized spaces. This is clear in the case of Contraction mapping theorem ([2], p. 2) which will be stated as

Lemma 1. *Let $(E, \|\,.\,\|_G)$ be a generalized Banach space and let $T: E \to E$ be such that for all $x, y \in E$ and for some positive integer p*

$$\| T^p x - T^p y \|_G \leqq L \| x - y \|_G,$$

where $L \in M_{d \times d}$ is a nonnegative matrix with $\varrho(L) < 1$ and T^p is the p-th iterate of T. Then T has a unique fixed point.

Theorem 1. *Let for all $(t, u_0, u_1, \dots, u_{n-1}), (t, v_0, v_1, \dots, v_{n-1}) \in D$ the function f satisfy Lipschitz condition*

$$(9) \qquad | f(t, u_0, u_1, \dots, u_{n-1}) - f(t, v_0, v_1, \dots, v_{n-1}) | \leqq \leqq \sum_{k=0}^{n-1} L_k | u_k - v_k |,$$

where $L_k \in M_{d \times d}$ are nonnegative matrices. Let $C_{n,k}$, $k = 0, 1, \dots, n-1$, be Poussin constants. Let

$$(10) \qquad \varrho\Big(\sum_{k=0}^{n-1} L_k C_{n,k} (b-a)^{n-k}\Big) < 1.$$

Then there exists a unique solution to (1), (2) for all $a_{i,j} \in R^d$, $i = 1, \dots, r_j$, $j = 1, 2, \dots, m$.

Proof. First we transform the problem (1), (2) to a simpler one. Let w be the unique solution of (2),

$$(1') \qquad x^{(n)} = 0.$$

Then x is a solution to (1), (2) iff the function $y(t) = x(t) - w(t)$, $a \leqq t \leqq b$, is a solution to boundary value problem

$$(11) \qquad y^{(n)} = f(t, y + w(t), y' + w'(t), \dots, y^{(n-1)} + w^{(n-1)}(t)) \equiv \equiv g(t, y, y', \dots, y^{(n-1)}),$$

$$(2''') \qquad y^{(i-1)}(t_j) = 0, \qquad i = 1, \dots, r_j, j = 1, 2, \dots, m.$$

The function $g \in C(D, R^d)$ and it satisfies Lipschitz condition (9). We shall show that problem (11), (2‴) has a unique solution.

Denote for any $x \in C([a, b], R^d)$, $x(t) = (x_1(t), \dots, x_d(t))^T$, $a \leqq t \leqq b$,

$$\max_{a \leqq t \leqq b} | x(t) | = (\max_{a \leqq t \leqq b} | x_1(t) |, \dots, \max_{a \leqq t \leqq b} | x_d(t) |)^T.$$

Let $S_1 = \{x \in C^n([a, b], R^d) : x \text{ satisfies } (2')\}$ where

$$(2') \qquad x^{(i-1)}(t_j) = 0, \qquad i = 1, \dots, r_j, j = 1, 2, \dots, m.$$

Then S_1 is a real vector space and define the generalized norm on S_1 by

$$\| x \|_1 = \max_{a \leq t \leq b} | x^{(n)}(t) | \qquad \text{for each } x \in S_1.$$

The properties of the generalized norm can be easily checked. E.g. $\| x \|_1 = 0$ implies that x is a solution to (1′), (2′) and hence $x = 0$. S_1 is a complete generalized space and thus a generalized Banach space. In fact if $\{x_p\}$ is a Cauchy sequence, then there is a function $y \in C([a, b], R^d)$ such that $x_p^{(n)}$ converge uniformly to y on $[a, b]$. The problem $x^{(n)} = y(t)$, (2′), has a unique solution $x \in S_1$ and $\lim_{p \to \infty} \| x_p - x \|_1 = 0$.

Define the mapping $T_1 : S_1 \to S_1$ as follows: If $y \in S_1$, let $T_1 y$ be the solution x to boundary value problem (2′),

$$x^{(n)} = g(t, y(t), y'(t), \dots, y^{(n-1)}(t)).$$

Then by (9), for any two functions $y, z \in S_1$ we have

$$(12) \qquad | (T_1 y)^{(n)}(t) - (T_1 z)^{(n)}(t) | \leq \sum_{k=0}^{n-1} L_k | y^{(k)}(t) - z^{(k)}(t) |, \qquad a \leq t \leq b.$$

Denote the j-th coordinate of the function y and z, respectively, by y_j and z_j, respectively. As by (3)

$$| y_j^{(k)}(t) - z_j^{(k)}(t) | \leq C_{n,k}(b - a)^{n-k} \max_{a \leq t \leq b} | y_j^{(n)}(t) - z_j^{(n)}(t) |$$

it follows that

$$(13) \qquad | y^{(k)}(t) - z^{(k)}(t) | \leq C_{n,k}(b - a)^{n-k} \| y - z \|_1.$$

(12) and (13) imply that

$$| (T_1 y)^{(n)}(t) - (T_1 z)^{(n)}(t) | \leq \sum_{k=0}^{n-1} L_k C_{n,k}(b - a)^{n-k} \| y - z \|_1$$

and

$$\| T_1 y - T_1 z \|_1 \leq \sum_{k=0}^{n-1} L_k C_{n,k}(b - a)^{n-k} \| y - z \|_1.$$

In view of the assumption (10), Lemma 1 yields the existence of a unique fixed point of T_1 in S_1. This means that the problem (11), (2‴) as well as the problem (1), (2) has a unique solution.

