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ULTRAPRODUCTS AND THE AXIOM OF CHOICE
To Professor HLAWKA on his 70th birthday

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Abstract. We construct a model of ZF set theory in which there is an amorphous set (an infinite set, every infinite subset of which is cofinite) which is the ultraproduct of a family of finite sets. We give an application to topology.

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1. Introduction

The main lemmata in the application of the ultraproduct method are the Boolean prime ideal theorem which guarantees for the existence of sufficiently many ultrafilters and the theorem of Łoš LT which describes the first order sentences of the ultraproduct. As was shown by Howard [6], BPI + LT is equivalent to the axiom of choice AC. On the other hand, in set theoretical independence proofs it is often helpful to use arguments from nonstandard-analysis in the absence of AC. For example, Pincus [10] modified Luxemburg's proof of the Hahn – Banach theorem in a model where BPI was false. So while it is possible to retain some of the applications of ultraproducts in models of ZF minus AC, AC cannot be avoided completely. In the model of Blass [1], for instance, there are no nonprincipal ultrafilters, whence there the method of ultraproducts becomes trivial. Here we show, that even if there are enough ultrafilters on ω, the formation of countable ultraproducts may result in sets without any structure.

1.1. Theorem. *There is a model of* ZF *in which* $P(\omega)$ *can be wellordered* (whence there are nonprincipal ultrafilters on ω) *and there is a family* $(F_n)_{n\in\omega}$ *of finite sets, such that their product* $X = \prod_{n\in\omega} F_n$ *contains no infinite, Dedekind-finite subset, but for every nonprincipal ultrafilter* F *on* ω*, the ultraproduct* X/F *is amorphous.*

We will prove 1.1 in section 2. In section 3 we will apply it and answer a question from [2]. The author did not check, whether in the above model there is a nonprincipal ultrafilter on every infinite set. Concerning the strength of this axiom

SPI which was first studied by Halpern [4], the following result (which was obtained in conversations with J. Truss) is interesting.

1.2. *If on every infinite set there is a nonprincipal ultrafilter, then* AC^{ω}_{fin} *holds, the axiom of choice for countable families of nonempty finite sets.*

Proof: Let $(F_n)_{n\in\omega}$ be a sequence of disjoint finite sets, C_n the set of choice functions on $(F_k)_{k\in n}$ and set $C = \bigcup_{n\in\omega} C_n$. **U** is a nonprincipal ultrafilter on C. We write $\bar{c} = \{d \in C\colon d \supseteq c\}$, $c \in C$, and observe, that $\bar{c} \cap \bar{d} = \Phi$ if $c \neq d \in C_n$ and that $\cup\{\bar{c}\colon c \in C_n\}$ is cofinite in C. Hence, since C_n is finite, there is exactly one $c^n \in C_n$ such that $\bar{c}^n \in \mathbf{U}$, whence $f(n) = c^{n+1}(n)$ defines a choice function. It follows, that AC_2 does not imply SPI (use a model of Levy [8], where multiple choice MC holds and AC_n for every n, but not AC^{ω}_{fin}). Other known results are $\neg$SPI $\Rightarrow AC_2$, $\neg$SPI $\Rightarrow AC^{\omega}$ (Halpern) and $\neg$DC $\Rightarrow$ SPI (Feferman), but PW + $AC^{\omega} \Rightarrow$ SPI (Halpern).

2. Proof of the Theorem

2.1. We construct a permutation model **M** of ZF° + PW (ZF° = Zermelo–Fraenkel set theory without the axioms of foundation F or choice, PW is Rubin's axiom, that the power set of an ordinal can be wellordered), where 1.1 holds, and then apply transfer to get a model of ZF with 1.1 (c.f. [7]; ZF = ZF° + F, in ZF PW = AC). The model **M** is given by a set U of urelements ($u = \{u\}$, if $u \in U$) which is countable, when viewed from outside the model. We set $U = \cup\{P_n\colon n \in \omega\}$, where $|P_n| = n + 1$ ($|X|$ is the cardinality of X) and $P_n \cap P_m = \Phi$, if $n \neq m$. Γ is the group of all bijective mappings $\pi\colon U \to U$ such that $\pi P_n = P_n$, each $n \in \omega$. Γ is extended to all sets $x \in \mathbf{M}$ by induction on the modified rank. The normal ideal I consists of all sets $e \subseteq U$ such that $\sup\{|e \cap P_n|\colon n \in \omega\} < \infty$. A set x is symmetric, if for some $e \in I$ sym $x \supseteq$ fix e (we write $x \in \Delta(e)$; sym $x = \{\pi \in \Gamma\colon \pi x = x\}$, fix $e = \{\pi \in \Gamma\colon \pi \mid e = \text{identity}\}$ – such an e is called a support of x), and **M** consists of the hereditary symmetric sets (i.e.: $x \in \mathbf{M}$ implies $x \subseteq \mathbf{M}$) with the usual $\in$-relation. We assume, that AC holds in the real world V. Then **M** is a model of ZF° + PW (c.f. [7]). Hence $P(\omega)$ can be wellordered and there is a nonprincipal ultrafilter **F** on $P(\omega)$.

