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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION

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Abstract. The linear differential equation $y^{(n)} + p(t) y^{(k)} + q(t) y = 0$ is concerned in this paper. Under the conditions that ratios of certain powers of the coefficients and some their derivatives of this equation are small, the asymptotic behaviour as $t \to \infty$ of the fundamental set of solutions are given.

Key words. Linear differential equation of $n$-th order, asymptotic behaviour, asymptotic formula, fundamental system.

MS Classification. 34064

1. Introduction

Asymptotic behaviour of the $n$-order linear differential equations under the conditions $\int t^k |p_k(t)| \, dt < \infty$ or $p_k(t) \to 0$, where $p_k(t)$ are coefficients of this equation, were studied in many papers and they can be found in the monographs E. A. Coddington and N. Levinson [1], P. Hartman [2]. The asymptotic behaviour of this equation under the weaker assumptions that $\int t^k p_k(t) \, dt < \infty$ we can find in [10, 11]. In 1947 A. Wintner [12] derived asymptotic formulae for the differential equation $y'' + q(t) y = 0$ (see them in the corollaries of this paper) which have wide application in quantum mechanics under conditions that ratios of certain powers $q(t)$ and $q'(t)$ are small (improper integrals on $[a, \infty)$ exist). The similar conditions have been used in other papers [3, 4, 5, 8, 9] for differential equations of the second, the third, the fourth and binomial of the $n$-th order.

Since some results of the $n$-order linear differential equations with two coefficients have been lately published [6, 7, 13], the aim of this paper will be investigation of asymptotic properties of the differential equation

$$y^{(n)} + p(t) y^{(k)} - (-1)^m q(t) y = 0.$$  

The results of this paper generalize the results for the third and fourth order and give new results for the second and the $n$-th order linear differential equation generally.
2. Preliminary results

Let us consider the equation (1), where $n, k, m$ are integers, $n > 1$, $1 \leq k < n$, $m = 1, 2, p(t), q(t) > 0$ are continuous functions including the derivatives that stand in theorems.

A vector-matrix form of the equation (1) is

\[(2) \quad z' = A(t) z,\]

where $A(t) = (a_{ij}(t))$ is $n \times n$ matrix defined as follows

\[a_{ij}(t) = \begin{cases} 1 & \text{if } j = i + 1 \\ (-1)^m q(t) & \text{if } i = n \text{ and } j = 1 \\ -p(t) & \text{if } i = n \text{ and } j = k + 1 \\ 0 & \text{otherwise} \end{cases}\]

and $z = [y, y', ..., y^{(n-1)}]^T$. If we make a linear transformation $w = Tz$ with continuously differentiable nonsingular matrix

\[T(t) = dia \left[ q(t)^{1 - \frac{1}{n}}, q(t)^{1 - \frac{2}{n}}, ..., q(t)^{1}, 1 \right],\]

we get the equation

\[(3) \quad w' = \left[ A_0 q(t)^{1 - \frac{1}{n}} + A_1 p(t) q(t)^{\frac{k+1}{n} - 1} + A_2 q'(t) q(t)^{-1} \right] w,\]

where $A_0 = (a_{ij}^0), A_1 = (a_{ij}^1), A_2$ are $n \times n$ constant matrix defined as follows

\[a_{ij}^0 = \begin{cases} 1 & \text{if } j = i + 1 \\ (-1)^m & \text{if } i = n \text{ and } j = 1 \\ 0 & \text{otherwise} \end{cases}\]

\[a_{ij}^1 = \begin{cases} -1 & \text{if } i = n \text{ and } j = k + 1 \\ 0 & \text{otherwise} \end{cases}\]

\[A_2 = \text{dia} \left[ 1 - \frac{1}{n}, 1 - \frac{2}{n}, ..., 1 \right].\]