ADMISSIBLE SYSTEM OF FUNCTIONS AND ASSOCIATED SYSTEM OF CONSTANTS

Another approach to solving the problem (1), (2) is based on the notion of an admissible system with respect to the Green function.

Let G be the Green function of the scalar problem

(1″) $x^{(n)} = 0$,

(2″) $x^{(i-1)}(t_j) = 0,\ i = 1, \dots, r_j, j = 1, 2, \dots, m.$

Then the functions

$$\Phi_j(t) = \int_a^b \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| ds, \qquad a \leqq t \leqq b, j = 0, 1, \dots, n-1 \tag{14}$$

are continuous in $[a, b]$.

Definition 2. The system of nonnegative continuous scalar functions $\mathscr{C}_j$ in $[a, b]$, $j = 0, 1, \dots, n-1$, is called admissible (with respect to the Green function G) if there exist positive constants k_j, $j = 0, 1, \dots, n-1$, such that

$$\Phi_j(t) \leqq k_j \mathscr{C}_j(t), \qquad a \leqq t \leqq b, \qquad j = 0, 1, \dots, n-1. \tag{15}$$

With respect to boundedness of the functions $\mathscr{C}_j$, $j = 0, 1, \dots, n-1$, there exist positive constants $\bar{k}_{l,j}$, $l, j = 0, 1, \dots, n-1$, such that

$$\int_a^b \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| \mathscr{C}_l(s)\, ds \leqq \bar{k}_{l,j} \mathscr{C}_j(t), \qquad a \leqq t \leqq b, \qquad l, j = 0, 1, \dots, n-1. \tag{16}$$

Let $k_{l,j} = \inf \bar{k}_{l,j}$, $l, j = 0, 1, \dots, n-1$. Then (16) is also true for $k_{l,j}$, l, $j = 0, 1, \dots, n-1$.

Denote

$$K_l = \max(k_{l,0}, k_{l,1}, \dots, k_{l,n-1}), l = 0, 1, \dots, n-1. \tag{17}$$

Hence

$$\int_a^b \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| \mathscr{C}_l(s)\, ds \leqq K_l \mathscr{C}_j(t), \qquad a \leqq t \leqq b, \qquad l, j = 0, 1, \dots, n-1. \tag{18}$$

By definition of K_l, for a constant $\bar{K}_l < K_l$ the inequality (18) cannot hold for all $t \in [a, b]$, $j = 0, 1, \dots, n-1$. The constants K_j, $j = 0, 1, \dots, n-1$, will be called the associated system of constants to the admissible system $\mathscr{C}_j$, $j = 0, 1, \dots, n-1$. Hence, the following definition will be of use.

Definition 3. The system of smallest nonnegative constants K_j, $j = 0, 1, \dots, n-1$, such that (18) are true for all $t \in [a, b]$, $l, j = 0, 1, \dots, n-1$, will be called associated system of constants to the admissible system $\mathscr{C}_j$, $j = 0, 1, \dots, n-1$.

Its meaning is explained in the following theorem.

Theorem 2. *Let $\mathscr{C}_j$, $j = 0, 1, \dots, n-1$, be an admissible system and K_j, $j = 0, 1, \dots, n-1$, the associated system of constants to that system. Let the function f satisfy Lipschitz condition* (9) *with nonnegative matrices $L_k \in M_{d \times d}$, $k =$*

$= 0, 1, \ldots, n-1$. *Then for any* $a_{i,j} \in R^d$, $i = 1, \ldots, r_j$, $j = 1, 2, \ldots, m$, *there exists a unique solution to* (1), (2) *provided*

$$\varrho\Big(\sum_{k=0}^{n-1} K_k L_k\Big) < 1. \tag{19}$$

Proof. The problem (1), (2) is equivalent to the equation

$$x(t) = w(t) + \int_a^b G(t, s)\, f[s, x(s), x'(s), \ldots, x^{(n-1)}(s)]\, \mathrm{d}s, \qquad a \leqq t \leqq b,$$

where $w : [a, b] \to R^d$ is the unique solution to (1′), (2). Now we define the operator T on $S = C^{n-1}([a, b], R^d)$ by

$$Tx(t) = w(t) + \int_a^b G(t, s)\, f[s, x(s), x'(s), \ldots, x^{(n-1)}(s)]\, \mathrm{d}s, \qquad a \leqq t \leqq b. \tag{20}$$

Clearly $T : S \to S$.

The space S will be provided by the generalized norm

$$\| x \| = \max\big(\max_{a \leqq t \leqq b} | x(t) |, \ldots, \max_{a \leqq t \leqq b} | x^{(n-1)}(t) |\big),$$

whereby $\max(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n \in R^d$ is defined componentwise, i.e.

if $x_i = (x_{1i}, \ldots, x_{di})^T$, $i = 1, \ldots, n$, then $\max(x_1, \ldots, x_n) = (\max_{i=1,\ldots,n} x_{1i}, \ldots, \ldots, \max_{i=1,\ldots,n} x_{di})^T$.

Clearly $\max(x_1, \ldots, x_n) \geqq x_i$ for each $i = 1, \ldots, n$.

