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ON A VECTOR MULTIPOINT BOUNDARY VALUE PROBLEM

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Abstract. Existence and uniqueness of the solution to a multipoint boundary value problem for nth-order nonlinear differential systems is proved by using the estimates for derivatives of the scalar functions and by introducing admissible system of functions with respect to the Green function. The theory of positive linear operators gives that the obtained result is the best in a certain sense.

Key words. Vector multipoint boundary value problem, generalized Banach space, Green function, admissible system of functions, associated system of constants, Lipschitz condition, ordered Banach space with positive cone, positive eigenvalue.

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In the paper a multipoint boundary value problem for n-th order nonlinear differential systems is studied. It is shown how the facts and methods from the scalar case can be applied to the vector one. Existence theorems for De la Vallée Poussin problem are obtained by means of the estimates for derivatives of scalar functions or by introducing admissible system of functions with respect to the Green function. Here the theory of positive linear operators is applied. The obtained results extend and generalize some theorems proved by R. P. Agarwal and J. Vosmanský in [2].

In the paper the following vector multipoint boundary value problem

\( x^{(n)} = f(t, x, x', ..., x^{(n-1)}) \),

\( x^{(i-1)}(t_j) = a_{i,j}, \quad i = 1, ..., r_i, \quad j = 1, 2, ..., m \)

will be considered where

\( n \geq 2, 2 \leq m \leq n, 1 \leq r_i \) are natural numbers such that \( r_1 + r_2 + ... + r_m = n \),

\( -\infty < a = t_1 < ... < t_m = b < \infty \) are real numbers, \( a_{i,j} \in \mathbb{R}^d \) are vectors, \( d \geq 1 \) and throughout the whole paper we assume that

\( f \in C(D, \mathbb{R}^d) \) \quad where \quad \( D = [a, b] \times \mathbb{R}^d \times \mathbb{R}^d \times ... \times \mathbb{R}^d \) \quad \( n \) times
The scalar case ($d = 1$) has been thoroughly studied. See the monograph [7] by I. T. Kiguradze, [4] by S. R. Bernfeld and V. Laksmikantham. As to the vector case ($d > 1$), there are substantially less papers devoted to problem (1), (2). Among them the paper [2] written by R. P. Agarwal, J. Vosmansky attracted the attention. The ideas from that paper are deepened and developed here.

**Preliminaries**

If $x = (x_1, \ldots, x_d)^T$ is a column vector, then we denote $|x| = (|x_1|, \ldots, |x_d|)^T$. A partial ordering in $R^d$ can be introduced by the relation:

If $x = (x_1, \ldots, x_d)^T$, $y = (y_1, \ldots, y_d)^T$ belong to $R^d$, then $x \leq y$ iff $x_j \leq y_j$ for $j = 1, \ldots, d$.

Further we denote $u_d = (1, \ldots, 1)^T \in R^d$.

The set of all real $d \times d$ matrices will be denoted as $M_{d \times d}$. Similarly as in the case of vectors, if $L = (l_{ij})$, then $|L| = (|l_{ij}|)$, $i, j = 1, \ldots, d$. Further $L \leq L$ iff $l_{ij} \leq l_{ij}$ for $i, j = 1, \ldots, d$ and $L = (l_{ij})$, $L = (l_{ij})$. $U_d (0_d)$ will mean the matrix from $M_{d \times d}$, all elements of which are 1 (0). $E_d$ will denote the unit matrix. As usual, the spectral radius $\rho(L)$ of the matrix $L \in M_{d \times d}$ $\rho(L) = \max_i |\lambda_i|$ where $\lambda_i$ are all eigenvalues of $L$.

**Poussin Constants**

The first result is based on the following estimates for scalar functions. If $x \in C^n([a, b], R)$ satisfies

$$x^{(i-1)}(t_j) = 0, \quad i = 1, \ldots, r_j, \quad j = 1, 2, \ldots, m,$$

then there exist positive numbers $C_{n,k}$, $k = 0, 1, \ldots, n - 1$, such that

$$|x^{(k)}(t)| \leq C_{n,k}(b - a)^{n-k}\max_{a \leq t \leq b} |x^{(n)}(t)|, \quad a \leq t \leq b, \quad k = 0, 1, \ldots, n - 1.$$

The constants $C_{n,k}$ have been determined by several authors. E. g. G. A. Bessmertnych, A. Ju. Levin in [6] found that

$$C_{n,0} = \frac{(n - 1)^{n-1}}{n! n^n}, \quad C_{n,k} = \frac{k}{(n - k)! n^n}, \quad k = 1, \ldots, n - 1.$$

G. A. Bessmertnych in [5] has proved:

Let $l = \min \{r_1, r_m\} > 1$. Then
ON A VECTOR MULTIPOINT BOUNDARY VALUE PROBLEM

\[ C_{n, k} = \begin{cases} 
\frac{(n - k - 1)^{n-k-1}}{(n - k)! (n - k)^{n-k}}, & k = 0, 1, \ldots, l - 1, \\
\frac{k - l + 1}{(n-k)! (n - l + 1)}, & k = l, l + 1, \ldots, n. 
\end{cases} \]

R. P. Agarwal in [1] derived:

Let \( l = \min \{r_1, r_m\} \). Then

\[ C_{n, k} = \begin{cases} 
\frac{(n - l)^{n-l} (l - k)^{l-k}}{(n - k)! (n - k)^{n-k}}, & k = 0, 1, \ldots, l - 1, \\
\frac{k - l + 1}{(n-k)! (n - l + 1)}, & k = l, l + 1, \ldots, n - 1. 
\end{cases} \]

For their meaning we shall call \( C_{n, k} \) Poussin constants. Thus the following definition will be of use.

**Definition 1.** Any nonnegative constants \( C_{n, k}, k = 0, 1, \ldots, n - 1, \) such that for all functions \( x \in C^n([a, b], \mathbb{R}) \) satisfying (2') the estimates (3) hold will be called Poussin constants.

