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APPROXIMATION RELATIVE TO AN ULTRA FUNCTION

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Abstract. Let $X$ be a non-empty set. A symmetric function $f : X \times X \to R$ is called an ultra function on $X$ if $f(x, y) \leq \max \{f(x, z), f(z, y)\}$ for all $x, y, z \in X$. If $G$ is a subset of a set $X$ with an ultra function $f$ then an element $g_0 \in G$ is said to be (i) an $f$-best approximation to $x \in X$ if $f(x, g_0) \leq f(x, g)$ for all $g \in G$ and (ii) an $f$-best co-approximation to $x$ if $f(g_0, g) \leq f(x, g)$ for all $g \in G$. In this paper we extend some of the known results on best approximation and best co-approximation in non-archimedean normed linear spaces to approximation relative to an ultra function which is defined either on an arbitrary set $X$ or on a Hausdorff topological vector space $X$ over a non-archimedean valued field $F$.

Key words. $f$-best approximation, $f$-best co-approximation, symmetric function and ultra function.

The main aim of the present study is to extend some known results on approximation in non-archimedean normed linear spaces to approximation relative to an ultra function which is defined either on an arbitrary set or on a Hausdorff topological vector space over a non-archimedean valued field.

1. Introduction

The notion of $f$-best approximation in a vector space $X$ was given by Breckner and Brosowski [1] and in a Hausdorff topological space $X$ by the author in [5]. Taking $X$ to be a Hausdorff locally convex topological vector space and $f$ to be a continuous sublinear functional on $X$, certain results on best approximation relative to the functional $f$ were proved in [1], [2] and [8]. We shall discuss $f$-best approximation, $f$-best coapproximation and $f$-orthogonality in Hausdorff topological vector spaces over non-archimedean valued fields relative to an ultra function $f$ in section 2, and in section 3 we shall discuss $f$-best approximation and $f$-best co-approximation for an ultra function $f$ defined on an arbitrary set $X$. When $X$ is a non-archimedean normed linear space and $f = ||.||$, the norm on $X$, we get some of the results of [3], [4] and [6].

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2. \( f \)-Approximation in Topological Vector Spaces

In this section we discuss \( f \)-best approximation, \( f \)-best co-approximation and \( f \)-orthogonality in Hausdorff topological vector spaces over non-archimedean valued fields relative to an ultra function \( f \).

Let \( X \) be a Hausdorff topological vector space over a non-archimedean (n.a.) valued field \( F \) and \( f \) a symmetric (i.e. \( f(-x) = f(x) \) for all \( x \in X \)) real-valued ultra function (i.e. \( f(x + y) \leq \max \{f(x), f(y)\} \) for all \( x, y \in X \)) on \( X \). Let \( K \) be a non-empty closed subset of \( X \) and \( x \in X \).

An element \( k_0 \in K \) is said to be an \( f \)-best approximation to \( x \) in \( K \) if

\[
f(x - k_0) = f_K(x) \equiv \inf \{f(x - k) : k \in K\}.
\]

We denote by \( P_{K,f}(x) \) the collection of all such \( k_0 \in K \). The set \( K \) is said to be \( f \)-proximinal if \( P_{K,f}(x) \) is non-empty for each \( x \in X \), \( f \)-semi-Chebyshev if \( P_{K,f}(x) \) is atmost singleton for any \( x \in X \) and \( f \)-Chebyshev if \( P_{K,f}(x) \) is exactly singleton for each \( x \in X \).

The set \( K \) is said to be \( f \)-infimum compact if for every \( x \in X \) and every minimizing net \( \{k_\alpha\} \) in \( K \) (i.e. \( f(x - k_\alpha) \to f_K(x) \)) has an \( f \)-convergent subset in \( K \).

An element \( g_0 \in K \) is said to be an \( f \)-best coapproximation of an element \( x \in X \) if

\[
f(g_0 - g) \leq f(x - g)
\]

for all \( g \in K \). The set of all such \( g_0 \in K \) is denoted by \( R_{K,f}(x) \).

For \( x, y \in X \), \( x \) is said to be \( f \)-orthogonal to \( y \), \( x \perp_f y \), if

\[
f(x) \leq f(x + \alpha y)
\]

for every scalar \( \alpha \).

\( x \) is said to be \( f \)-orthogonal to \( K \), \( x \perp_f K \), if \( x \perp_f y \) for all \( y \in K \).

The following theorem gives existence of \( f \)-best approximation for a non-negative \( f \).