$(S, \| . \|)$ is a generalized Banach space. We shall show that Lemma 1 can be applied to the operator T given by (20). To that aim denote

$$K = \max_{j=0,1,\ldots,n-1} k_j, \tag{21}$$

where k_j, $j = 0, 1, \ldots, n-1$, are arbitrarily chosen but fixed numbers satisfying (15).

Let $u, v \in S$ and let $j \in \{0, 1, \ldots, n-1\}$. Then, on the basis of (20), (9), (15) and (21) we obtain the following inequalities. First

$$| T^{(j)}(u)(t) - T^{(j)}(v)(t) | \leqq$$
$$\leqq \int_a^b \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| \Big(\sum_{k=0}^{n-1} L_k\, | u^{(k)}(s) - v^{(k)}(s) |\Big)\, \mathrm{d}s \leqq$$
$$\leqq K \mathscr{C}_j(t) \sum_{k=0}^{n-1} L_k\, \| u - v \|, \qquad a \leqq t \leqq b.$$

Suppose that for a natural p the inequality

$$(22)\qquad |(T^p)^{(j)}(u)(t) - (T^p)^{(j)}(v)(t)| \leqq$$
$$\leqq K\mathscr{C}_j(t)(\sum_{k=0}^{n-1} K_kL_k)^{p-1}\sum_{k=0}^{n-1} L_k \|u - v\|, \qquad a \leqq t \leqq b$$

is true. Then using (20), (9), (22), (16), (17) we come to the inequalities

$$|(T^{p+1})^{(j)}(u)(t) - (T^{p+1})^{(j)}(v)(t)| \leqq$$
$$\leqq \int_a^b \left| \frac{\partial^j G(t,s)}{\partial t^j} \right| (\sum_{k=0}^{n-1} L_k K\mathscr{C}_k(s)) [(\sum_{k=0}^{n-1} K_kL_k)^{p-1} \sum_{k=0}^{n-1} L_k \|u - v\|] \,\mathrm{d}s =$$
$$= K\mathscr{C}_j(t)(\sum_{k=0}^{n-1} K_kL_k)^p \sum_{k=0}^{n-1} L_k \|u - v\|, \qquad a \leqq t \leqq b.$$

Hence, by induction, we get that (22) is true for all natural p. The inequality (22) implies

$$\|T^p(u) - T^p(v)\| \leqq$$
$$\leqq K[\max_{j=0,1,\dots,n-1} (\max_{a\leqq t\leqq b} \mathscr{C}_j(t))](\sum_{k=0}^{n-1} K_kL_k)^{p-1} \sum_{k=0}^{n-1} L_k \|u - v\|.$$

By (19), $\lim_{p\to\infty} (\sum_{k=0}^{n-1} K_kL_k)^{p-1} = 0_d$ and hence there exists a number P_0 such that for all $p \geqq P_0$

$$\varrho(K[\max_{j=0,1,\dots,n-1} (\max_{a\leqq t\leqq b} \mathscr{C}_j(t))](\sum_{k=0}^{n-1} K_kL_k)^{p-1} \sum_{k=0}^{n-1} L_k \|u - v\|) < 1.$$

Lemma 1 then implies that the operator T has a unique fixed point in S which gives the statement of the theorem.

Corollary 1. *Let the function f satisfy Lipschitz condition* (9) *with nonnegative matrices* $L_k \in M_{d\times d}$, $k = 0, 1, \dots, n - 1$. Let

$$(23)\qquad c_k = \max_{a\leqq t\leqq b} \int_a^b \left| \frac{\partial^k G(t,s)}{\partial t^k} \right| \mathrm{d}s, \qquad k = 0, 1, \dots, n - 1.$$

Then for any $a_{i,j} \in R^d$, $i = 1, \dots, r_j$, $j = 1, 2, \dots, m$, there exists a unique solution of (1), (2) provided

$$(24)\qquad \varrho(\sum_{k=0}^{n-1} c_kL_k) < 1.$$

Proof. Clearly the functions Φ_j, $j = 0, 1, \dots, n - 1$, given by (14), form an admissible system of functions. As

$$(25)\qquad \int_a^b \left| \frac{\partial^j G(t,s)}{\partial t^j} \right| \Phi_l(s)\,\mathrm{d}s \leqq c_l\Phi_j(t), \qquad a \leqq t \leqq b, j, l = 0, 1, \dots, n - 1,$$

the associated system of constants K_l, $l = 0, 1, \ldots, n-1$ to that admissible system of functions fulfils the relation

$$K_l \leqq c_l, l = 0, 1, \ldots, n-1.$$

Thus $\varrho(\sum_{k=0}^{n-1} K_k L_k) \leqq \varrho(\sum_{k=0}^{n-1} c_k L_k) < 1$ and, by Theorem 2, the statement of Corollary 1 follows.

OPTIMAL VALUES OF THE ASSOCIATED SYSTEM OF CONSTANTS

As we have seen the set of all admissible systems of functions is not ampty and in view of Theorem 2 the problem arises what are the best (smallest) constants K_k, $k = 0, 1, \ldots, n-1$, for this set. The answer to this question can be given by applying the theory of positive linear operators developed by M. A. Krasnosel'skij and others. This theory will be taken from the survey paper by H. Amann [3], books by M. A. Krasnosel'skij and others [8], [9].