2.2. In **M** we consider the set $X = \prod_{n\in\omega} P_n$. Since I has infinite elements, X is nonempty. In fact, X has the same elements in **M** as it has in V (where AC holds). If **F** is a nonprincipal ultrafilter on ω, we set $x = y \bmod \mathbf{F}$, if and only if $\{n \in \omega\colon x(n) = y(n)\} \in \mathbf{F}$.

$\Phi(x) = \{y \in X\colon x = y \bmod \mathbf{F}\}$ and $\Phi'X = X \mid \mathbf{F}$ is the ultraproduct (X, $\Phi \in \Delta(\Phi)$ are in **M**). We will show, that $\Phi'X$ is amorphous and hence Dedekind-finite (each countable subset is finite), while no infinite subset of X can be Dedekind-finite.

Since AC does not hold in **M**, we cannot use LT (we do not know, if LT is true in **M**).

2.3. $X \mid \mathbf{F}$ is infinite.
Proof: Assume for the converse, that $X \mid \mathbf{F} = \{\Phi(x_i): i \in n\}$. In V (and hence in **M**) there is a $x \in X$ such that for $j > n$, $x(j) \in P_j \setminus \{x_i(j): i \in n\}$. Therefore $\{j \in \omega: x(j) = x_i(j)\} \subseteq n + 1$, whence $x \neq x_i \bmod \mathbf{F}$ and $\Phi(x) \neq \Phi(x_i)$ for all $i \in n$, a contradiction.

2.4. For $e \in I$ set $e_n = e \cap P_n$ and $N = \sup\{|e_n| : n \in \omega\}$: $|\Phi' \prod_{n \in \omega} e_n| \leqq N$.

Proof. Since this is obvious, if some $e_n = \Phi$, assume that all $e_n \neq \Phi$. In V– and hence in **M**–there are N mappings $x_i \in X$ such that $e_n = \{x_i(n): i \in N\}$ for every n. If $x \in \prod_{i \in \omega} e_n$ is arbitrary, we consider the set $x^i = \{n \in \omega: x(n) = x_i(n)\}$ and observe, that since **F** is an ultrafilter and $\cup \{x^i: i \in N\} = \omega$ there is an $i \in N$ such that $x^i \in \mathbf{F}$. Therefore $\Phi(x) = \Phi(x_i)$ for this i.

2.5. With the same notation as above, $X \mid \mathbf{F} \cap \Delta(e) = \Phi' \prod_{n \in \omega} e_n$ and if $\mathbf{x} \in X \mid \mathbf{F} \setminus \Delta(e)$, then $\mathrm{orb}_e\, \mathbf{x} = X \mid \mathbf{F} \setminus \Delta(e)$, where $\mathrm{orb}_e\, \mathbf{x} = \{\pi \mathbf{x}: \pi \in \mathrm{fix}\, e\}$.