Suppose $\int_{\alpha}^{\infty} q(t)^{\frac{1}{n}} dt = \infty$. By putting the substitution $t = \alpha(s)$ into (3), where $\alpha(s)$ is the inverse function with $\omega(t) = \int_{\alpha}^{t} q(s)^{\frac{1}{n}} ds$, the equation (3) reduces to

\[(4) \quad x' = [A_0 + A_1 f(s) + A_2 q(s)] x,\]

where $x(s) = (s)$,

\[f(s) = \frac{p[\alpha(s)]}{q[\alpha(s)]^{1 - \frac{k}{n}}}, \quad g(s) = \frac{q'[\alpha(s)]}{q[\alpha(s)]^{\frac{k}{n} + 1}}.\]
An asymptotic behaviour of solutions of the equation

\[ x' = [A + V(s) + R(s)]x \]

was proved in Theorem 8.1 in [1]. Now we proceed to apply this theorem for the equation (4) in two ways. In the first case we put \( V(s) = A_1f(s) + A_2g(s) \) and \( R(s) = 0 \), in the second one we put \( V(s) = A_1f(s) \) and \( R(s) = A_2g(s) \).

Throughout by \( L[a, \infty) \) we denote the Banach space of all complex valued functions which are Lebesque integrable on \([a, \infty)\). The next Lemma will be needed.

**Lemma.** (D. B. Hinton [5].) Let \( h(t) > 0 \) on \([a, \infty)\) and \( h'(t) \cdot h(t)^{-\frac{1}{n}} \in L[a, \infty) \). Then

(i) \( h(t)^{\frac{1}{n}} \notin L[a, \infty) \),

(ii) \( \left[ h'(t) \cdot h(t)^{-\frac{1}{n}} \right] ' \in L[a, \infty) \),

(iii) \( \left[ h'(t) \cdot h(t)^{-\frac{1}{2n}} \right]^2 \in L[a, \infty) \).

3. Main results

**Theorem 1.** Let the functions \( p'(t) \) and \( q'(t) \) be continuous on \([a, \infty)\). Let

\[
\frac{q''(t)}{q(t)^{1 + \frac{1}{n}}} , \quad \frac{p'(t)}{q(t)^{1 - \frac{k}{n}}} \quad \text{and} \quad \frac{p(t)^2}{q(t)^{2 - \frac{1+2k}{n}}}
\]

be in \( L[a, \infty) \). Then there exists a fundamental system \( z_i(t) \) of the equation (2) such that

\[
Tz_iq(t)^{\frac{1-n}{2n}} \cdot \exp \left\{ -\lambda_i \int_{\alpha}^{t} \left[ q(\tau)^{\frac{1}{n}} - (-1)^m \frac{\lambda_i^k}{n} \frac{p(\tau)}{q(\tau)^{1 - \frac{k+1}{n}}} \right] d\tau \right\} \to p_i,
\]

where \( \lambda_i \) are the roots of the equation \( \lambda^n + (-1)^m = 0 \) and \( p_i = [1, \lambda_1, \lambda_1, \ldots, \lambda_i^{-1}]^T \).

**Proof.** To apply Theorem 8.1 of [1] denote \( A_0 = A \) and \( V(s) = A_1f(s) + A_2g(s) \). Since \( \det [\lambda E - A_0] = \lambda^n + (-1)^m \), the characteristic roots \( \lambda_i \) of \( A_0 \) are distinct and \( p_i = [1, \lambda_1, \ldots, \lambda_i^{-1}]^T \) are characteristic vectors of \( A_0 \) corresponding to \( \lambda_i \).

By change of variable \( t = \alpha(s) \) we obtain

\[
\int_{\alpha}^{t} f(\alpha) \, d\alpha = \int_{\alpha}^{t} \left[ \frac{p(\alpha)}{q(\alpha)} \right] \, d\alpha \leq \int_{\alpha}^{t} \left| \frac{p'(t)}{q(t)^{1 - \frac{k}{n}}} \right| \, dt + \left( 1 - \frac{k}{n} \right) \int_{\alpha}^{t} \left| \frac{p(t)q'(t)}{q(t)^{2 - \frac{k}{n}}} \right| \, dt \leq
\]

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= \int_{a}^{\infty} \frac{p'(t)}{q(t)^{1-k/n}} \, dt + \left(1 - \frac{k}{n}\right) \left[ \int_{a}^{\infty} \frac{p^2(t)}{q(t)^{2-1+2k/n}} \, dt \right]^{1/2} \cdot \left[ \int_{a}^{\infty} \frac{q'(t)}{q(t)^{1+1/2n}} \, dt \right]^{1/2}. \\

