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BOUNDED, ALMOST-PERIODIC AND PERIODIC SOLUTIONS OF CERTAIN SINGULARLY PERTURBED SYSTEMS WITH DELAY

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Abstract. In this paper we shall show that the problem of the existence of bounded (almost-periodic and periodic) solutions of a certain singularly perturbed system with delay is equivalent to the solvability in the formerly mentioned classes of functions, of certain differential-operational equation, in which the small parameter appears in a regular manner.

Key words. Singular perturbation, differential-operational equation, bounded, almost-periodic and periodic solutions, system with delay, asymptotic expansion.

MS Classification. 34 K 25

1. INTRODUCTION

There is not a unifying method for attacking problems in the theory of singular perturbations. Particularly, in this paper we mention the method of differential-operational equation proposed by V. I. Rozhkov in [4], which had been employed successfully in the problem of the existence of periodic solutions for singularly perturbed ordinary differential equations. In this work we shall establish the validity of the above mentioned method for the following system with delay:

$$\begin{aligned} x'(t) &= G(t, x(t), x(t - \varepsilon), y(t)) \\ \varepsilon y'(t) &= F(t, x(t - \varepsilon), y(t)), \end{aligned} \tag{1}$$

where $\varepsilon > 0$, $x(t) \in R^n$, $y(t) \in R^m$. More precisely, we shall show that the problem of the existence of bounded solutions (almost-periodic or periodic) of system (1) is equivalent to the solvability, in the formerly mentioned classes of functions, of certain differential-operational equation. In spite of the fact that the equivalent equation is a differential-operational one, its study is simpler than the original system, because the parameter ε appears in a regular manner. Moreover, the equivalent differential-operational equation can be transformed in such a way that the methods developed for the case in which the right-hand sides are regular functions of ε are applicable to this transformed equation.

2. PRELIMINARIES

Throughout the following $B(R) = \{\varphi\colon R \to C^n$, φ-uniformly continuous and bounded$\}$ and for $\varphi \in B(R)$, $\| \varphi \| = \sup \{| \varphi(t) |, t \in R\}$. The subset AP of $B(R)$ denotes the set of almost-periodic functions. The subset P_ω of AP denotes the set of periodic functions of period ω. The spaces $B(R)$, AP and P_ω with $\| \cdot \|$ defined as above are Banach spaces. An $n \times n$ matrix function on R is said to belong to one of these spaces if each column belongs to the space.

In what follows we shall suppose D is one of the classes $B(R)$, AP or P_ω. We shall say that a solution $(x(\cdot, \varepsilon), y(\cdot, \varepsilon))$ in D of system (1) is **asymptotically close to** $(\bar{x}, \bar{y}) \in D$, if there is a $\varepsilon_0 > 0$ such that for each $\varepsilon, \in (0, \varepsilon_0]$

$$\| x(\cdot, \varepsilon) - \bar{x} \| \leqq \varrho, \qquad \| y(\cdot, \varepsilon) - \bar{y} \| \leqq \sigma,$$

for ϱ and σ sufficiently small.

Consider the system

$$\varepsilon x'(t) = A(t)\, x(t), \tag{3}$$

where ε is a small positive parameter.

Lemma 1. *Suppose that the following conditions hold:*

i) The matrix $A : R \to R^{nxn}$ belongs to $B(R)$.

ii) $| Re\lambda_i(A(t)) | \geqq \mu > 0$ for each $t \in R$, and $i = 1, \dots, n$, where $Re\lambda_i(A(t))$ denotes the real part of the i-th eigenvalue of the matrix $A(t)$.

Then there exist constant $\varepsilon_0 > 0$, $\alpha > 0$, $\beta > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$ system (3) *has a fundamental matrix $U(t)$ such that the matrix*

$$G(t, s) = \begin{cases} U(t)\, P U^{-1}(s), & t > s \\ -U(t)\,(I - P)\, U^{-1}(s), & t < s \end{cases} \tag{4}$$

verifies the following inequality:

$$| G(t, s) | \leqq \beta \exp\left\{-\frac{\alpha}{\varepsilon} | t - s |\right\}, \qquad \forall t, s \in R; \tag{5}$$

where P is a projection operator.

The assertion of Lemma 1 certainly follows from Lemma 3 in [1], except for obvious modifications.

Consider now the nonhomogeneous linear system

$$\varepsilon y'(t) = A(t)\, y(t) + f(t). \tag{6}$$

The following result holds:

Lemma 2. *Assume that the hypotheses of Lemma 1 hold. If A and f belong to D, then there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ the function*

(7) $$(K_\varepsilon f)(t) := \varepsilon^{-1} \int_R G(t, s) f(s) \, ds,$$

is the unique solution of (6) *in* D, *where* $G(t, s)$ *is given by the expression* (4). *Furthermore,* K_ε *is a continuous linear operator from* D *into itself and there is a constant* $K > 0$ (*independent of* ε) *such that*

(8) $$| K_\varepsilon f |_D \leqq K | f |_D,$$

for every $f \in D$.