Proof. We claim, that if $\mathbf{x} \in X \mid \mathbf{F} \setminus \Phi' \prod_{n \in \omega} e_n$, then $\mathrm{orb}_e\, \mathbf{x} = X \mid \mathbf{F} \setminus \Phi' \prod_{n \in \omega} e_n$. It then follows from 2.3 and 2.4, that $\mathrm{orb}_e\, \mathbf{x}$ is infinite, whence $\mathbf{x} \notin \Delta(e)$ and $\mathbf{X} \mid F \cap \Delta(e) \subseteq \Phi' \prod_{n \in \omega} e_n$; since "$\supseteq$" is obvious, we get "$=$" here. For the proof of the claim we assume, that $\Phi(x)$, $\Phi(y)$ are arbitrary in $X \mid \mathbf{F} \setminus \Phi' \prod_{n \in \omega} e_n$. Since **F** is an ultrafilter, $E = \{n \in \omega: x(n) \notin e_n \text{ and } y(n) \notin e_n\}$ is in **F**. In the real world V we define a permutation $\pi \in \mathrm{fix}\, e$ through $\pi x(n) = y(n)$, $\pi y(n) = x(n)$, if $n \in E$, and $\pi u = u$ for all other $u \in U$. Since $\pi x = y \bmod \mathbf{F}$, $\pi \mathbf{x} = \pi \Phi(x) = \Phi(y) = \mathbf{y}$; $\mathbf{y} \in \mathrm{orb}_e\, \mathbf{x}$.

2.6. $X \mid \mathbf{F}$ is amorphous.
Proof. If $A \in \Delta(e)$ is an infinite subset of $X \mid \mathbf{F}$, then $A \setminus \Delta(e) \neq \Phi$ by 2.4 and so $A \supseteq X \mid F \setminus \Delta(e)$ by 2.5., whence the complement of A consists of at most N elements.

2.7. The same argument as in 2.5. shows, that if E, F are k-element subsets of $X \mid \mathbf{F} \setminus \Delta(e)$, then there is a $\pi \in \mathrm{fix}\, e$ such that $\pi E = F$. From this remark the following strong form of Ramsey's theorem follows: If $C \subseteq [X \mid \mathbf{F}]^{<\omega}$, the family of finite subsets of $X \mid \mathbf{F}$, then there is a cofinite set $H \subseteq X \mid \mathbf{F}$ such that for each $n [H]^n \subseteq C$ or $[H]^n \cap C = \Phi$. For if $C \in \Delta(e)$, then $H = X \mid \mathbf{F} \setminus \Delta(e)$ is the desired homogeneous set. Since by [3] every mapping f on such a set satisfies $f(x) \in \{x\} \cup E$, some finite set E, there is no group operation on $X \mid \mathbf{F}$. On the other hand, each factor of the ultraproduct carries even abelian groups. 2.6. alone does not suffice to prove this result, since in a model of Hickman [5] there is an amorphous group.

We conclude that in ZF the property G: "X carries a group structure" is not invariant under the formation of ultraproducts. This is related to a result of Hajnal and Kertesz, that AC holds, iff every set satisfies G (c.f. [11]). This does not contradict LT.

2.8. Every Dedekind-finite subset of X is finite.

Proof. Be $Y \subseteq X$ Dedekind-finite, $Y \in \Delta(e)$, $e_n = e \cap P_n \neq \Phi$ and $N = \sup\{|e_n|: n \in \omega\}$. We set $E(x) = \{n: x(n) \in P_n \backslash e_n\}$, $x \in X$. We will show, that $E = \cup \{E(y): y \in Y\}$ is finite, whence $Y \subseteq \Delta(f)$, $f = e \cup \cup \{P_i: i \in E\}$, is a wellorderable Dedekind-finite set, hence finite. For if E is infinite, we apply PW in order to obtain a sequence $(F_n)_{n \in E}$ in $\Delta(\varphi)$, $F_n \in P(\omega)$, such that $n \in F_n$ and $F_n = E(y)$ for some y. Since $\{y \mid \omega\backslash F_n: E(y) = F_n, y \in Y\} \subseteq \prod_{n \in \omega\backslash F_n} e_n \subseteq \Delta(e)$, there is a sequence $(z_n)_n$ in $\Delta(\bar{e})$ such that each $z_n = y \mid \omega\backslash F_n$, some $y \in Y$ with $E(y) = F_n$. Using AC in V, we get x, x' in X (and in **M**) such that $Q_n = \{x(n), x'(n)\} \subseteq P_n\backslash e_n$ and $|Q_n| = 2$, if $n \geq N + 2$. We define a sequence $y_n \in \Delta(e)$, $n \in E$, by $y_n(i) = z_n(i)$, if $i \in \omega\backslash F_n$, $y_n(n) = x(n)$ and $y_n(i) = x'(i)$, if $i \in F_n\backslash\{n\}$. Since there is a $y \in Y$ with $E(y) = F_n$ and $y \mid \omega\backslash F_n = z_n$, the following permutation π is in fix $\bar{e}$: $\pi y(i) = y_n(i)$, $\pi y_n(i) = y(i)$, if $i \in F_n$, and $\pi u = u$ for all other $u \in U$. As $Y \in \Delta(\bar{e})$ and $\pi y = y_n$, $y_n \in Y$. Since $y_n(n) \neq y_m(n)$ if $n \neq m$ are in E, this sequence is injective, contradicting Dedekind-finiteness.

