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ON THE ONE PRINCIPAL CONGRUENCE IDENTITY

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Abstract. If two congruences $\Theta(x, 0)$, $\Theta(y, 0)$ permute, then clearly $\Theta(x, 0) . \Theta(y, 0)$ is a congruence and $\Theta(x, y) \subseteq \Theta(x, 0) . \Theta(y, 0)$. The paper gives sufficient conditions under which this relation identity is satisfied also in the case of non permutable congruences $\Theta(x, 0)$, $\Theta(y, 0)$.

Key words. Congruence relation, binary polynomial, variety.

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It is well-known fact that the relational product of two congruences $\Theta_1, \Theta_2 \in Con\ A$ is a congruence on A if and only if $\Theta_1 . \Theta_2 = \Theta_2 . \Theta_1$. An algebra A is *congruence permutable* if this equality is true for each $\Theta_1, \Theta_2 \in Con\ A$; a variety $\mathscr{V}$ is *congruence permutable* if each $A \in \mathscr{V}$ has this property. Denote *by* $\Theta(a, b)$ the principal congruence on A containing the pair $\langle a, b \rangle \in A \times A$.

Let A be an algebra with a nullary operation 0. Since $\langle x, y \rangle \in \Theta(x, 0). \Theta(y, 0)$ for each $x, y \in A$, it is clear that

$$(*) \qquad \Theta(x, y) \subseteq \Theta(x, 0) . \Theta(y, 0)$$

if $\Theta(x, 0) . \Theta(y, 0) = \Theta(y, 0) . \Theta(x, 0)$. However, $(*)$ can be satisfied also if $\Theta(x, 0)$, $\Theta(y, 0)$ do not permute. The investigation of $(*)$ in this case is the aim of this short note. We say that $(*)$ is satisfied in A if it is true for each $x, y \in A$.

By a *tolerance* on an algebra A is meant a reflexive and symmetrical binary relation on A satisfying the substitution property with respect to all operations of A. Since the set of all tolerances on A forms a complete lattice with respect to set inclusion [2], there exists the least tolerance on A containing the given pair $\langle a, b \rangle \in A \times A$; denote it by $T(a, b)$. It is called the *principal tolerance on A* (generated by $\langle a, b \rangle$). An algebra A is *tolerance trivial* if each tolerance on A is a congruence on A. A is *principal tolerance trivial* if $T(a, b) = \Theta(a, b)$ for each a, b of A. A variety $\mathscr{V}$ is *(principal) tolerance trivial* if each $A \in \mathscr{V}$ has this property. A variety $\mathscr{V}$ is tolerance trivial if and only if $\mathscr{V}$ is congruence permutable, [3], [9]. Principal tolerance trivial algebras and varieties were characterized in [3], [4], [8].

Since (*) contains a principal congruence generated by the pair containing 0 and since other statements presented here are formulated for principal tolerances, we can firstly repeat the following assertion (see [1] or [4]):

Theorem 1. *Let $\mathcal{V}$ be a variety with a nullary operation* 0. *The following conditions are equivalent:*

(1) $T(x, 0) = \Theta(x, 0)$ *is the relation identity in* $\mathcal{V}$;

(2) $T(x, 0) . T(y, 0) . T(x, 0) = T(y, 0) . T(x, 0) . T(y, 0)$

is the relation identity in $\mathcal{V}$.

It implies that the relation identity $T(x, 0) = \Theta(x, 0)$ is not equal to the relation identity $T(x, 0) . T(y, 0) = T(y, 0) . T(x, 0)$. However, varieties satisfying the last identity have a special polynomial property which will be used for characterizing of (*) and which is satisfied in each permutable variety with 0. Namely, if $\mathcal{V}$ is a congruence permutable variety, then there exists a ternary polynomial $p(x, y, z)$ such that $p(x, z, z) = x$ and $p(x, x, z) = z$. Let $\mathcal{V}$ be congruence permutable and contain a nullary operation 0. Put $b(x, y) = p(x, 0, y)$. Then clearly

$$b(x, 0) = x \quad \text{and} \quad b(0, x) = x.$$

(Note that varieties with a binary polynomial $b(x, y)$ satisfying $b(x, x) = 0$, $b(x, 0) = x$ are "permutable at 0", see [1], [6], [7] and varieties with $b(x, y)$ satisfying $b(x, x) = 0$, $b(0, x) = 0$, $b(x, 0) = x$ are "arithmetical at 0", see [6]).