Consider the Banach space $E = C([a, b], R)$ with the supnorm, partially ordered by the relation $x \leqq y$ iff $x(t) \leqq y(t)$ for all $t \in [a, b]$. Then $(E, \leqq)$ is an ordered Banach space with positive cone $P = \{x \in E : x(t) \geqq 0, a \leqq t \leqq b\}$. P is normal, i.e. every order interval $[x, y] = \{z \in E : x \leqq z \leqq y\}$ is bounded, and P is generating, i.e. $E = P - P$ ([3], pp. 625–628).

Let $k \in \{0, 1, \ldots, n-1\}$ and let G be the Green function of the scalar problem (1″), (2″). Define the operator $A_k : E \to E$ by

$$A_k x(t) = \int_a^b \left| \frac{\partial^k G(t, s)}{\partial t^k} \right| x(s) \, ds, \qquad a \leqq t \leqq b, \ x \in E. \tag{26}$$

A_k is a positive linear operator and using Ascolli lemma we can easily prove that it is completely continuous. If $\mathscr{C}_k$ belongs to an admissible system of functions $\mathscr{C}_j$, $j = 0, 1, \ldots, n-1$, then the operator A_k is $\mathscr{C}_k$-bounded from above ([9], p. 78), because for any $x \in P$, $x \neq 0$, there exists a constant $c = c(x) > 0$ such that

$$\begin{aligned} A_k x(t) = \int_a^b \left| \frac{\partial^k G(t, s)}{\partial t^k} \right| x(s) \, ds &\leqq (\max_{a \leqq t \leqq b} x(t)) \, \Phi_k(t) \leqq \\ &\leqq (\max_{a \leqq t \leqq b} x(t)) \, k_k \mathscr{C}_k(t) = c(x) \, \mathscr{C}_k(t), \qquad a \leqq t \leqq b. \end{aligned} \tag{27}$$

At the same time we have shown that A_k is Φ_k-bounded from above.

Further we need theorems on the estimate of spectral radius of a positive linear operator. A part of Theorem 5.3 from [9], p. 85, will be stated here as

Lemma 2. *Let $(E_1, \leqq)$ be an ordered Banach space with positive cone P_1 which is normal and generating. Let $A : E_1 \to E_1$ be a positive linear operator such that there* *is an element* $x_0 \in P_1$, $x_0 \neq 0$, *and a constant* $K \geqq 0$ *with*

$$Ax_0 \leqq Kx_0.$$

Let A *be* x_0*-bounded from above or* x_0 *be an inner point of* P_1.

Then the spectral radius $\varrho(A)$ *of* A *satisfies*

$$\varrho(A) \leqq K.$$

Remark 1. In fact, the lemma has been proved for $K > 0$, but by a limit process we get its validity also in the case $K = 0$.

With respect to this lemma, $\varrho(A_k)$ can be estimated from above where A_k is the operator defined by (26), by the relation

$$\varrho(A_k) \leq \max_{a \leq t \leq b} \left[\left(\int_a^b \left| \frac{\partial^k G(t, s)}{\partial t^k} \right| x(s) \, ds \right) / x(t) \right]$$

for any function $x \in C([a, b], R)$, $x(t) > 0$ in $[a, b]$, because such a function is an inner point of P.

Especially,

$$\varrho(A_k) \leqq c_k, \tag{28}$$

c_k being defined by (23).

Lemma 2 also implies that the following lemma is true.

Lemma 3. *Let* $\mathscr{C}_j$, $j = 0, 1, \ldots, n - 1$, *be an admissible system of functions and let* K_j, $j = 0, 1, \ldots, n - 1$, *be the associated system of constants to that admissible system. Let* A_k *be the operator given by* (26). *Then*

$$K_j \geqq \varrho(A_j), \qquad j = 0, 1, \ldots, n - 1. \tag{29}$$

We shall show that for each $k \in \{0, 1, \ldots, n - 1\}$ there is an admissible system of functions such that the constant K_k from the associated system of constants is equal to $\varrho(A_k)$. In the proof we need some theorems on positive linear operators. First we state Theorem 5.4 from [9], p. 81, as Lemma 4.

Lemma 4. *Let* $(E_1, \leqq)$ *be an ordered Banach space with positive cone* P_1 *which is generating. Let* $A : E_1 \to E_1$ *be a positive linear operator such that there is an element* $x_1 \in P_1$, $x_1 \neq 0$, and a constant $\bar{K} \geqq 0$ with

$$Ax_1 \geqq \bar{K}x_1.$$

Then the spectral radius $\varrho(A)$ *of* A *satisfies*

$$\varrho(A) \geqq \bar{K}.$$

In view of this lemma, $\varrho(A_k) > 0$ will hold for the operator A_k defined by (26) if

$$\min_{a \leqq t \leqq b} \int_a^b \left| \frac{\partial^k G(t, s)}{\partial t^k} \right| ds > 0,$$

since $\varrho(A_k) \geqq \min_{a \leqq t \leqq b} \left[\left(\int_a^b \left| \frac{\partial^k G(t, s)}{\partial t^k} \right| x(s) \, ds \right) / x(t) \right]$ for any $x \in C([a, b], R), x(t) > 0$ in $[a, b]$.

Especially,

$$\varrho(A_k) \geqq \min_{a \leqq t \leqq b} \int_a^b \left| \frac{\partial^k G(t, s)}{\partial t^k} \right| ds. \tag{30}$$

Part of the famous Krejn—Rutman theorem ([3], p. 632, Theorem 3.1) is stated here as Lemma 5.