Proof. Let $y(t) := (K_\varepsilon f)(t)$. Rewriting $y(t)$ as

$$y(t) = \varepsilon^{-1}\left(\int_{-\infty}^{t} G(t, s) f(s) \, ds + \int_{t}^{+\infty} G(t, s) f(s) \, ds \right),$$

and differentiating with respect to t, we get that y is a solution of (6). The uniqueness follows from the fact that under the conditions of the Lemma $x = 0$ is the unique solution of (3) belonging to D.

In order to prove (8) assume first that A and f belong to $B(R)$. It is clear that K_ε is a linear operator, and for $D = B(R)$ (8) follows from (5) and (7), with $K = 2\beta/\alpha$.

Assume now that A and f belong to AP. Since $AP \subset B(R)$, therefore it remains only to show that $K_\varepsilon f$ belongs to AP. Let (τ_j) be any sequence of real numbers for which $f(t - \tau_j) - f(t) \to 0$ as $j \to \infty$ uniformly for $t \in R$. Let $f_j(t) = f(t + \tau_j)$. By virtue of the definition of y, we obtain:

$$\begin{aligned} y(t + \tau_j) - y(t) &= (K_\varepsilon f)(t + \tau_j) - K_\varepsilon f)(t) = (K_\varepsilon f_j)(t) - (K_\varepsilon f)(t) = \\ &= K_\varepsilon(f_j - f)(t). \end{aligned}$$

This together with (8) imply $| y(t + \tau_j) - y(t) | \to 0$ as $j \to \infty$ uniformly on R, so that $K_\varepsilon f \in AP$ (see Thm. 8, p. 320 in [2]).

Finally, let A and f in P_ω. From the expression (7) and the fact that $G(t + \omega, s + \omega) = G(t, s)$, we get that $K_\varepsilon f \in P_\omega$. This completes the proof.

3. THE EQUIVALENT DIFFERENTIAL-OPERATIONAL EQUATION FOR SYSTEM (1)

We shall suppose that the so-called degenerate system

$$\begin{aligned} x'(t) &= G(t, x(t), x(t), y(t)) \\ 0 &= F(t, x(t), y(t)) \end{aligned}$$

has at least a solution $(\bar{x}, \bar{y}) \in D$, and that for each $(x, y) \in D$, $F(\cdot, x(\cdot), y(\cdot)) \in D$. Let us fix a solution $(\bar{x}, \bar{y}) \in D$ of the degenerate system, and let $Q^\varrho_{\bar{x}} = \{x(\cdot) \in D : \| x - \bar{x} \| \leqq \varrho\}$. Let x be an arbitrarily given function belonging to the int $Q^\varrho_{\bar{x}}$ and consider the following differential equation:

$$\varepsilon z'(t) = F(t, x(t-\varepsilon), z(t)). \tag{9}$$

Lemma 3. *Suppose that the conditions of Lemma 1 hold, with* $A(t) := \frac{\partial F}{\partial y}(t, \bar{x}(t), \bar{y}(t))$. *Then there exist constants* $\sigma_0 > 0$ *and* $\varrho_0 > 0$ *such that for each* $\varrho \in (0, \varrho_0]$, *there exists* $\varepsilon_0 > 0$ *such that for every* $\varepsilon \in (0, \varepsilon_0]$ *and* $\sigma \in (0, \sigma_0]$ *the system (9) has a unique solution* $z(\cdot, \varepsilon) \in Q_{\bar{y}}^{\sigma} = \{z(\cdot) \in D : \| z - \bar{y} \| \leqq \sigma\}$.

Proof. Setting $v = z - \bar{y}$, we reduce the problem to the existence in D of small solutions of the following system:

$$\varepsilon v'(t) = A(t)\, v(t) + R(t, v(t), \varepsilon), \tag{10}$$

where $R(t, v, \varepsilon) := F(t, x(t-\varepsilon), v + \bar{y}(t)) - A(t)\, v - \varepsilon \bar{y}'(t)$.

For a given $\sigma > 0$, let $S_\sigma = \{v(\cdot) \in D : \| v \| \leqq \sigma\}$. For any $v \in S_\sigma$ the function $R(\cdot, v(\cdot), \varepsilon)$ belongs to D. By the conditions of the Lemma, we may consider the transformation $w := T_\varepsilon v$, $v \in D$, defined by $T_\varepsilon v := K_\varepsilon R(\cdot, v(\cdot), \varepsilon)$, where K_ε is the operator defined in Lemma 2. Furthermore, the fixed points of $T_\varepsilon \in S_\sigma$ coincide with the solutions of (10) which are in S_σ. We now use the contraction principle to show that T_ε has a unique fixed point in $S\sigma$ for σ, ϱ and ε sufficiently small.