Further we shall use the concept of a generalized norm ([4], pp. 225–228). If \( E \) is a real vector space, then the generalized norm for \( E \) is a mapping \( \| . \|_G : E \to \mathbb{R}^d \) denoted by

\[ \| x \|_G = (\alpha_1(x), \ldots, \alpha_d(x))^T, \]

such that

(a) \( \| x \|_G \geq 0 \), that is \( \alpha_j(x) \geq 0 \) for \( j = 1, \ldots, d, x \in E; \)
(b) \( \| x \|_G = 0 \) iff \( x = 0; \)
(c) \( c \| x \|_G = \| c \| \| x \|_G, c \in \mathbb{R}, x \in E; \)
(d) \( \| x + y \|_G \leq \| x \|_G + \| y \|_G, x, y \in E. \)

The couple \((E, \| . \|_G)\) is then called a generalized normed space. The topology in this space is given in the following way. For each \( x \in E, \) and \( \delta > 0, \) let \( B_\delta(x) = \{ y \in E : \| y - x \|_G < \delta \mu_d \}. \) Then \( \{ B_\delta(x) : x \in E, \delta > 0 \} \) forms a basis for a topology on \( E. \) The same topology can be induced by the norm \( \| . \| \) which is defined in this way.

If \( \| x \|_G \) is given by (7), then

\[ \| x \| = \max (\alpha_1(x), \ldots, \alpha_d(x)), \quad x \in E. \]

\( \| . \| \) has all properties of the norm. Since the topology of the normed space \((E, \| . \|)\) is given by the basis of neighbourhoods \( V_\delta(x) = \{ y \in E : \| y - x \| < \delta \}, x \in E, \delta > 0, \) and \( V_\delta(x) = B_\delta(x), \) both (7) and (8) define the same topology on \( E \) and in this sense they are equivalent. From the topological point of view there is no need for introducing the generalized norm. Yet we have more flexibility when
working with generalized spaces. This is clear in the case of Contraction mapping theorem ([2], p. 2) which will be stated as

**Lemma 1.** Let \((E, \| \cdot \|_G)\) be a generalized Banach space and let \(T : E \to E\) be such that for all \(x, y \in E\) and for some positive integer \(p\)
\[
\| T^p x - T^p y \|_G \leq L \| x - y \|_G,
\]
where \(L \in M_{d \times d}\) is a nonnegative matrix with \(\varrho(L) < 1\) and \(T^p\) is the \(p\)-th iterate of \(T\). Then \(T\) has a unique fixed point.

**Theorem 1.** Let for all \((t, u_0, u_1, \ldots, u_{n-1}), (t, v_0, v_1, \ldots, v_{n-1}) \in D\) the function \(f\) satisfy Lipschitz condition
\[
|f(t, u_0, u_1, \ldots, u_{n-1}) - f(t, v_0, v_1, \ldots, v_{n-1})| \leq \sum_{k=0}^{n-1} L_k |u_k - v_k|,
\]
where \(L_k \in M_{d \times d}\) are nonnegative matrices. Let \(C_{n,k}, k = 0, 1, \ldots, n - 1,\) be Poussin constants. Let
\[
\varrho\left(\sum_{k=0}^{n-1} L_k C_{n,k}(b - a)^{n-k}\right) < 1.
\]

Then there exists a unique solution to (1), (2) for all \(a_{i,j} \in \mathbb{R}^d, \ i = 1, \ldots, r_j, j = 1, 2, \ldots, m.\)

**Proof.** First we transform the problem (1), (2) to a simpler one. Let \(w\) be the unique solution of (2),
\[
(1') \quad x^{(n)} = 0.
\]
Then \(x\) is a solution to (1), (2) iff the function \(y(t) = x(t) - w(t), \ a \leq t \leq b,\) is a solution to boundary value problem
\[
(11) \quad y^{(n)} = f(t, y + w(t), y' + w'(t), \ldots, y^{(n-1)} + w^{(n-1)}(t)) \equiv\]
\[
\equiv g(t, y, y', \ldots, y^{(n-1)}),
\]
\[
(2'') \quad y^{(i-1)}(t_j) = 0, \quad i = 1, \ldots, r_j, j = 1, 2, \ldots, m.
\]
The function \(g \in C(D, \mathbb{R}^d)\) and it satisfies Lipschitz condition (9). We shall show that problem (11), (2'') has a unique solution.

Denote for any \(x \in C([a, b], \mathbb{R}^d), x(t) = (x_1(t), \ldots, x_d(t))^T, \ a \leq t \leq b,\)
\[
\max_{a \leq t \leq b} |x(t)| = (\max_{a \leq t \leq b} |x_1(t)|, \ldots, \max_{a \leq t \leq b} |x_d(t)|)^T.
\]

Let \(S_1 = \{x \in C^n([a, b], \mathbb{R}^d) : x\) satisfies (2')\} where
\[
(2') \quad x^{(i-1)}(t_j) = 0, \quad i = 1, \ldots, r_j, j = 1, 2, \ldots, m.
\]
Then \(S_1\) is a real vector space and define the generalized norm on \(S_1\) by
ON A VECTOR MULTIPLE POINT BOUNDARY VALUE PROBLEM

\[ \| x \|_1 = \max_{a \leq t \leq b} |x^{(n)}(t)| \quad \text{for each } x \in S_1. \]

The properties of the generalized norm can be easily checked. E.g. \( \| x \|_1 = 0 \) implies that \( x \) is a solution to \((1'), (2')\) and hence \( x = 0. S_1 \) is a complete generalized space and thus a generalized Banach space. In fact if \( \{x_p\} \) is a Cauchy sequence, then there is a function \( y \in C([a, b], \mathbb{R}^n) \) such that \( |x_p^{(n)}| \) converge uniformly to \( y \) on \([a, b]\). The problem \( x^{(n)} = y(t), (2') \), has a unique solution \( x \in S_1 \) and \( \lim_{p \to \infty} \| x_p - x \|_1 = 0. \)

Define the mapping \( T_1 : S_1 \to S_1 \) as follows: If \( y \in S_1 \), let \( T_1 y \) be the solution \( x \) to boundary value problem \((2')\),

\[ x^{(n)} = g(t, y(t), y'(t), \ldots, y^{(n-1)}(t)). \]

Then by (9), for any two functions \( y, z \in S_1 \) we have

\[ (12) \quad |(T_1 y)^{(n)}(t) - (T_1 z)^{(n)}(t)| \leq \sum_{k=0}^{n-1} L_k |y^{(k)}(t) - z^{(k)}(t)|, \quad a \leq t \leq b. \]