Lemma 5. *Let $(E_1, \leqq)$ be an ordered Banach space with positive cone P_1. Let P_1 be total,* i.e. *the closure $\overline{P_1 - P_1} = E_1$. Suppose that $A : E_1 \to E_1$ is a positive linear, completely continuous operator and it has a positive spectral radius $\varrho(A)$. Then $\varrho(A)$ is an eigenvalue of A with an eigenvector in P_1.*

We also need the definition of a u_0-positive operator ([9], p. 77). If $(E_1, \leqq)$ is an ordered Banach space with positive cone P_1, $A : E_1 \to E_1$ is a linear positive operator, and $u_0 \in P_1$, $u_0 \neq 0$, then we say that A is u_0-positive, if for each nonzero element $x \in P_1$ there exist constants $\alpha(x) > 0$, $\beta(x) > 0$ such that

$$\alpha(x)\, u_0 \leqq Ax \leqq \beta(x)\, u_0 .$$

In other words, A is u_0-bounded from below as well as u_0-bounded from above.

Lemmas 4, 5 and Theorem 2.11 in [8], p. 80, imply the following lemma on existence and uniqueness of the positive eigenvector.

Lemma 6. *Let $(E_1, \leqq)$ be an ordered Banach space with positive cone P_1. Let P_1 be generating and let $A : E_1 \to E_1$ be a positive linear, completely continuous, operator which is u_0-positive for an element $u_0 \in P_1$, $u_0 \neq 0$. Then there exists up to a multiplicative positive constant a unique eigenvector of A in P_1 and that vector corresponds to $\varrho(A)$.*

Proof. As A is u_0-bounded from below, by Lemma 4 the spectral radius $\varrho(A)$ of A is positive. Hence, by Lemma 5, there exists an eigenvector of A belonging to P_1 which corresponds to $\varrho(A)$. Now we apply Theorem 2.11 in [8], all assumptions of which are satisfied. By this theorem the uniqueness follows.

By Lemma 2.1 (p. 77) as well as by Theorem 2.16 (p. 90) in [8] we come to the following statement.

Lemma 7. *Let the assumptions of Lemma 6 be satisfied. Then for every $y \in P_1$, $y \neq 0$, the equation*

$$\lambda x - Ax = y$$

has exactly one solution $x \in P_1$ if $\lambda > \varrho(A)$ and no solution in P_1 for $\lambda \leqq \varrho(A)$.

Let $k \in \{0, 1, \ldots, n - 1\}$. We summon fundamental properties of the operator A_k which is defined by (26). E, P, $\varrho(A_k)$, Φ_k have the same meaning as above. We remind that P is normal and generating.

Lemma 8. $A_k : E \to E$ *is a positive linear, completely continuous operator.* Further:

(a) A_k *is* Φ_k*-positive.*

(b) $\varrho(A_k) > 0$.

(c) *There exists a unique, up to a multiplicative positive constant, eigenfunction* $\mathscr{C}_k$ *of* A_k *belonging to* P. $\mathscr{C}_k$ *corresponds to* $\varrho(A_k)$ *and* A_k *is* $\mathscr{C}_k$*-positive.*

(d) $\mathscr{C}_k \equiv 0$ *is not true on any subinterval of* $[a, b]$.

Proof. Our considerations will be based on the following property of the Green function G for the scalar problem (1″), (2″) (see [10], p. 375). It is true that

$$(31) \qquad \frac{\partial^j G(t_0, s)}{\partial t^j} \equiv 0 \qquad \text{for a } t_0 \in [a, b] \text{ and all } s, a \leqq s \leqq b, j \in \{0, 1, \dots, n-1\}$$

if

(32) there is an $l \in \{1, 2, \dots, m\}$ such that $t_0 = t_l$, $0 \leqq j \leqq r_l - 1$. Conversely, if (31) is valid, then for all functions $y \in C^n([a, b], R)$ satysfying (2″) we must have $y^{(j)}(t_0) = 0$ which is only so possible when (32) is true. Thus (31), (32) are equivalent each to other.

Statement (a). We have already found (see (27)) that A_k is Φ_k-bounded from above. Since for each $x \in P$, $x \not\equiv 0$, $A_k x(t) \equiv 0$ does not hold on any subinterval $[a_1, b_1] \subset [a, b]$, the Φ_k-boundedness of A_k from below will be shown if to any function $x \in P$, $x \not\equiv 0$ on any subinterval $[a_1, b_1] \subset [a, b]$ there exists a constant $\alpha(x) > 0$ such that

$$(33) \qquad A_k x(t) \geqq \alpha(x)\, \Phi_k(t), \qquad a \leqq t \leqq b.$$

By (14), $\Phi_k(t) = (A_k\, 1)(t)$, $a \leqq t \leqq b$. Hence, the equivalence between (31) and (32) implies that $\Phi_k(t_0) = 0$ iff there exists an $l \in \{1, 2, \dots, m\}$ such that $t_0 = t_l$ and $k \leqq r_l - 1$. Therefore (34) $\Phi_k(t) > 0$ for all $t \in [a, b]$, $t \neq t_j$, $j = 1, 2, \dots, m$, as well as for $t = t_l$ such that $k \geqq r_l$.

If we show that any point t_l with $k \leqq r_l - 1$, $l = 1, 2, \dots, m$, there exists a one-sided limit

$$\lim_{t \to t_l+} A_k x(t)/\Phi_k(t), \qquad \lim_{t \to t_l-} A_k x(t)/\Phi_k(t)$$

and it is different from 0 (it is positive), then (33) is true with a positive constant $\alpha(x)$ and the proof of (a) is complete.