From the conditions of the theorem and from the Lemma we get \( f'(s) \in L[0, \infty) \). Similarly we deduce
\[
\int_{0}^{\infty} |g'(s)| \, ds \leq \int_{a}^{\infty} \left| \frac{q''(t)}{q(t)^{1+1/n}} \right| \, dt + \left(1 + \frac{1}{n}\right) \int_{a}^{\infty} \left[ \frac{q'(t)}{q(t)^{1+1/2n}} \right]^2 \, dt < \infty,
\]
\[
\int_{0}^{\infty} f(s)^2 \, ds \leq \int_{a}^{\infty} \frac{p(t)^2}{q(t)^{2-1+2k/n}} \, dt < \infty,
\]
\[
\int_{0}^{\infty} g(s)^2 \, ds \leq \int_{a}^{\infty} \left[ \frac{q'(t)}{q(t)^{1+1/2n}} \right]^2 \, dt < \infty.
\]
So we obtained \( \int_{0}^{\infty} |V'(s)| \, ds < \infty \) and \( V(s) \to 0 \) as \( s \to \infty \).

Now we investigate the characteristic roots \( \lambda(s) \) of the matrix \( A_0 + V(s) \). The characteristic equation has the form

\[
(6) \quad P[\lambda(s)] = -(-1)^m + f(s) \prod_{i=1}^{k} \left[ \lambda - \frac{n-i}{n} \, g(s) \right] + \prod_{i=1}^{n} \left[ \lambda - \frac{n-i}{n} \, g(s) \right] = 0.
\]

It is evident that \( P[\lambda(s)] \to \lambda^n - (-1)^m \), because \( f(s) \to 0 \) and \( g(s) \to 0 \) as \( s \to \infty \).

In the notation of Theorem 8.1 of [1] all \( j, 1 \leq j \leq n \) for a given \( i \) are supposed to fall into one of two classes \( I_1 \) and \( I_2 \), where
\[
j \in I_1, \quad \text{if} \quad \int_{0}^{s_1} D_{ij}(s) \, ds \to \infty \quad \text{and} \quad \int_{s_1}^{s_2} D_{ij}(s) \, ds > -K,
\]
\[
j \in I_2, \quad \text{if} \quad \int_{s_1}^{s_2} D_{ij}(s) \, ds < K, \quad \text{for} \quad (s_2 \geq s_1 \geq 0),
\]
where \( K > 0 \) is a constant and \( D_{ij}(s) = \text{Re} \left[ \lambda_i(s) - \lambda_j(s) \right] \).

To prove this fact we express \( \lambda(s) \) in the form
\[
\lambda(s) = \lambda + \beta(s) + \gamma(s),
\]
where \( \beta(s) \to 0, \gamma(s) \to 0 \) as \( s \to \infty \) and \( \gamma(s) \in L[0, \infty) \). For this aim we look for numbers \( c_1, c_2 \) such that
\[
\beta(s) = c_1 f(s) + c_2 g(s)
\]
and \( P[\lambda + \beta(s)] \in L[0, \infty) \). From (6) it follows

\[
(7) \quad P[\lambda + \beta(s)] = -(-1)^m + f(s) \prod_{i=1}^{k} \left[ \lambda + \beta(s) - \frac{n-i}{n} \, g(s) \right] +
\]

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\[ + \prod_{i=1}^{n} \left[ \lambda + \beta(s) - \frac{n-i}{n} g(s) \right]. \]

All terms of the first product in (7), except \([\lambda + \beta(s)]f(s)\), contain \(f^2(s), g^2(s)\) or \(f(s) g(s)\) and so they are in \(L[0, \infty)\). Since we choose \(\beta(s) = c_1 f(s) + c_2 g(s)\), all terms of \([\lambda + \beta(s)]^2 f(s)\) are in \(L[0, \infty)\). Therefore, if we put

\[-(-1)^m + f(s) \lambda^k + \lambda^n - \lambda^{n-1} g(s) \frac{n-1}{2} + n \beta(s) \lambda^{n-1} = 0,\]

i.e.

\[\beta(s) = -\frac{1}{n} \lambda^k f(s) + \frac{1}{2} \frac{n-1}{n} g(s),\]

we obtain that each term of \(P[\lambda + \beta(s)]\) contains \(f^2(s), g^2(s)\) or \(f(s) g(s)\) and hence \(P[\lambda + \beta(s)] \in L[0, \infty)\). Evidently \(\beta(s) \to 0\) as \(s \to \infty\).