**Theorem 1.** Let \( K \) be a non-empty \( f \)-infimum compact subset of \( X \). Then \( K \) is \( f \)-proximinal.

**Proof.** Let \( x \in X \) then by the definition of \( f_K(x) \), there exists a net \( \{k_\alpha\} \) in \( K \) such that

\[
f(x - k_\alpha) \to f_K(x).
\]

Since \( \{k_\alpha\} \) is a minimizing net in \( K \) and \( K \) is \( f \)-infimum compact, there exists a subnet \( \{k_\beta\} \) of \( \{k_\alpha\} \) and \( k_0 \in K \) such that \( \lim f(k_\beta - k_0) = 0 \). Consider

\[
f(x - k_0) \leq \max \{f(x - k_\beta), f(k_\beta - k_0)\}.
\]

In the limiting case this gives
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\[ f(x - k_0) \leq f_K(x), \]
\[ \leq f(x - k_0), \]

i.e. \( f(x - k_0) = f_K(x) \) and so \( K \) is \( f \)-proximinal.

**Remark.** It will be interesting to study conditions under which \( K \) is \( f \)-semi-Chebyshev and \( f \)-Chebyshev. One such conditions under which \( K \) is \( f \)-Chebyshev is given in section 3, Theorem 1.

The following theorem characterizes elements of \( f \)-best approximation.

**Theorem 2.** For a linear subspace \( G \) of \( X \), \( g_0 \in P_{G,f}(x) \) if and only if \( (x - g_0) \perp_f G \)

**Proof.** \( (x - g_0) \perp_f G \) if and only if \( f(x - g_0 + ag) \geq f(x - g_0) \) for all \( g \in G \), \( a \in F \),
\[ \iff g_0 \in P_{G,f}(x). \]

**Corollary.** For a linear subspace \( G \), \( P_{G,f}(x) \) is empty for every \( x \in X \) if there exist no \( y \in X \setminus \{0\} \) such that \( y \perp_f G \).

**Proof.** Suppose \( P_{G,f}(x) \neq \emptyset \) for some \( x \in X \setminus G \). Let \( g_0 \in P_{G,f}(x) \). Then \( (x - g_0) \perp_f G \).

Take \( y = x - g_0 \). Then \( y \in X \setminus \{0\} \) and \( y \perp_f G \), a contradiction.

The following theorem characterizes elements of \( f \)-best coapproximation when \( f \) is sublinear (a symmetric sublinear functional is homoneous i.e. \( f(\alpha x) = |\alpha| f(x) \)).

**Theorem 3.** For a linear subspace \( G \), \( g_0 \in R_{G,f}(x) \) if and only if \( G \perp_f (x - g_0) \).

**Proof.** \( G \perp_f (x - g_0) \) if and only if \( f(x - g_0 + \alpha g) \geq f(x - g_0) \) for all \( g \in G \), \( \alpha \in F \),
\[ \iff f(x - g_0 + \alpha^{-1} g) \geq f(x^{-1} g) \] for all \( g \in G \), \( \alpha \in F \), \( \alpha \neq 0 \),
\[ \iff f(x - g_0 + g') \geq f(g') \] for all \( g' \in G \),
\[ \iff f(x - g'') \geq f(g_0' - g'') \] for all \( g'' \in G \),
\[ \iff g_0 \in R_{G,f}(x). \]

**Corollary.** For a linear subspace \( G \), \( R_{G,f}(x) \) is empty for every \( x \in X \) if there exist no \( y \in X \setminus \{0\} \) such that \( G \perp_f y \) when \( f \) is sublinear.

**Proof.** It is similar to Corollary to Theorem 2.

The following result shows that for a sublinear \( f \) the \( f \)-orthogonality is symmetric in \( X \).

**Theorem 4.** For a sublinear \( f \), the \( f \)-orthogonality is symmetric.

**Proof.** Let \( x \perp_f y \). Then
\[ f(x + ay) \geq f(x) \] for every scalar \( \alpha \),
we are to show that \( y \perp_f x \) i.e.
\[ f(y + \beta x) \geq f(y) \] for every scalar \( \beta \).

Suppose that for some \( \beta \neq 0 \in F \),
\[ f(y + \beta x) < f(y). \]
This implies
\[(2) \quad f(x + \beta^{-1}y) < f(\beta^{-1}y),\]
as \(f\) is homogeneous. Then
\[f(x) = f(x + \beta^{-1}y - \beta^{-1}y) = \max \{f(x + \beta^{-1}y), f(\beta^{-1}y)\},\]
as \(f\) is symmetric (if \(f(x) < f(y)\))
\[\text{then } f(x + y) = \max \{f(x), f(y)\} = f(\beta^{-1}y).\]
Then (2) gives
\[f(x + \beta^{-1}y) < f(x),\]
a contradiction to (1). Hence \(y \perp_f x\).

The following theorem shows that for a subspace \(G\), elements of \(f\)-best approximation and \(f\)-best coapproximation coincide and so there is no need to study, \(f\)-best co-approximation separately for a sublinear \(f\).