2.9. The condition, that X is a product of finite sets, ensures, that in ZF° every subset of X with a Dedekind-finite power set is finite. But 2.8. is not provable in ZF°. For example, in the Cohen–Halpern–Levy model of ZF 2^ω contains an infinite, Dedekind-finite subset. Also ZF° + PW is too weak to give 2.8. In the Fraenkel–Levy permutation model, where $U = \cup \{P_n: n \in \omega\}$, $|P_n| = 2$, $P_n \cap P_m = \Phi$ if $n \neq m$, Γ is the group of all bijective mappings on U which preserve $(P_n)_n$ and $I = [U]^{<\omega}$, we set $X_n = \cup \{P_i: i \in n+1\}$ and $X = \prod_{i \in \omega} X_i$. Fix $u_0 \in P_0$ and set $f(u)(i) = u_0$ if $u \notin P_i$, $f(u)(i) = u$, if $u \in P_i$. This defines an one-to-one mapping $f: U \to X$ and $f'U$ is an infinite, Dedekind-finite subset of X.

3. Application

As was observed in [2], in the Cohen–Halpern–Levy model of ZF the space $\boldsymbol{R}$ has both a countable base and a Dedekind-finite base, but no isolated points. The question appeared, if there are spaces, where "Dedekind-finite" can be weakened to "Dedekind-finite power set" or even "amorphous". We start with a negative result.

3.1. Let X be a topological space, **X** its topology, $x \in X$, $\mathbf{B} \subseteq \mathbf{X}$ a neighbourhood base at x and $\{x\} = \cap \mathbf{A}$, where $\mathbf{A} \subseteq \mathbf{X}$. If one of **A**, **B** is wellorderable as a set and the other one has a Dedekind-finite power set, then x is isolated.

Proof: First assume, that $P(\mathbf{B})$ is D-finite and $\mathbf{A}$ is wellorderable. For $A \subseteq X$ we set $\mathbf{B}(A) = \{B \in \mathbf{B}: B \subseteq A\}$. From our hypotheses $\mathbf{B}'\mathbf{A}$ is finite, e.g. $\mathbf{B}'\mathbf{A} = \{\mathbf{B}(A_i): i \in n\}$. We set $S = \bigcap_{i \in n} A_n$: Then $x \in S \in \mathbf{X}$ but not necessarily $S \in \mathbf{A}$. For $B \in \mathbf{B}$ such that $x \in B \subseteq S$ we get: $B \in \mathbf{B}(S) \subseteq \mathbf{B}(A_i)$, $i \in n$, whence $B \in \mathbf{B}(A)$ for each $A \in \mathbf{A}$ and $x \in B \subseteq \cap \mathbf{A} = \{x\}$. If $P(\mathbf{A})$ is D-finite and $\mathbf{B}$ is wellorderable, consider the sets $\mathbf{A}(B) = \{A \in \mathbf{A}: B \subseteq A\}$. Again, $\mathbf{A}'\mathbf{B}$ is finite, say $\mathbf{A}'\mathbf{B} = \{\mathbf{A}(B_i): i \in n\}$. $0 = \bigcap_{i \in n} B_i$ is an open neighborhood of x. Since for each $A \in \mathbf{A}$ there is a B and some $i \in n$, such that $A \in \mathbf{A}(B) = \mathbf{A}(B_i)$, $0 \subseteq B_i \subseteq A$ whence $x \in 0 \subseteq \cap \mathbf{A} = \{x\}$.