Now we are to prove \(\gamma(s) \to 0\) as \(s \to \infty\) and \(\gamma(s) \in L[0, \infty)\). Since \(\lambda(s) = \lambda + \beta(s) + \gamma(s)\) is the characteristic root of \(A_0 + V(s)\), hence

\[P[\lambda + \beta(s) + \gamma(s)] = A(s) \gamma(s) + P[\lambda + \beta(s)] = 0\]

and so

\[(9) \quad |A(s) \gamma(s)| = |P[\lambda + \beta(s)]|.

By the same way as (7) we see that

\[P[\lambda + \beta(s) + \gamma(s)] = -(-1)^m + f(s) \prod_{i=1}^{k} \left[ \lambda + \beta(s) + \gamma(s) - \frac{n-1}{n} g(s) \right] + \prod_{i=1}^{n} \left[ \lambda + \beta(s) + \gamma(s) - \frac{n-1}{n} g(s) \right].\]

(10)

From (10) it follows that \(A(s)\) consists of the terms which tend to zero except \(n\lambda^{n-1}\), i.e.

\[\lim_{s \to \infty} A(s) = n\lambda^{n-1}.\]

Then

\[|A(s)| - |n\lambda^{n-1}| < \frac{1}{2}\]

and

\[|A(s)| > n\lambda^{n-1} - \frac{1}{2} = n - \frac{1}{2} \geq \frac{1}{2}\]

for sufficiently large \(s\). Then (9) gives

\[|\gamma(s)| \leq |2P[\lambda + \beta(s)]|\]

and hence \(\gamma(s) \in L[0, \infty), \gamma(s) \to 0\) as \(s \to \infty\). Consequently we obtain that the characteristic roots \(\lambda_i(s)\) of \(A_0 + V(s)\) may be written as
(11) \[ \lambda_i(s) = \lambda_i + \frac{1}{n} \sum_{k=1}^{L_{i-1}} f(s) + \frac{1}{2} \frac{n-1}{n} g(s) + \gamma_i(s), \]

where \( \lambda_i \) are the roots of \( \lambda^n - (\text{-}1)^m = 0 \), \( \gamma_i(s) \in L[0, \infty) \) and \( \gamma_i(s) \to 0 \) as \( s \to \infty \).

From (11) it follows that \( D_{ij}(s) \) for all \( i, j = 1, 2, \ldots, n \) may have the following forms

a) \( D_{ij}(s) = G(s) \),

b) \( D_{ij}(s) = c + F(s) + G(s) \),

d) \( D_{ij}(s) = -c + F(s) + G(s) \),

where \( c > 0 \) is a number, \( F(s) \), \( G(s) \) are continuous functions on \([0, \infty)\), \( F(s) \to 0 \), \( G(s) \to 0 \) as \( s \to \infty \) and \( G(s) \in L[0, \infty) \).

a) If \( G(s) \in L[0, \infty) \), then there exists a number \( K > 0 \) such that

\[ \int_{s_1}^{s_2} D_{ij}(s) \, ds < K \quad (s_2 \geq s_1 \geq 0) \]

and hence \( j \in I_2 \).

b) If \( F(s) \to 0 \) as \( s \to \infty \), then there exists a number \( s' \in [0, \infty) \) such that \( c + F(s) + G(s) \geq \frac{c}{2} + G(s) \) for all \( s > s' \). Then

\[ \int_{0}^{\infty} D_{ij}(s) \, ds = \int_{0}^{\infty} [c + F(s) + G(s)] \, ds = \infty \]

and

\[ \int_{s_1}^{s_2} D_{ij}(s) \, ds > -K \quad (s_2 \geq s_1 \geq 0), \quad K > 0 \]

because of \( c + F(s) + G(s) \to c \) as \( s \to \infty \). Hence \( j \in I_1 \).

c) From the condition \( F(s) \to 0 \) as \( s \to \infty \) it yields that there exists a number \( s^* > 0 \) such that

\[ -c + F(s) + G(s) < -\frac{c}{2} + G(s) \]

for all \( s > s^* \) and hence

\[ \int_{s_1}^{s_2} D_{ij}(s) \, ds < K \quad (s_2 \geq s_1 \geq 0), \quad K > 0. \]

So \( j \in I_2 \).