Denote the \( j \)-th coordinate of the function \( y \) and \( z \), respectively, by \( y_j \) and \( z_j \), respectively. As by (3)

\[ |y_j^{(k)}(t) - z_j^{(k)}(t)| \leq C_{n, k}(b - a)^{n-k} \max_{a \leq t \leq b} |y_j^{(n)}(t) - z_j^{(n)}(t)| \]

it follows that

\[ (13) \quad |y^{(k)}(t) - z^{(k)}(t)| \leq C_{n, k}(b - a)^{n-k} \| y - z \|_1. \]

(12) and (13) imply that

\[ |(T_1 y)^{(n)}(t) - (T_1 z)^{(n)}(t)| \leq \sum_{k=0}^{n-1} L_k C_{n, k}(b - a)^{n-k} \| y - z \|_1 \]

and

\[ \| T_1 y - T_1 z \|_1 \leq \sum_{k=0}^{n-1} L_k C_{n, k}(b - a)^{n-k} \| y - z \|_1. \]

In view of the assumption (10), Lemma 1 yields the existence of a unique fixed point of \( T_1 \) in \( S_1 \). This means that the problem \((11), (2'')\) as well as the problem \((1), (2)\) has a unique solution.

ADMISSIBLE SYSTEM OF FUNCTIONS
AND ASSOCIATED SYSTEM OF CONSTANTS

Another approach to solving the problem \((1), (2)\) is based on the notion of an admissible system with respect to the Green function.
Let $G$ be the Green function of the scalar problem

\[ x^{(m)} = 0, \]

\[ x^{(i-1)}(t_j) = 0, \quad i = 1, \ldots, r_j, j = 1, 2, \ldots, m. \]

Then the functions

\[ \Phi_f(t) = \int_a^b \left| \frac{\partial^l G(t, s)}{\partial t^l} \right| ds, \quad a \leq t \leq b, \quad j = 0, 1, \ldots, n - 1 \]

are continuous in $[a, b]$.

**Definition 2.** The system of nonnegative continuous scalar functions $\mathcal{E}_j$ in $[a, b]$, $j = 0, 1, \ldots, n - 1$, is called admissible (with respect to the Green function $G$) if there exist positive constants $k_j, j = 0, 1, \ldots, n - 1$, such that

\[ \Phi_f(t) \leq k_j \mathcal{E}_j(t), \quad a \leq t \leq b, \quad j = 0, 1, \ldots, n - 1. \]

With respect to boundedness of the functions $\mathcal{E}_j$, $j = 0, 1, \ldots, n - 1$, there exist positive constants $\mathcal{K}_{l,j}, l = 0, 1, \ldots, n - 1$, such that

\[ \int_a^b \left| \frac{\partial^l G(t, s)}{\partial t^l} \right| \mathcal{E}_j(s) ds \leq \mathcal{K}_{l,j} \mathcal{E}_j(t), \quad a \leq t \leq b, \quad l, j = 0, 1, \ldots, n - 1. \]

Let $k_{l,j} = \inf \mathcal{K}_{l,j}, l = 0, 1, \ldots, n - 1$. Then (16) is also true for $k_{l,j}, l = 0, 1, \ldots, n - 1$.

Denote

\[ K_1 = \max (k_{0,0}, k_{1,1}, \ldots, k_{l,n-1}), l = 0, 1, \ldots, n - 1. \]

Hence

\[ \int_a^b \left| \frac{\partial^l G(t, s)}{\partial t^l} \right| \mathcal{E}_j(s) ds \leq K_1 \mathcal{E}_j(t), \quad a \leq t \leq b, \quad l, j = 0, 1, \ldots, n - 1. \]

By definition of $K_1$, for a constant $K_1 < K_l$ the inequality (18) cannot hold for all $t \in [a, b], j = 0, 1, \ldots, n - 1$. The constants $K_l, j = 0, 1, \ldots, n - 1$, will be called the associated system of constants to the admissible system $\mathcal{E}_j, j = 0, 1, \ldots, n - 1$. Hence, the following definition will be of use.

**Definition 3.** The system of smallest nonnegative constants $K_l, j = 0, 1, \ldots, n - 1$, such that (18) are true for all $t \in [a, b], l, j = 0, 1, \ldots, n - 1$, will be called associated system of constants to the admissible system $\mathcal{E}_j, j = 0, 1, \ldots, n - 1$.

Its meaning is explained in the following theorem.

**Theorem 2.** Let $\mathcal{E}_j, j = 0, 1, \ldots, n - 1$, be an admissible system and $K_l, j = 0, 1, \ldots, n - 1$, the associated system of constants to that system. Let the function $f$ satisfy Lipschitz condition (9) with nonnegative matrices $L_k \in M_{d \times d}, k =$
ON A VECTOR MULTIPOINT BOUNDARY VALUE PROBLEM

= 0, 1, ..., n – 1. Then for any \( a_{i,j} \in \mathbb{R}^d \), \( i = 1, ..., r_j \), \( j = 1, 2, ..., m \), there exists a unique solution to (1), (2) provided

\[
\phi \left( \sum_{k=0}^{n-1} K_k L_k \right) < 1.
\]

**Proof.** The problem (1), (2) is equivalent to the equation

\[
x(t) = w(t) + \int_{a}^{b} G(t, s) f[s, x(s), x'(s), ..., x^{(n-1)}(s)] ds, \quad a \leq t \leq b,
\]

where \( w : [a, b] \to \mathbb{R}^d \) is the unique solution to (1'), (2). Now we define the operator \( T \) on \( S = C^{n-1}([a, b], \mathbb{R}^d) \) by

\[
Tx(t) = w(t) + \int_{a}^{b} G(t, s) f[s, x(s), x'(s), ..., x^{(n-1)}(s)] ds, \quad a \leq t \leq b.
\]

Clearly \( T : S \to S \).

The space \( S \) will be provided by the generalized norm

\[
\| x \| = \max (\max_{a \leq s \leq b} | x(t) |, ..., \max_{a \leq s \leq b} | x^{(n-1)}(t) |),
\]

whereby \( \max (x_1, ..., x_n) \) for \( x_1, ..., x_n \in \mathbb{R}^d \) is defined componentwise, i.e.

- if \( x = (x_1, ..., x_n)^T, i = 1, ..., n \), then \( \max (x_1, ..., x_n) = \max_{i=1,...,n} x_i \).