Suppose $1 \leqq l < m$, $k \leqq r_l - 1$, $t_l < t$. By Taylor's formula for any $a \leqq s \leqq b$ there exists a $\tau(s)$, $t_l < \tau(s) < t$, such that

$$\frac{\partial^k G(t, s)}{\partial t^k} = \frac{\partial^k G(t_l, s)}{\partial t^k} + \frac{\partial^{k+1} G(t_l, s)}{\partial t^{k+1}} \frac{(t - t_l)}{1!} + \dots + \\ + \frac{\partial^{r_l} G(\tau(s), s)}{\partial t^{r_l}} \frac{(t - t_l)^{r_l - k}}{(r_l - k)!}$$

and, in view of (31),

$$\frac{\partial^k G(t, s)}{\partial t^k} = \frac{\partial^{r_l} G(\tau(s), s)}{\partial t^{r_l}} \frac{(t - t_l)^{r_l - k}}{(r_l - k)!}, \qquad a \leqq s \leqq b.$$

Hence

$$\lim_{t \to t_l +} \left[\left(\int_a^b \left| \frac{\partial^k G(t, s)}{\partial t^k} \right| x(s)\, ds \right) \Big/ \left(\int_a^b \left| \frac{\partial^k G(t, s)}{\partial t^k} \right| ds \right) \right] =$$

$$= \int_a^b \left| \frac{\partial^{r_l} G(t_1, s)}{\partial t^{r_l}} \right| x(s)\, ds \Big/ \int_a^b \left| \frac{\partial^{r_l} G(t_l, s)}{\partial t^{r_l}} \right| ds$$

exists, and by (34), it is finite and different from zero. A similar result follows for $1 < l \leqq m$ and for $\lim_{t \to t_l -} A_k x(t)/\Phi_k(t)$.

Statement (b). By (a), there exists a $\bar{K} > 0$ such that $A_k \Phi_k \geqq \bar{K} \Phi_k$. On the basis of Lemma 4, this implies that $\varrho(A_k) > 0$.

Statement (c). The first part of this statement follows from Lemma 6, all assumptions of which are satisfied. By (a), for each $x \in P$, $x \not\equiv 0$, there exist two constants $\beta(x) \geqq \alpha(x) > 0$ such that

$$\alpha(x)\, \Phi_k \leqq A_k x \leqq \beta(x)\, \Phi_k. \tag{35}$$

Hence, there are $\beta_k \geqq \alpha_k$ such that

$$\alpha_k \Phi_k \leqq A_k \mathscr{C}_k \leqq \beta_k \Phi_k.$$

Since $A_k \mathscr{C}_k = \varrho(A_k) \mathscr{C}_k$, we have

$$\frac{\alpha_k}{\varrho(A_k)} \Phi_k \leqq \mathscr{C}_k \leqq \frac{\beta_k}{\varrho(A_k)} \Phi_k \tag{36}$$

and thus, with respect to (35), (36), we get that

$$\frac{\alpha(x)\, \varrho(A_k)}{\beta_k} \mathscr{C}_k \leqq A_k x \leqq \frac{\beta(x)\, \varrho(A_k)}{\alpha_k} \mathscr{C}_k$$

which means that A_k is $\mathscr{C}_k$-positive.

Statement (d). If $\mathscr{C}_k(t) \equiv 0$ were true on a subinterval of $[a, b]$, then by (36) the same would hold for Φ_k. But this is in contradiction with (34) which proves (d).

On the basis of Lemma 8 we prove the following theorem.

Theorem 3. *Let $k \in \{0, 1, \ldots, n - 1\}$ and let $\mathscr{C}_k$ be a nonnegative eigenfunction of the operator A_k. Then the functions*

$$\mathscr{C}_j(t) = \frac{1}{\varrho(A_k)} \int_a^b \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| \mathscr{C}_k(s)\, ds, \tag{37}$$

$$a \leqq t \leqq b, \qquad j = 0, 1, \ldots, n - 1,$$

form an admissible system of functions with respect to G such that for the associated system of constants K_j, $j = 0, 1, \ldots, n-1$,

$$K_k = \varrho(A_k) \tag{38}$$

is true.

Proof. The functions $\mathscr{C}_j$ determined by (37) are all continuous and nonnegative in $[a, b]$. Clearly $\mathscr{C}_k$ satisfies (37) for $j = k$.

First we show that the functions $\mathscr{C}_j$, $j = 0, 1, \ldots, n-1$, form an admissible system of functions with respect to G. In agreement with (26), we define the operator $A_j : E \to E$ by

$$A_j x(t) = \int_a^b \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| x(s)\, ds, \qquad a \leqq t \leqq b, x \in E, j = 0, 1, \ldots, n-1. \tag{39}$$

By (14), (15), (37) and (39) we have to find such constants $k_j > 0, j = 0, 1, \ldots, n-1$, that

$$\Phi_j(t) = (A_j 1)(t) \leqq \frac{k_j}{\varrho(A_k)} A_j \mathscr{C}_k(t), \qquad a \leqq t \leqq b. \tag{40}$$

But the proof of (40) runs in the same way as the proof of (33). Hence we can assert that the existence of $k_j > 0$, $j = 0, 1, \ldots, n-1$, with property (40) is guaranteed.