**Theorem 5.** Let \(G\) be a subspace of \(X\) and \(x \in X\). Then an element of \(f\)-best approximation to \(x\) in \(G\) is an element of \(f\)-best coapproximation and vice-versa i.e. \(P_{f,G}(x) = R_{f,G}(x)\).

**Proof.** The proof follows from Theorems 2, 3 and 4.

### 3. \(f\)-Approximation in Arbitrary Sets

In this section we discuss \(f\)-best approximation and \(f\)-best co-approximation where \(f\) is an ultra function defined on an arbitrary set \(X\).

To start with we restate a few definitions of section 2 in the context of an ultra function defined on an arbitrary set.

Let \(X\) be any set. A symmetric function \(f: X \times X \rightarrow \mathbb{R}\) is called an ultra function on \(X\) [7] if
\[f(x, y) \leq \max \{f(x, z), f(z, y)\}\]
for all \(x, y, z \in X\).

Let \(G\) be a subset of a set \(X\) with an ultra function \(f\).

An element \(g_0 \in G\) is said to be \(f\)-best approximation to \(x \in X\) if
\[f(x, g_0) \leq f(x, g)\]
for all \(g \in G\).

An element \(k_0 \in G\) is said to be \(f\)-best co-approximation of \(x\) if
\[f(k_0, g) \leq f(x, g)\]
for all \(g \in G\).

Regarding the uniqueness of best approximation the following result was proved in [3]:
For a linear subspace $G$ of a n.a. normed linear space $X$, best approximation of $x \in X$, $x \notin G$ in $G$ when it exists is never uniquely determined unless $G = \{0\}$.

The following example shows that in our case, $f$-best approximation may be unique.

Let $X = N$, the set of natural numbers,

$$f: N \times N \to R,$$

defined by

$$f(m, n) = \max \left\{ \frac{1}{m}, \frac{1}{n} \right\},$$

$$G = \{1, 2, 3, \ldots, n : n > 1\},$$

and $n_0 \in X$, $n_0 \notin G$. Then it is easy to see that $n$ is $f$-best approximation for $n_0$ and is unique.

It is interesting to note that every element of $X$ which is not in $G$ has $n$ as $f$-best approximation in $G$.

The following theorem characterizes the uniqueness of $f$-best approximation:

**Theorem 1.** Let $E$ be a subset of a set $X$ with an ultra function $f$ and $x \in X$. An $f$-best approximation $z \in E$ to $x$ is unique if and only if there exist no $t \in E$ such that $f(t, z) \leq f(x, z)$.

**Proof.** Firstly, suppose there exist $t \in E$ such that

$$f(t, z) \leq f(x, z).$$

Then

$$f(x, t) \leq \max \{f(x, z), f(z, t)\} = f(x, z),$$

implies that $t$ is also an $f$-best approximation to $x$, a contradiction.

Conversely, suppose there exist no such $t$. Then $z$ is unique $f$-best approximation to $x$. For, let if possible, there exist $\Theta \in E$, $\Theta \neq z$ such that $\Theta$ is also an $f$-best approximation to $x$. Then

$$f(x, \Theta) = f(x, z) = \inf_{y \in E} f(x, y).$$

Therefore

$$f(\Theta, z) \leq \max \{f(\Theta, x), f(x, z)\}$$

gives

$$f(\Theta, z) \leq f(x, z),$$

a contradiction.

The following result shows that as in section 2, there is no need to study best co-approximation separately in this case too.

**Theorem 2.** Let $G$ be a subset of $X$ and $x \in X$. Then an element of $f$-best approximation to $x$ in $G$ is an element of $f$-best co-approximation and vice-versa.
Proof. Let \( g_0 \in G \) be an \( f \)-best approximation to \( x \). Then

\[
f(x, g_0) \leq f(x, g)
\]

for all \( g \in G \). Consider

\[
f(g_0, g) \leq \max \{f(g_0, x), f(x, g)\} = f(x, g).
\]

Thus \( g_0 \in G \) is \( f \)-best co-approximation to \( x \).

Conversely, suppose \( g_0 \in G \) is \( f \)-best co-approximation to \( x \). Then

\[
f(g_0, g) \leq f(x, g)
\]

for all \( g \in G \). Consider

\[
f(x, g_0) \leq \max \{f(x, g), f(g, g_0)\} = f(x, g).
\]

Thus \( g_0 \in G \) is \( f \)-best approximation to \( x \).

Remark 1. When \( f = d \), the metric on \( X \), we get: In an ultra metric space elements of best approximation and best co-approximation coincide.

Remark 2. The notions of \( \varepsilon \)-approximation, best simultaneous approximation, proximal points of pairs of sets, strong approximation, strong co-approximation, farthest points and strong farthest points, available in literature can be discussed relative to an ultra function defined on an arbitrary set.

REFERENCES


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