Clearly \( \max (x_1, ..., x_n) \geq x_i \) for each \( i = 1, ..., n \).

\((S, \| \cdot \|)\) is a generalized Banach space. We shall show that Lemma 1 can be applied to the operator \( T \) given by (20). To that aim denote

\[
K = \max_{j=0,1, ..., n-1} k_j,
\]

where \( k_j, j = 0, 1, ..., n - 1 \), are arbitrarily chosen but fixed numbers satisfying (15).

Let \( u, v \in S \) and let \( j \in \{0, 1, ..., n - 1\} \). Then, on the basis of (20), (9), (15) and (21) we obtain the following inequalities. First

\[
| T^{(j)}(u)(t) - T^{(j)}(v)(t) | \leq \int_{a}^{b} \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| \left( \sum_{k=0}^{n-1} L_k | u^{(k)}(s) - v^{(k)}(s) | \right) ds \leq K \psi_j(t) \sum_{k=0}^{n-1} L_k \| u - v \|, \quad a \leq t \leq b.
\]

Suppose that for a natural \( p \) the inequality

\[
| u^{(p)}(t) | - | v^{(p)}(t) | \leq K \psi_j(t) \sum_{k=0}^{n-1} L_k \| u - v \|, \quad a \leq t \leq b.
\]
is true. Then using (20), (9), (22), (16), (17) we come to the inequalities
\[
| (T^{p+1}(u)(t) - (T^{p+1}(v)(t)) | \leq \\
K\mathcal{C}_j(t) \left( \sum_{k=0}^{n-1} L_k \| u - v \| \right)^{p-1} \sum_{k=0}^{n-1} L_k \| u - v \|, \quad a \leq t \leq b.
\]
Hence, by induction, we get that (22) is true for all natural \( p \). The inequality (22) implies
\[
\| T^p(u) - T^p(v) \| \leq \\
K \left[ \max_{j=0,1,\ldots,n-1} \left( \max_{a \leq t \leq b} \mathcal{C}_j(t) \right) \left( \sum_{k=0}^{n-1} L_k \| u - v \| \right)^{p-1} \sum_{k=0}^{n-1} L_k \| u - v \| \right].
\]
By (19), \( \lim_{p \to \infty} \left( \sum_{k=0}^{n-1} L_k \right)^{p-1} = 0 \) and hence there exists a number \( P_0 \) such that for all \( p \geq P_0 \)
\[
q(K \left[ \max_{j=0,1,\ldots,n-1} \left( \max_{a \leq t \leq b} \mathcal{C}_j(t) \right) \left( \sum_{k=0}^{n-1} L_k \| u - v \| \right)^{p-1} \sum_{k=0}^{n-1} L_k \| u - v \| \right]) < 1.
\]
Lemma 1 then implies that the operator \( T \) has a unique fixed point in \( S \) which gives the statement of the theorem.

**Corollary 1.** Let the function \( f \) satisfy Lipschitz condition (9) with nonnegative matrices \( L_k \in M_{d \times d}, k = 0, 1, \ldots, n - 1 \). Let
\[
c_k = \max_{j=0,1,\ldots,n-1} \left( \max_{a \leq t \leq b} \frac{\partial^k G(t,s)}{\partial t^k} \right) ds, \quad k = 0, 1, \ldots, n - 1.
\]
Then for any \( a_i, f \in R^d, i = 1, \ldots, r_f, j = 1, 2, \ldots, m \), there exists a unique solution of (1), (2) provided
\[
q \left( \sum_{k=0}^{n-1} c_k L_k \right) < 1.
\]
**Proof.** Clearly the functions \( \Phi_j, j = 0, 1, \ldots, n - 1 \), given by (14), form an admissible system of functions. As
\[
\int_a^b \left| \frac{\partial^l G(t,s)}{\partial t^l} \right| \Phi_l(s) ds \leq c_l \Phi_j(t), \quad a \leq t \leq b, j = 0, 1, \ldots, n - 1,
\]
ON A VECTOR MULTIPOINT BOUNDARY VALUE PROBLEM

the associated system of constants \( K_l, l = 0, 1, \ldots, n - 1 \) to that admissible system of functions fulfills the relation

\[
K_l \leq c_l, l = 0, 1, \ldots, n - 1.
\]

Thus \( q(\sum_{k=0}^{n-1} K_k L_k) \leq q(\sum_{k=0}^{n-1} c_k L_k) < 1 \) and, by Theorem 2, the statement of Corollary 1 follows.

OPTIMAL VALUES OF THE ASSOCIATED SYSTEM OF CONSTANTS

As we have seen the set of all admissible systems of functions is not empty and in view of Theorem 2 the problem arises what are the best (smallest) constants \( K_k, k = 0, 1, \ldots, n - 1 \), for this set. The answer to this question can be given by applying the theory of positive linear operators developed by M. A. KrasnoseIskij and others. This theory will be taken from the survey paper by H. Amann [3], books by M. A. KrasnoseIskij and others [8], [9].

Consider the Banach space \( E = C([a, b], \mathbb{R}) \) with the supnorm, partially ordered by the relation \( x \leq y \) iff \( x(t) \leq y(t) \) for all \( t \in [a, b] \). Then \( (E, \leq) \) is an ordered Banach space with positive cone \( P = \{ x \in E : x(t) \geq 0, a \leq t \leq b \} \). \( P \) is normal, i.e. every order interval \([x, y] = \{ z \in E : x \leq z \leq y \} \) is bounded, and \( P \) is generating, i.e. \( E = P - P \) ([3], pp. 625–628).

Let \( k \in \{0, 1, \ldots, n - 1\} \) and let \( G \) be the Green function of the scalar problem (1*), (2*). Define the operator \( A_k : E \rightarrow E \) by

\[
A_k x(t) = \int_a^b \frac{\partial^k G(t, s)}{\partial t^k} x(s) \, ds, \quad a \leq t \leq b, \ x \in E.
\]

\( A_k \) is a positive linear operator and using Ascoli lemma we can easily prove that it is completely continuous. If \( \mathcal{C}_k \) belongs to an admissible system of functions \( \mathcal{C}_j, j = 0, 1, \ldots, n - 1 \), then the operator \( A_k \) is \( \mathcal{C}_k \)-bounded from above ([9], p. 78), because for any \( x \in P, x \neq 0 \), there exists a constant \( c = c(x) > 0 \) such that

\[
A_k x(t) = \int_a^b \frac{\partial^k G(t, s)}{\partial t^k} x(s) \, ds \leq (\max_{a \leq t \leq b} x(t)) \Phi_k(t) \leq (\max_{a \leq t \leq b} x(t)) k_k \mathcal{C}_k(t) = c(x) \mathcal{C}_k(t), \quad a \leq t \leq b.
\]

At the same time we have shown that \( A_k \) is \( \Phi_k \)-bounded from above.