Theorem 2. *Let $\mathcal{V}$ be a variety with a nullary operation* 0. *If $T(x, 0) . T(y, 0) = T(y, 0) . T(x, 0)$ is an relation identity in $\mathcal{V}$, then there exists a binary polynomial $b(x, y)$ such that*

$$b(x, 0) = x, \quad b(0, x) = x.$$

Proof. Let $\mathcal{V}$ be a variety with 0 and $F_2(x, y)$ be the free algebra of $\mathcal{V}$ generated by x, y. Suppose $T(x, 0) . T(y, 0) = T(y, 0) . T(x, 0)$. Since $\langle x, y \rangle \in T(x, 0) . T(y, 0)$, thus $\langle x, y \rangle \in T(y, 0) . T(x, 0)$, i.e. there exists an element $v \in F_2(x, y)$ with $\langle x, v \rangle \in T(y, 0)$ and $\langle v, y \rangle \in T(x, 0)$. Hence $v = b(x, y)$ for some binary polynomial b and

$$\langle x, b(x, y) \rangle \in T(y, 0) \quad \text{implies} \quad b(x, 0) = x$$

and

$$\langle b(x, y), y \rangle \in T(x, 0) \quad \text{implies} \quad b(0, y) = y.$$

Example. There exists a wide class of varieties having a binary polynomial $b(x, y)$ with $b(x, 0) = x = b(0, x)$. If $\mathcal{V}$ is a variety of $\vee$-*semilattices with the least element* 0, then $b(x, y) = x \vee y$. If $\mathcal{V}$ is a *variety of additive groupoids with* 0 (i.e. $x + 0 = x = 0 + x$), we can put $b(x, y) = x + y$.

Theorem 3. *Let $\mathcal{V}$ be a variety with a nullary operation* 0. *The following conditions are equivalent:*

(1) $$\langle y, x\rangle \in T(x,0) \,.\, T(y,0);$$

(2) $$T(x,y) \subseteq T(x,0) \,.\, T(y,0);$$

(3) $$T(x,y) \subseteq \Theta(x,0) \,.\, \Theta(y,0);$$

(4) *there exists a binary polynomial* $b(x, y)$ *with*

$$b(x,0) = x, \qquad b(0,x) = x.$$

Proof. (1) $\Rightarrow$ (2): Let $\mathscr{V}$ satisfy (1) and $A \in \mathscr{V}$, $x, y \in A$. Clearly the relation $T(x,0) \,.\, T(y,0)$ is reflexive and has the substitution property. Thus

$$\langle x, y\rangle \in T(x,0) \,.\, T(y,0)$$

and

$$\langle y, x\rangle \in T(x,0) \,.\, T(y,0)$$

imply also

$$\langle \varphi(x,y), \varphi(y,x)\rangle \in T(x,0) \,.\, T(y,0)$$

for every binary algebraic function φ over A. By Lemma 2 of [2], we have (2) The implication (2) $\Rightarrow$ (3) is evident. Prove (3) $\Rightarrow$ (4): Let $F_2(x, y)$ be a free algebra of a variety $\mathscr{V}$ with 0 satisfying (3) and $\Theta(x,0)$, $\Theta(y,0) \in \text{Con}\, F_2(x,y)$. Clearly

$$\langle y, x\rangle \in \Theta(x,0) \,.\, \Theta(y,0).$$

We obtain (4) in the way completely analogous to that in the proof of Theorem 2.

(4) $\Rightarrow$ (1): Suppose $A \in \mathscr{V}$, $x, y \in A$ and $\mathscr{V}$ satisfies (4). Then

$$\langle y, b(x,y)\rangle = \langle b(0,y), b(x,y)\rangle \in T(x,0)$$
$$\langle b(x,y), x\rangle = \langle b(x,y), b(x,0)\rangle \in T(y,0),$$

thus $\langle y, x\rangle \in T(x,0) \,.\, T(y,0)$.

Corollary 1. *Let* $\mathscr{V}$ *be a variety with* 0 *satisfying the relation identity*

$$T(x,0) \,.\, T(y,0) \,.\, T(x,0) = T(y,0) \,.\, T(x,0) \,.\, T(y,0).$$

$\mathscr{V}$ *satisfies* $\Theta(x,y) \subseteq \Theta(x,0) \,.\, \Theta(y,0)$ *if and only if there exists a binary polynomial* $b(x,y)$ *with* $b(x,0) = x = b(0,x)$.

It follows directly from Theorem 1 and Theorem 3.

Corollary 2. *If* $\mathscr{V}$ *is principal tolerance trivial variety with* 0, *then* $\mathscr{V}$ *satisfies* $\Theta(x,y) \subseteq \Theta(x,0) \,.\, \Theta(y,0)$ if and *only if there exists a binary polynomial* $b(x,y)$ *such that*

$$b(x,0) = x = b(0,x).$$

Corollary 3. *The variety of all distributive lattices with the least element* 0 *satisfies the relation identity*

$$\Theta(x,y) \subseteq \Theta(x,0) \,.\, \Theta(y,0).$$

Proof. By [5], the variety of all distributive lattices is principal tolerance trivial. By the Example, there exists a binary polynomial $b(x, y) = x \vee y$ satisfying (4) of Theorem 3.

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