Finally we prove (38). In virtue of (17), (16) and (37)

$$K_k = \max(k_{k,0}, k_{k,1}, \ldots, k_{k,n-1}) \geqq \varrho(A_k).$$

On other hand, (29) gives an opposite inequality and hence, (38) is true.

APPLICATION TO SECOND ORDER SYSTEMS

The obtained results in Theorems 2, 3 will be applied to the vector boundary value problem of the second order

$$x'' = f(t, x, x'), \tag{41}$$

$$x(a) = a_{1,1}, \qquad x(b) = a_{1,2}. \tag{42}$$

Suppose that $f \in C([a, b] \times R^{2d}, R^d)$ satisfies the Lipschitz condition

$$|f(t, u_0, u_1) - f(t, v_0, v_1)| \leqq L_0 |u_0 - v_0| + L_1 |u_1 - v_1| \tag{43}$$

where $L_0, L_1 \in M_{d \times d}$ are nonnegative matrices.

Let G_1 be the Green function for the corresponding scalar homogeneous problem

$$x'' = 0, \qquad x(a) = 0, \qquad x(b) = 0.$$

Then

$$G_1(t, s) = -\begin{cases} \dfrac{(b-t)(s-a)}{b-a}, & a \leqq s \leqq t \leqq b, \\ \dfrac{(b-s)(t-a)}{b-a}, & a \leqq t \leqq s \leqq b \end{cases}$$

$$\frac{\partial G_1(t, s)}{\partial t} = \begin{cases} \dfrac{s-a}{b-a}, & a \leqq s \leqq t \leqq b, \\ \dfrac{s-b}{b-a}, & a \leqq t \leqq s \leqq b. \end{cases}$$

Consider the functions

$$\bar{\Phi}_0(t) = \int_a^b |G_1(t, s)|\, ds, \qquad \bar{\Phi}_1(t) = \int_a^b \left|\frac{\partial G_1(t, s)}{\partial t}\right| ds, \qquad a \leqq t \leqq b.$$

Similarly as the functions (14), they form an admissible system of functions with respect to G_1. By [2], p. 2,

$$\bar{\Phi}_0(t) = \frac{1}{2}(t-a)(b-t), \tag{44}$$

$$\bar{\Phi}_1(t) = [(t-a)^2 + (b-t)^2]/[2(b-a)], \qquad a \leqq t \leqq b.$$

The constants $\bar{c}_0, \bar{c}_1$ determined by (23), are

$$\bar{c}_0 = \max_{a \leqq t \leqq b} \bar{\Phi}_0(t) = \frac{1}{8}(b-a)^2, \qquad \bar{c}_1 = \max_{a \leqq t \leqq b} \bar{\Phi}_1(t) = \frac{1}{2}(b-a).$$

Hence, similarly as in the proof of Corollary 1, we get that the associated system of constants $\bar{K}_0, \bar{K}_1$ satisfies

$$\bar{K}_0 \leqq \frac{1}{8}(b-a)^2, \qquad \bar{K}_1 \leqq \frac{1}{2}(b-a). \tag{45}$$

Comparing the inequalities (iv), (ix), (vi), (xi) in [2], p. 3, with (16), we get that

$$\bar{k}_{0,0} = \frac{5}{48}(b-a)^2, \qquad \bar{k}_{0,1} = \frac{\sqrt{3}-1}{4\sqrt{3}}(b-a)^2,$$

$$\bar{k}_{1,0} = \frac{8}{25}(b-a), \qquad \bar{k}_{1,1} = \frac{1}{3}(b-a)$$

and

$$\max(\bar{k}_{0,0}, \bar{k}_{0,1}) = \frac{\sqrt{3}-1}{4\sqrt{3}}(b-a)^2,$$

$$\max(\bar{k}_{1,0}, \bar{k}_{1,1}) = \frac{1}{3}(b-a).$$

Hence, in view of (17),

$$K_0 \leqq \frac{\sqrt{3}-1}{4\sqrt{3}}(b-a)^2, \qquad K_1 \leqq \frac{1}{3}(b-a). \tag{46}$$

Because the estimates in (46) are better than the estimates in (45), on the basis of Theorem 2 we get condition (8) in [2], p. 1, i.e. if

$$\varrho\left(\frac{\sqrt{3}-1}{4\sqrt{3}}(b-a)^2 L_0 + \frac{1}{3}(b-a) L_1\right) < 1, \tag{47}$$

then the problem (41), (42) has a unique solution for any two vectors $a_{1,1}, a_{1,2} \in \in R^d$.

Let E, P have the same meaning as above. Consider, now, the operators $\bar{A}_0, \bar{A}_1 : E \to E$, which according to (39) are defined as follows:

$$\bar{A}_0 x(t) = \int_a^b |G_1(t,s)| \, x(s) \, ds, \qquad a \leqq t \leqq b,$$

$$\bar{A}_1 x(t) = \int_a^b \left| \frac{\partial G_1(t,s)}{\partial t} \right| x(s) \, ds, \qquad a \leqq t \leqq b.$$