Further we need theorems on the estimate of spectral radius of a positive linear operator. A part of Theorem 5.3 from [9], p. 85, will be stated here as

Lemma 2. Let \((E_1, \leq)\) be an ordered Banach space with positive cone \( P \), which is normal and generating. Let \( A : E_1 \rightarrow E_1 \) be a positive linear operator such that there
is an element \( x_0 \in P_1, x_0 \neq 0 \), and a constant \( K \geq 0 \) with
\[
Ax_0 \leq Kx_0.
\]
Let \( A \) be \( x_0 \)-bounded from above or \( x_0 \) be an inner point of \( P_1 \).
Then the spectral radius \( \rho(A) \) of \( A \) satisfies
\[
\rho(A) \leq K.
\]

**Remark 1.** In fact, the lemma has been proved for \( K > 0 \), but by a limit process we get its validity also in the case \( K = 0 \).

With respect to this lemma, \( \rho(A_k) \) can be estimated from above where \( A_k \) is the operator defined by (26), by the relation
\[
\rho(A_k) \leq \max \left[ \left( \int_a^b \frac{\partial^k G(t, s)}{\partial t^k} \right) x(s) ds \right]/x(t)
\]
for any function \( x \in C([a, b], \mathbb{R}) \), \( x(t) > 0 \) in \( [a, b] \), because such a function is an inner point of \( P \).
Especially,
\[
\rho(A_k) \leq c_k,
\]
c_\( k \) being defined by (23).

Lemma 2 also implies that the following lemma is true.

**Lemma 3.** Let \( \mathcal{G}_j, j = 0, 1, \ldots, n - 1 \), be an admissible system of functions and let \( K_j, j = 0, 1, \ldots, n - 1 \), be the associated system of constants to that admissible system. Let \( A_k \) be the operator given by (26). Then
\[
K_j \geq \rho(A_j), \quad j = 0, 1, \ldots, n - 1.
\]

We shall show that for each \( k \in \{0, 1, \ldots, n - 1\} \) there is an admissible system of functions such that the constant \( K_k \) from the associated system of constants is equal to \( \rho(A_k) \). In the proof we need some theorems on positive linear operators. First we state Theorem 5.4 from [9], p. 81, as Lemma 4.

**Lemma 4.** Let \( (E_1, \leq) \) be an ordered Banach space with positive cone \( P_1 \) which is generating. Let \( A : E_1 \rightarrow E_1 \) be a positive linear operator such that there is an element \( x_1 \in P_1, x_1 \neq 0 \), and a constant \( K \geq 0 \) with
\[
Ax_1 \leq Kx_1.
\]
Then the spectral radius \( \rho(A) \) of \( A \) satisfies
\[
\rho(A) \geq K.
\]
In view of this lemma, \( \rho(A_k) > 0 \) will hold for the operator \( A_k \) defined by (26) if
\[
\min_{a \leq t \leq b} \left| \int_a^b \frac{\partial^k G(t, s)}{\partial t^k} \right| ds > 0,
\]
since \( q(A_k) \geq \min_{a \leq t \leq b} \left[ \left( \int_{a}^{b} \frac{\partial^k G(t, s)}{\partial t^k} |x(s)| ds \right) / |x(t)| \right] \) for any \( x \in C([a, b], R), x(t) > 0 \) in \([a, b]\).

Especially,

\[
q(A_k) \geq \min_{a \leq t \leq b} \int_{a}^{b} \frac{\partial^k G(t, s)}{\partial t^k} ds.
\]

Part of the famous Krejn—Rutman theorem ([3], p. 632, Theorem 3.1) is stated here as Lemma 5.

**Lemma 5.** Let \((E_1, \leq)\) be an ordered Banach space with positive cone \(P_1\). Let \(P_1\) be total, i.e. the closure \(\overline{P_1 - P_1} = E_1\). Suppose that \(A : E_1 \to E_1\) is a positive linear, completely continuous operator and it has a positive spectral radius \(q(A)\). Then \(q(A)\) is an eigenvalue of \(A\) with an eigenvector in \(P_1\).

We also need the definition of a \(u_0\)-positive operator ([9], p. 77). If \((E_1, \leq)\) is an ordered Banach space with positive cone \(P_1\), \(A : E_1 \to E_1\) is a linear positive operator, and \(u_0 \in P_1, u_0 \neq 0\), then we say that \(A\) is \(u_0\)-positive, if for each nonzero element \(x \in P_1\) there exist constants \(a(x) > 0, \beta(x) > 0\) such that

\[
a(x) u_0 \leq Ax \leq \beta(x) u_0.
\]

In other words, \(A\) is \(u_0\)-bounded from below as well as \(u_0\)-bounded from above.

Lemmas 4, 5 and Theorem 2.11 in [8], p. 80, imply the following lemma on existence and uniqueness of the positive eigenvector.

**Lemma 6.** Let \((E_1, \leq)\) be an ordered Banach space with positive cone \(P_1\). Let \(P_1\) be generating and let \(A : E_1 \to E_1\) be a positive linear, completely continuous, operator which is \(u_0\)-positive for an element \(u_0 \in P_1, u_0 \neq 0\). Then there exists up to a multiplicative positive constant a unique eigenvector of \(A\) in \(P_1\) and that vector corresponds to \(q(A)\).

**Proof.** As \(A\) is \(u_0\)-bounded from below, by Lemma 4 the spectral radius \(q(A)\) of \(A\) is positive. Hence, by Lemma 5, there exists an eigenvector of \(A\) belonging to \(P_1\) which corresponds to \(q(A)\). Now we apply Theorem 2.11 in [8], all assumptions of which are satisfied. By this theorem the uniqueness follows.

By Lemma 2.1 (p. 77) as well as by Theorem 2.16 (p. 90) in [8] we come to the following statement.