Thus all assumptions of Theorem 8.1 of [1] are fulfilled. Therefore there exist \( n \) linearly independent solutions \( x_i(s), i = 1, 2, \ldots, n \) of (4) such that

\[ x_i(s) \exp \left[ -\int_{s_0}^{s} \lambda_i(\xi) \, d\xi \right] \to p_i, \]

i.e.

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(12) \[ x_\lambda(s) \exp \left( -\int_0^s \left[ \lambda_i - \frac{1}{n} \lambda^{k}_{i-1} f(\xi) + \frac{1}{2} n - \frac{1}{n} g(\xi) + \gamma(\xi) \right] d\xi \right) \to \rho_1. \]

By substituting \( \xi = \omega(\tau) \) in (12) and putting
\[ L_i = q[\alpha(s_0)] \frac{1}{2} n - \frac{1}{n} \exp \left[ -\int_{s_0}^\infty \gamma(\tau) d\tau \right] \]
we have
\[ L_i w_i(t) q(t) \frac{1}{n} \exp \left( -\lambda_i \int_{t_0}^t \left[ q(\tau) \frac{1}{n} - (-1)^n \frac{\lambda_i}{n} \frac{p(\tau)}{q(\tau)^{k+1}} \right] d\tau \right) \to \rho_1. \]

Since \( w_i = Tz_i \) and the equation (2) is linear, we have the assertion of Theorem 1.

**Theorem 2.** Let \( p(t) \) and \( q^*(t) \) be continuous functions on \([a, \infty)\). Let
\begin{align*}
\frac{q''(t)}{q(t)^{k+1}} & \quad \text{and} \quad \frac{p(t)}{q(t)^{k+1}} \\
\frac{1-n}{2n} \exp \left[ -\lambda_i \int_{t_0}^t \left( q(\tau) \frac{1}{n} - (-1)^n \frac{\lambda_i}{n} \frac{p(\tau)}{q(\tau)^{k+1}} \right) d\tau \right] \to \rho_1,
\end{align*}
be in \( L(a, \infty) \). Then there exists a fundamental system \( z_i(t) \) of the equation (2) such that
\[ Tz_i q(t) \frac{1-n}{2n} \exp \left[ -\lambda_i \int_{t_0}^t q(\tau) \frac{1}{n} d\tau \right] \to \rho_1, \]
where \( \lambda_i \) are roots of \( \lambda^n - (-1)^m = 0 \) and \( \rho_1 = [1, \lambda_i, \ldots, \lambda^n_{i-1}]^T \).

Proof. In the notation of Theorem 8.1 in [1] we denote \( V(s) = A_2 g(s) \) and \( R(s) = A_1 f(s) \) in (4). Then
\[ \int_{0}^\infty |g'(s)| ds < \infty \quad \text{and} \quad \int_{0}^\infty g^2(s) ds < \infty \]
by the same arguments used in the proof of Theorem 1. So \( \int_{0}^\infty |V'(s)| ds < \infty \) and \( V(s) \to 0 \) as \( s \to \infty \). Since
\[ \int_{0}^\infty |f(s)| ds = \int_{a}^\infty \left| \frac{p(t)}{q(t)^{k+1}} \right| dt < \infty \]
it holds that \( \int_{0}^\infty |R(s)| ds < \infty \).