As $G_1(t,s) \leqq 0$ in $[a,b] \times [a,b]$, the eigenvalue problem

$$\bar{A}_0 x = \lambda x, \qquad \lambda \neq 0$$

is equivalent to the problem

$$x'' = -\frac{1}{\lambda} x, \qquad x(a) = 0, \qquad x(b) = 0,$$

the eigenvalues of which are

$$\lambda_k = \frac{(b-a)^2}{k^2 \pi^2}, \qquad k = 1, 2, \ldots,$$

and the corresponding eigenfunctions uniquely determined up to a multiplicative constant are

$$x_k(t) = \sin \frac{k\pi(t-a)}{b-a}, \qquad k = 1, 2, \ldots$$

By Lemma 8, $\varrho(\bar{A}_0) = \lambda_1 = (b-a)^2/\pi^2$. The corresponding eigenfunction is

$$\bar{\mathscr{C}}_0(t) = \sin \frac{\pi(t-a)}{b-a}, \qquad a \leqq t \leqq b.$$

By (33) and (x) in [2], pp. 2–3,

$$\bar{\mathscr{C}}_1(t) = \frac{\pi^2}{(b-a)^2} \int_a^b \left| \frac{\partial G_1(t,s)}{\partial t} \right| \bar{\mathscr{C}}_0(s) \, ds =$$

$$= \frac{2}{b-a} \sin \frac{\pi(t-a)}{b-a} + \frac{\pi(b-2t+a)}{(b-a)^2} \cos \frac{\pi(t-a)}{b-a}.$$

By Theorem 3, $\bar{\mathscr{C}}_0, \bar{\mathscr{C}}_1$ form an admissible system of functions with respect to G_1 and for the associated system of constants $\bar{\bar{K}}_0, \bar{\bar{K}}_1$

$$\bar{\bar{K}}_0 = \varrho(\bar{A}_0) = \frac{(b-a)^2}{\pi^2}$$

is true. By the inequalities (v), (x), (vii), (xii) in [2], p. 3, we get that

$$\bar{\bar{k}}_{0,0} = \frac{b-a)^2}{\pi^2}, \qquad \bar{\bar{k}}_{0,1} = \frac{(b-a)^2}{\pi^2},$$

$$\bar{\bar{k}}_{1,0} = \frac{4}{\pi^2}(b-a), \qquad \bar{\bar{k}}_{1,1} = \frac{4}{\pi^2}(b-a).$$

Hence

$$\bar{\bar{K}}_0 = \frac{1}{\pi^2}(b-a)^2, \qquad \bar{\bar{K}}_1 = \frac{4}{\pi^2}(b-a).$$

Using these results we get condition (6) in Theorem in [2], p. 1, i.e.

$$\varrho\left(\frac{1}{\pi^2}(b-a)^2 L_0 + \frac{4}{\pi^2}(b-a) L_1\right) < 1, \tag{48}$$

which is sufficient for the existence of a unique solution to (41), (42) for any two vectors $a_{1,1}, a_{1,2} \in R^d$.

Consider now the operator $\bar{A}_1$. By (30), (28) we obtain that the spectral radius $\varrho(\bar{A}_1)$ of $\bar{A}_1$ satisfies

$$\frac{1}{4}(b-a) = \min_{a \leq t \leq b} \bar{\Phi}_1(t) \leq \varrho(\bar{A}_1) = \max_{a \leq t \leq b} \bar{\Phi}_1(t) = \frac{1}{2}(b-a), \tag{49}$$

where $\bar{\Phi}_1$ is given by (44). By Lemma 8, there is a uniquely determined (up to a positive constant) positive eigenfunction $\bar{\bar{\mathscr{C}}}_1$ of $\bar{A}_1$ corresponding to $\varrho(\bar{A}_1)$. By (37), we define

$$\bar{\bar{\mathscr{C}}}_0(t) = \frac{1}{\varrho(\bar{A}_1)} \int_a^b |G_1(t,s)| \bar{\bar{\mathscr{C}}}_1(s) \, ds, \qquad a \leq t \leq b.$$

Then, by Theorem 3, $\bar{\bar{\mathscr{C}}}_0, \bar{\bar{\mathscr{C}}}_1$ form an admissible system of functions whereby for the associated system of constants K_0^*, K_1^* we have $K_1^* = \varrho(\bar{A}_1)$. The inequality (49) can be improved. By the inequality (xi) in [2], p. 3, as well as by the fact that $\min_{a \leq t \leq b} \bar{\Phi}_1(t) = \frac{1}{4}(b-a) > 0$, Lemma 2 implies that $\bar{\varrho}(A_1) \leq \frac{1}{3}(b-a)$. By (44), $\bar{\Phi}_1$ cannot be an eigenfunction of $\bar{A}_1$. Hence in (xi) the sign of equality cannot hold. Thus the difference $\frac{1}{3}(b-a)\bar{\Phi}_1 - \bar{A}_1\bar{\Phi}_1 = y \geq 0$, where $y \in P, y \neq 0$. By Lemma 7, $\varrho(\bar{A}_1) < \frac{1}{3}(b-a)$. Thus

$$\frac{1}{4}(b-a) \leqq \varrho(\bar{A}_1) < \frac{1}{3}(b-a) \tag{50}$$

and we can state the result:

There exists a constant $K_0^* > 0$ such that

$$\varrho(K_0^* L_0 + \varrho(\bar{A}_1) L_1) < 1 \tag{51}$$

is a sufficient condition for the existence of a unique solution to (41), (42) for any two vectors $a_{1,1}, a_{1,2} \in R^d$.

The condition (51) is not contained among conditions (6), (7), (8) of Theorem in [2], p. 1.

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