**Lemma 7.** Let the assumptions of Lemma 6 be satisfied. Then for every \(y \in P_1, y \neq 0\), the equation

\[
\lambda x - Ax = y
\]

has exactly one solution \(x \in P_1\) if \(\lambda > q(A)\) and no solution in \(P_1\) for \(\lambda \leq q(A)\).

Let \(k \in \{0, 1, \ldots, n - 1\}\). We summon fundamental properties of the operator \(A_k\) which is defined by (26). \(E, P, q(A_k), \Phi_k\) have the same meaning as above. We remind that \(P\) is normal and generating.
Lemma 8. $A_k : E \rightarrow E$ is a positive linear, completely continuous operator. Further:

(a) $A_k$ is $\Phi_k$-positive.
(b) $q(A_k) > 0$.
(c) There exists a unique, up to a multiplicative positive constant, eigenfunction $\xi_k$ of $A_k$ belonging to $P$. $\xi_k$ corresponds to $q(A_k)$ and $A_k$ is $\xi_k$-positive.
(d) $\xi_k \equiv 0$ is not true on any subinterval of $[a, b]$.

Proof. Our considerations will be based on the following property of the Green function $G$ for the scalar problem (1'), (2') (see [10], p. 375). It is true that

\begin{equation}
\frac{\partial^j G(t_0, s)}{\partial t^j} = 0 \quad \text{for } t_0 \in [a, b] \text{ and all } s, a \leq s \leq b, j \in \{0, 1, ..., n - 1\}
\end{equation}

if

\begin{equation}
\text{there is an } l \in \{1, 2, ..., m\} \text{ such that } t_0 = t_l, 0 \leq j \leq r_l - 1. \text{ Conversely, if (31) is valid, then for all functions } y \in C^n([a, b], R) \text{ satisfying (2')} \text{ we must have } y^{(j)}(t_0) = 0 \text{ which is only so possible when (32) is true. Thus (31), (32) are equivalent each to other.}
\end{equation}

Statement (a). We have already found (see (27)) that $A_k$ is $\Phi_k$-bounded from above. Since for each $x \in P, x \not= 0, A_kx(t) \equiv 0$ does not hold on any subinterval $[a_1, b_1] \subset [a, b]$, the $\Phi_k$-boundedness of $A_k$ from below will be shown if to any function $x \in P, x \not= 0$ on any subinterval $[a_1, b_1] \subset [a, b]$ there exists a constant $\alpha(x) > 0$ such that

\begin{equation}
A_kx(t) \geq \alpha(x) \Phi_k(t), \quad a \leq t \leq b.
\end{equation}

By (14), $\Phi_k(t) = (A_k 1)(t), a \leq t \leq b$. Hence, the equivalence between (31) and (32) implies that $\Phi_k(t_0) = 0$ iff there exists an $l \in \{1, 2, ..., m\}$ such that $t_0 = t_l$ and $k \leq r_l - 1$. Therefore (34) $\Phi_k(t) > 0$ for all $t \in [a, b], t \neq t_j, j = 1, 2, ..., m$, as well as for $t = t_l$ such that $k \geq r_l$.

If we show that any point $t_l$ with $k \leq r_l - 1, l = 1, 2, ..., m$, there exists a one-sided limit

$$
\lim_{t \rightarrow t_l^+} A_kx(t)/\Phi_k(t), \quad \lim_{t \rightarrow t_l^-} A_kx(t)/\Phi_k(t)
$$

and it is different from 0 (it is positive), then (33) is true with a positive constant $\alpha(x)$ and the proof of (a) is complete.

Suppose $1 \leq l < m, k \leq r_l - 1, t_l < t$. By Taylor's formula for any $a \leq s \leq b$ there exists a $\tau(s), t_l < \tau(s) < t$, such that

$$
\frac{\partial^k G(t, s)}{\partial t^k} = \frac{\partial^k G(t_l, s)}{\partial t^k} + \frac{\partial^{k+1} G(t_l, s)}{\partial t^{k+1}} \frac{(t - t_l)}{1!} + ... + \frac{\partial^{r_l} G(\tau(s), s)}{\partial t^{r_l}} \frac{(t - t_l)^{r_l - k}}{(r_l - k)!}
$$
and, in view of (31),
\[ \frac{\partial^k G(t, s)}{\partial t^k} = \frac{\partial^r G(\tau(s), s)}{\partial t^r} \frac{(t - t_i)^{r_i - k}}{(r_i - k)!}, \quad a \leq s \leq b. \]

Hence
\[
\lim_{t \to t_i} \left[ \left( \int_a^b \left| \frac{\partial^k G(t, s)}{\partial t^k} \right| x(s) \, ds \right) \left( \int_a^b \left| \frac{\partial^r G(t, s)}{\partial t^r} \right| \, ds \right) \right] = 
\int_a^b \left| \frac{\partial^r G(t_1, s)}{\partial t^r} \right| x(s) \, ds \left( \int_a^b \left| \frac{\partial^r G(t_1, s)}{\partial t^r} \right| \, ds \right)
\]
exists, and by (34), it is finite and different from zero. A similar result follows for
\[ 1 < I \leq m \] and for \( \lim \limits_{t \to t_i} A_k x(t) / \Phi_k(t) \).

**Statement (b).** By (a), there exists a \( K > 0 \) such that \( A_k \Phi_k \leq K \Phi_k \). On the basis of Lemma 4, this implies that \( q(A_k) > 0 \).

**Statement (c).** The first part of this statement follows from Lemma 6, all assumptions of which are satisfied. By (a), for each \( x \in P, x \neq 0 \), there exist two constants \( \beta(x) \geq \alpha(x) > 0 \) such that
\[ \alpha(x) \Phi_k \leq A_k x \leq \beta(x) \Phi_k. \]
Hence, there are \( \beta_k \geq \alpha_k \) such that
\[ \alpha_k \Phi_k \leq A_k \xi_k \leq \beta_k \Phi_k. \]
Since \( A_k \xi_k = q(A_k) \xi_k \), we have
\[ \frac{\alpha_k}{q(A_k)} \Phi_k \leq \xi_k \leq \frac{\beta_k}{q(A_k)} \Phi_k \]
and thus, with respect to (35), (36), we get that
\[ \frac{\alpha(x) q(A_k)}{\beta_k} \xi_k \leq A_k x \leq \frac{\beta(x) q(A_k)}{\alpha_k} \xi_k \]
which means that \( A_k \) is \( \xi_k \)-positive.