3.2. The only remaining interpretation of the question which is not ruled out by 3.1 is the following: $\mathbf{A}, \mathbf{B} \subseteq \mathbf{X}$, $\{x\} = \cap \mathbf{A} = \cap \mathbf{B}$, $\mathbf{A}$ amorphous and $\mathbf{B}$ wellorderable. And indeed, there exists a model of ZF with a Hausdorff-space without isolated points which has such families at each of its points. For example, the following model $\mathbf{M}$ of Monro [9] works. $\mathbf{C}$ is the Cohen–Halpern–Levy model with a generic set G of reals. Monro extends $\mathbf{C}$ by adding a generic function g: $G \to A$. He shows, that A is amorphous and $g^{-1}(a)$ is dense in G in its natural topology (inherited from $\boldsymbol{R}$). We induce a topology $\mathbf{G}$ on G by defining as neighbourhoods of $a \in G$ the sets $U(a, \varepsilon, E) = \{b \in G: b = a \text{ or } |a - b| < \varepsilon \text{ and } g(b) \notin E \cup \{g(a)\}\}$ where $\varepsilon > 0$ and $E \subseteq A$ is finite. $(G, \mathbf{G})$ is T_2, without isolated points (since $f^{-1}(c) \cap U(a, \varepsilon, E) \neq \Phi$ for $c \in A \setminus (E \cup \{g(a)\})$ by the density of $f^{-1}(c)$), $\mathbf{A} = \{U(a, 1, \{b\}): b \in A\}$ and $\mathbf{B} = \left\{U\left(a, \frac{1}{n}, \Phi\right): n \geqq 1\right\}$. In this model the space $(G, \mathbf{G})$ is Dedekind-finite and the question appears, if every such space has an infinite, D-finite subset. Using our ultraproduct from section 2, we can answer it as follows (this problem was the original inspiration for this paper).

3.3. There is a model $\mathbf{M}$ of ZF in which there is a T_2 space $(X, \mathbf{X})$ without isolated points which has at each point x the following property: There are families $\mathbf{A}$, $\mathbf{B}$ in $\mathbf{X}$ such that $\{x\} = \cap \mathbf{A} = \cap \mathbf{B}$, $\mathbf{A}$ amorphous, $\mathbf{B}$ countable. Moreover, X has no infinite, Dedekind-finite subsets.

Proof. With the notation of section 2, $X = \prod_{n \in \omega} P_n$ and neighbourhoods of x are defined through $U(x, \varepsilon, E) = \{y \in X: \operatorname{dist}(x, y) < \varepsilon \text{ and } \Phi(y) \notin E \cup \{\Phi(x)\}\}$, where $E \subseteq X/\mathbf{F} = \Phi'X$ is finite and dist is the Baire metric on X: $\operatorname{dist}(x, y) = \frac{1}{n+1}$, if $x \mid n = y \mid n$ and $x(n) \neq y(n)$, while $\operatorname{dist}(x, x) = 0$. In view of 3.2 it suffices to observe, that each $\Phi^{-1}(a)$ is dense in (X, dist). Be $x \in X$, $\varepsilon > 0$ arbitrary and $\frac{1}{n} \leqq \varepsilon$, $\Phi(y) = a$. If we define y' through $y' \mid n = x \mid n$, $y' \mid \omega \setminus n = y \mid \omega \setminus n$, then $\operatorname{dist}(x, y') \leqq \frac{1}{n+1} < \varepsilon$ and $y' = y \bmod \mathbf{F}$, i.e.: $\Phi(y') = a$.

REFERENCES

[1] A. Blass: *A model without ultrafilters*, Bull. Acad. Polon. 25 (1977), 329–331.

[2] N. Brunner: *Lindelöf-Räume und Auswahlaxiom*, Anzeiger Akad. Wiss. Wien 119 (1982), 161–165.

[3] N. Brunner: *Hilberträume mit amorphen Basen*, Compositio Math. 52 (1984) 381–387.

[4] J. D. Halpern: *On a question of Tarski and a theorem of Kurepa*, Pacific J. Math. 41 (1972)-111–121.

[5] J. L. Hickman: *Groups in models of set theory*, Bull. Australian Math. Soc. 14 (1976), 199–232.

[6] P. E. Howard: *Łoš theorem and the BPI imply AC*, Proc. A.M.S. 49 (1975), 426–428.

[7] T. Jech: *The Axiom of Choice*, New York 1973.

[8] A. Levy: *Axioms of Multiple Choice*, Fundamenta Math. 59 (1962), 475–483.

[9] G. P. Monro: *Generic Extensions without AC*, J. Symbolic Logic 48 (1983), 39–52.

[10] D. Pincus: *Strength of the Hahn Banach Theorem*, in Victoria Symp. Nonstandard Analysis, Springer LN 369 (1972).

[11] H. Rubin, J. E. Rubin: *Equivalents of AC II*, North Holland P.C. 1985).

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