The characteristic equation of \( A_0 + V(s) \) is
\[ P[\lambda(s)] = -(-1)^m + \prod_{i=1}^n \left[ \lambda - \frac{n-1}{n} g(s) \right] = 0. \]

Similarly as in the proof of Theorem 1 we get that the characteristic roots of (15) may be expressed in the form
\[ \lambda_i(s) = \lambda_i + \frac{1}{2} \frac{n - 1}{n} g(s) + \gamma_i(s), \]

where \( \gamma_i(s) \in L[0, \infty) \) and \( \gamma_i(s) \to 0 \) as \( s \to \infty \). All the other assumptions of Theorem 8.1 in [1] are fulfilled, therefore there exists a fundamental system \( x_i(s) \) of (4) such that

\[ x_i(s) \exp \left[ -\int_{s_0}^{s} \left[ \lambda_i + \frac{1}{2} \frac{n - 1}{n} g(\xi) + \gamma_i(\xi) \right] \, d\xi \right] \to p_i. \]

If we put \( \xi = \omega(\tau) \) and consider \( \gamma_i(s) \in L[0, \infty) \) we get the assertion (14).

4. Corollaries

**Corollary 1.** Suppose the assumptions of Theorem 1 are fulfilled. Then the equation (1) has a fundamental system \( y_i(t) \), \( i = 1, 2, \ldots, n \) such that

\[ y_i^{(j)}(t) = \lambda_i^j q(t) \left( \frac{2j+1-n}{2n} \right) \times \exp \left( \lambda_i \int_{t_0}^{t} \frac{1}{q(r)} \, dr \right) \left[ 1 + o(1) \right] \]

where \( j = 0, 1, \ldots, n - 1 \) and \( \lambda_i \) are the roots of \( \lambda^n - (-1)^m = 0 \).

**Corollary 2.** Suppose the assumptions of Theorem 2 are fulfilled. Then the equation (1) has a fundamental system \( y_i(t) \), \( i = 1, 2, \ldots, n \) such that

(17) \[ y_i^{(j)}(t) = \lambda_i^j q(t) \left( \frac{2j+1-n}{2n} \right) \exp \left( \lambda_i \int_{t_0}^{t} \frac{1}{q(r)} \, dr \right) \left[ 1 + o(1) \right]. \]

**Proof of Corollary 2.** If we put the matrix \( T \) into (14) we have

\[ \text{diag} \left[ q(t) \left( \frac{n-1}{2n} \right), q(t) \left( \frac{n-3}{2n} \right), \ldots, q(t) \left( \frac{n-(2n-1)}{2n} \right) \right] \times \exp \left( -\int_{t_0}^{t} \gamma_i(r) \, dr \right) \left[ y_i, y'_i, \ldots, y_i^{(n-1)} \right]^T \to [1, \lambda_i, \ldots, \lambda_i^{n-1}]^T. \]

From this equality evidently follows (17).

If in the Corollary 2 we put \( n = 2 \) and \( p(t) = 0 \) we obtain

**Corollary 3.** Let \( q(t) > 0 \) and \( q'(t) \) be continuous on \([a, \infty)\). Let

(18) \[ q^*(t) q(t)^{-3/2} \in L[0, \infty). \]

Then the equation

\[ y'' + q(t) y = 0 \]

has the general solution

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(19) \( y(t) = q(t)^{-1/4} \left[ \cos \left( \int_{f_0}^t q(\tau)^{1/2} \, d\tau \right) (c_1 + o(1)) + \sin \left( \int_{f_0}^t q(\tau)^{1/2} \, d\tau \right) (c_2 + o(1)) \right] \)

and for \( y'(t) \) it yields

(20) \( y'(t) = q(t)^{1/4} \left[ -\sin \left( \int_{f_0}^t q(\tau)^{1/2} \, d\tau \right) (c_1 + o(1)) + \cos \left( \int_{f_0}^t q(\tau)^{1/2} \, d\tau \right) (c_2 + o(1)) \right] \).

A. Wintner [12] proved the assertions (19) and (20) under the conditions

(21) \( \int_0^\infty q(t)^{1/2} \, dt = \infty \) and \( \int_0^\infty \left| \frac{5q'(t)^2}{16q(t)^3} - \frac{q''(t)}{4q(t)^2} \right| q(t)^{1/2} \, dt < \infty. \)

To compare the assumptions (21) and (18) we easily verify that (18) implies (21), so Wintner theorem is a little general. However the Theorems 1 and 2 give other asymptotic formulae for differential equations of the second order.

If in Theorems 1 and 2 we put \( n = 3 \), resp. \( n = 4 \) and \( k = 1 \) we obtain the results of the papers [8] and [9].

REFERENCES


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