**Statement (d).** If \( \xi_k(t) \equiv 0 \) were true on a subinterval of \( [a, b] \), then by (36) the same would hold for \( \Phi_k \). But this is in contradiction with (34) which proves (d).

On the basis of Lemma 8 we prove the following theorem.

**Theorem 3.** Let \( k \in \{0, 1, \ldots, n - 1\} \) and let \( \xi_k \) be a nonnegative eigenfunction of the operator \( A_k \). Then the functions
\[ \xi_j(t) = \frac{1}{q(A_k)} \int_a^b \left| \frac{\partial^j G(t, s)}{\partial t^j} \right| \xi_k(s) \, ds, \]
\[ a \leq t \leq b, \quad j = 0, 1, \ldots, n - 1, \]
form an admissible system of functions with respect to $G$ such that for the associated system of constants $K_j, j = 0, 1, \ldots, n - 1$,

\begin{equation}
K_k = \varrho(A_k)
\end{equation}

is true.

Proof. The functions $\mathcal{C}_j$ determined by (37) are all continuous and non-negative in $[a, b]$. Clearly $\mathcal{C}_k$ satisfies (37) for $j = k$.

First we show that the functions $\mathcal{C}_j, j = 0, 1, \ldots, n - 1$, form an admissible system of functions with respect to $G$. In agreement with (26), we define the operator $A_j : E \to E$ by

\begin{equation}
A_jx(t) = \int_a^b \frac{\partial^j G(t, s)}{\partial t^j} x(s) \, ds, \quad a \leq t \leq b, x \in E, j = 0, 1, \ldots, n - 1.
\end{equation}

By (14), (15), (37) and (39) we have to find such constants $k_j > 0, j = 0, 1, \ldots, n - 1$, that

\begin{equation}
\Phi_j(t) = (A_j 1)(t) \leq \frac{k_j}{\varrho(A_k)} A_j \mathcal{C}_k(t), \quad a \leq t \leq b.
\end{equation}

But the proof of (40) runs in the same way as the proof of (33). Hence we can assert that the existence of $k_j > 0, j = 0, 1, \ldots, n - 1$, with property (40) is guaranteed.

Finally we prove (38). In virtue of (17), (16) and (37)

\begin{equation}
K_k = \max (k_{k,0}, k_{k,1}, \ldots, k_{k,n-1}) \geq \varrho(A_k).
\end{equation}

On other hand, (29) gives an opposite inequality and hence, (38) is true.

**APPLICATION TO SECOND ORDER SYSTEMS**

The obtained results in Theorems 2, 3 will be applied to the vector boundary value problem of the second order

\begin{align}
(41) \quad x'' &= f(t, x, x'), \\
(42) \quad x(a) &= a_{1,1}, \quad x(b) = a_{1,2}.
\end{align}

Suppose that $f \in C([a, b] \times \mathbb{R}^2, \mathbb{R}^d)$ satisfies the Lipschitz condition

\begin{equation}
|f(t, u_0, u_1) - f(t, v_0, v_1)| \leq L_0 |u_0 - v_0| + L_1 |u_1 - v_1|
\end{equation}

where $L_0, L_1 \in M_{d \times d}$ are nonnegative matrices.

Let $G_1$ be the Green function for the corresponding scalar homogeneous problem

\begin{equation}
x'' = 0, \quad x(a) = 0, \quad x(b) = 0.
\end{equation}
ON A VECTOR MULTIPONT BOUNDARY VALUE PROBLEM

Then

\[
G_1(t, s) = \begin{cases} 
\frac{(b-t)(s-a)}{b-a}, & a \leq s \leq t \leq b, \\
\frac{(b-s)(t-a)}{b-a}, & a \leq t \leq s \leq b
\end{cases}
\]

\[
\frac{\partial G_1(t, s)}{\partial t} = \begin{cases} 
\frac{s-a}{b-a}, & a \leq s \leq t \leq b, \\
\frac{s-b}{b-a}, & a \leq t \leq s \leq b.
\end{cases}
\]

Consider the functions

\[
\bar{\Phi}_0(t) = \int_a^b |G_1(t, s)| \, ds, \quad \bar{\Phi}_1(t) = \int_a^b \left| \frac{\partial G_1(t, s)}{\partial t} \right| \, ds, \quad a \leq t \leq b.
\]

Similarly as the functions (14), they form an admissible system of functions with respect to \(G_1\). By [2], p. 2,

\[(44) \quad \bar{\Phi}_0(t) = \frac{1}{2} (t-a)(b-t), \quad \bar{\Phi}_1(t) = \left[ (t-a)^2 + (b-t)^2 \right] \left[ 2(b-a) \right], \quad a \leq t \leq b.
\]

The constants \(\bar{c}_0, \bar{c}_1\) determined by (23), are

\[
\bar{c}_0 = \max_{a \leq t \leq b} \bar{\Phi}_0(t) = \frac{1}{8} (b-a)^2, \quad \bar{c}_1 = \max_{a \leq t \leq b} \bar{\Phi}_1(t) = \frac{1}{2} (b-a).
\]

Hence, similarly as in the proof of Corollary 1, we get that the associated system of constants \(K_0, K_1\) satisfies

\[(45) \quad K_0 \leq \frac{1}{8} (b-a)^2, \quad K_1 \leq \frac{1}{2} (b-a).
\]

Comparing the inequalities (iv), (ix), (vi), (xi) in [2], p. 3, with (16), we get that

\[
k_{0,0} = \frac{5}{48} (b-a)^2, \quad k_{0,1} = \frac{\sqrt{3} - 1}{4 \sqrt{3}} (b-a)^2,
\]

\[
k_{1,0} = \frac{8}{25} (b-a), \quad k_{1,1} = \frac{1}{3} (b-a)
\]

and

\[
\max (k_{0,0}, k_{0,1}) = \frac{\sqrt{3} - 1}{4 \sqrt{3}} (b-a)^2,
\]

\[
\max (k_{1,0}, k_{1,1}) = \frac{1}{3} (b-a).
\]

Hence, in view of (17),

89
Because the estimates in (46) are better than the estimates in (45), on the basis of Theorem 2 we get condition (8) in [2], p. 1, i.e. if

\[ q \left( \frac{\sqrt{3}}{4} \right) \left( \frac{b - a}{4} \right)^2 L_0 + \frac{1}{3} (b - a) L_1 \right) < 1, \]

then the problem (41), (42) has a unique solution for any two vectors \( a_1, a_2 \in \mathbb{R}^d \).

Let \( E, P \) have the same meaning as above. Consider, now, the operators \( \bar{A}_0, \bar{A}_1 : E \to E \), which according to (39) are defined as follows:

\[
\bar{A}_0 x(t) = \int_a^b G_1(t, s) | x(s) | ds, \quad a \leq t \leq b, \]
\[
\bar{A}_1 x(t) = \int_a^b \left| \frac{\partial G_1(t, s)}{\partial t} \right| x(s) ds, \quad a \leq t \leq b. \]

As \( G_1(t, s) \leq 0 \) in \([a, b] \times [a, b]\), the eigenvalue problem

\[ \bar{A}_0 x = \lambda x, \quad \lambda \neq 0 \]

is equivalent to the problem

\[ x'' = -\frac{1}{\lambda} x, \quad x(a) = 0, \quad x(b) = 0, \]

the eigenvalues of which are

\[ \lambda_k = \frac{(b - a)^2}{k^2 \pi^2}, \quad k = 1, 2, \ldots, \]

and the corresponding eigenfunctions uniquely determined up to a multiplicative constant are

\[ x_k(t) = \sin \frac{kn(t - a)}{b - a}, \quad k = 1, 2, \ldots. \]

By Lemma 8, \( q(\bar{A}_0) = \lambda_1 = (b - a)^2 / \pi^2 \). The corresponding eigenfunction is

\[ \bar{x}_0(t) = \frac{\pi}{b - a} \sin \frac{\pi(t - a)}{b - a}, \quad a \leq t \leq b. \]

By (33) and (x) in [2], pp. 2-3,

\[ \bar{x}_1(t) = \frac{\pi}{(b - a)^2} \int_a^b \left| \frac{\partial G_1(t, s)}{\partial t} \right| \bar{x}_0(s) ds = \]
\[
= \frac{2}{b - a} \sin \frac{\pi(t - a)}{b - a} + \frac{\pi(b - 2t + a)}{(b - a)^2} \cos \frac{\pi(t - a)}{b - a}. \]
By Theorem 3, \( \tilde{\zeta}_0, \tilde{\zeta}_1 \) form an admissible system of functions with respect to \( G_1 \) and for the associated system of constants \( \overline{K}_0, \overline{K}_1 \)

\[
\overline{K}_0 = \varrho(\overline{A}_0) = \frac{(b - a)^2}{\pi^2}
\]
is true. By the inequalities (v), (x), (vii), (xii) in [2], p. 3, we get that

\[
\overline{K}_{0,0} = \frac{b - a)^2}{\pi^2}, \quad \overline{K}_{0,1} = \frac{(b - a)^2}{\pi^2}, \quad \overline{K}_{1,0} = \frac{4}{\pi^2} (b - a), \quad \overline{K}_{1,1} = \frac{4}{\pi^2} (b - a).
\]

Hence

\[
\overline{K}_0 = \frac{1}{\pi^2} (b - a)^2, \quad \overline{K}_1 = \frac{4}{\pi^2} (b - a).
\]

Using these results we get condition (6) in Theorem in [2], p. 1, i.e.

\[
(48) \quad q \left( \frac{1}{\pi^2} (b - a)^2 L_0 + \frac{4}{\pi^2} (b - a) L_1 \right) < 1,
\]

which is sufficient for the existence of a unique solution to (41), (42) for any two vectors \( a_{1,1}, a_{1,2} \in \mathbb{R}^d \).

Consider now the operator \( \overline{A}_1 \). By (30), (28) we obtain that the spectral radius \( \varrho(\overline{A}_1) \) of \( \overline{A}_1 \) satisfies

\[
(49) \quad \frac{1}{4} (b - a) = \min_{s \in \mathbb{R}^d} \Phi_1(t) \leq \varrho(\overline{A}_1) = \max_{s \in \mathbb{R}^d} \Phi_1(t) = \frac{1}{2} (b - a),
\]

where \( \Phi_1 \) is given by (44). By Lemma 8, there is a uniquely determined (up to a positive constant) positive eigenfunction \( \overline{\zeta}_1 \) of \( \overline{A}_1 \) corresponding to \( \varrho(\overline{A}_1) \).

By (37), we define

\[
\overline{\zeta}_0(t) = \frac{1}{\varrho(\overline{A}_1)} \int_s^b |G_1(t, s)| \overline{\zeta}_1(s) \, ds, \quad a \leq t \leq b.
\]

Then, by Theorem 3, \( \overline{\zeta}_0, \overline{\zeta}_1 \) form an admissible system of functions whereby for the associated system of constants \( K_0^*, K_1^* \) we have \( K_1^* = \varrho(\overline{A}_1) \). The inequality (49) can be improved. By the inequality (xi) in [2], p. 3, as well as by the fact that

\[
\min_{s \in \mathbb{R}^d} \Phi_1(t) = \frac{1}{4} (b - a) > 0, \quad \text{Lemma 2 implies that } \varrho(\overline{A}_1) \leq \frac{1}{3} (b - a). \quad \text{By (44), } \Phi_1 \text{ cannot be an eigenfunction of } \overline{A}_1. \text{ Hence in (xi) the sign of equality cannot hold. Thus the difference } \frac{1}{3} (b - a) \overline{\Phi}_1 - \overline{A}_1 \overline{\Phi}_1 = y \geq 0, \text{ where } y \in \mathcal{P}, y \neq 0.
\]

By Lemma 7, \( \varrho(\overline{A}_1) < \frac{1}{3} (b - a) \). Thus
and we can state the result: There exists a constant $K_0^* > 0$ such that

$$\frac{1}{4} (b - a) \leq \varrho(\bar{A}_1) < \frac{1}{3} (b - a)$$

is a sufficient condition for the existence of a unique solution to (41), (42) for any two vectors $a_{1,1}, a_{1,2} \in \mathbb{R}^d$.

The condition (51) is not contained among conditions (6), (7), (8) of Theorem in [2], p. 1.

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