Since $x(\cdot) \in \operatorname{int} Q_{\bar{x}}^{\varrho}$, there exists an $\varepsilon_0(x) > 0$ such that $x(\cdot - \varepsilon) \in Q_{\bar{x}}^{\varrho}$, for each $\varepsilon \in [0, \varepsilon_0]$. Taking into account this remark and the fact that $x \to \bar{x}$ as $\varrho \to 0$, we get the existence of a continuous function $\nu(\varrho) \geqq 0$, $\nu(\varrho) \to 0$ as $\varrho \to 0$ such that

$$| F(t, x(t-\varepsilon), v(t) + \bar{y}(t)) - F(t, \bar{x}(t), v(t) + \bar{y}(t)) | \leqq \nu(\varrho), \tag{11}$$

for every $\varepsilon \in [0, \varepsilon_0]$ and $t \in \boldsymbol{R}$.

Let $q(t, v) := F(t, \bar{x}(t), v(t) + \bar{y}(t)) - A(t)\, v(t)$. By virtue of the hypotheses of Lemma 3 and the definition of $A(t)$, it follows that $q(t, v) \to 0$, $\frac{\partial q}{\partial v}(t, v) \to 0$ as $v \to 0$, uniformly in $t \in \boldsymbol{R}$. Hence, there is a continuous monotone increasing function $\eta(\sigma) \geqq 0$ such that

$$| F(t, \bar{x}(t), v(t) + \bar{y}(t)) - A(t)\, v(t) | \leqq \sigma \eta(\sigma), \qquad t \in \boldsymbol{R}. \tag{12}$$

Let $v \in S\sigma$. From (8), the definition of T_ε, (11) and (12), it follows that for every $\sigma > 0$ and $\varrho > 0$ there exists $\varepsilon_0 > 0$:

$$\| T_\varepsilon v \| \leqq K^*(\sigma \eta(\sigma) + \nu(\varrho) + \varepsilon), \tag{13}$$

for each $\varepsilon \in (0, \varepsilon_0]$, where K^* is a positive constant independent of ε.

An analogous argument yields

$$\| T_\varepsilon v_1 - T_\varepsilon v_2 \| \leqq K^*(\eta(\sigma) + \nu(\varrho)) \| v_1 - v_2 \|, \tag{14}$$

for every v_1 and $v_2 \in S_\sigma$.

Finally, from (13) and (14) our claim certainly follows.

Let $\sigma_0 > 0$, $\varrho_0 > 0$ and $\varepsilon_0 > 0$ be the numbers given by Lemma 3. For each

$\sigma \in (0, \sigma_0]$ and $\varepsilon \in (0, \varepsilon_0]$ with $0 < \varrho \leqq \varrho_0$ define a transformation $\Pi_\varepsilon : Q_{\bar{x}}^{\varrho} \to Q_{\bar{y}}^{\sigma}$ as follows: $\forall\, x \in \operatorname{int} Q_{\bar{x}}^{\varrho}$, set $\Pi_\varepsilon x := z(\cdot, \varepsilon)$, where $z(\cdot, \varepsilon)$ is the unique solution of system (9) belonging to $Q_{\bar{y}}^{\sigma}$ given by Lemma 3. With this definition of Π_ε, we can state the following:

Lemma 4. *Under conditions of Lemma 3, the existence of a solution* $(x(\cdot, \varepsilon), y(\cdot, \varepsilon)) \in D$ *of system (1) asymptotically close to* $(\bar{x}, \bar{y})$ *is equivalent to the existence of a solution in D of the following differential-operational equation:*

$$x'(t) = G(t, x(t), x(t-\varepsilon), \Pi_\varepsilon x), \qquad y = \Pi_\varepsilon x. \tag{15}$$

For a proof see the analogous Lemma 2 in [4].

In spite of the fact that (15) is a differential-operational equation, it is simpler to study than the original system, since the parameter ε appears in (15) in a regular manner.

4. ASYMPTOTIC EXPANSION OF Π_ε

Assume the functions F and x are sufficiently smooth with derivatives belonging to D. Let us seek an approximate solution of (9) in the form of a Poincaré-type asymptotic expansion $\sum_{i=0}^{\infty} \varepsilon^i z_i$. Substitutuing into (9), expanding, and equating coefficients of equal powers of ε leads to:

$$F(t, x(t), z_0(t)) = 0, \tag{16_0}$$

$$z_0'(t) = -\frac{\partial F}{\partial x}(t, x, z_0)\, x'(t) + \frac{\partial F}{\partial y}(t, x, z_0)\, z_1(t), \tag{16_1}$$

....

$$z_{k-1}'(t) = \frac{\partial F}{\partial y}(t, x, z_0)\, z_k(t) + R_k(t, x, z_0, \dots, z_{k-1}). \tag{16_k}$$

Assuming x close to $\bar{x}$, we wish to solve (16_0) for $z_0(t, x)$. In order to do this, let us define an operator $H : Q_{\bar{x}}^{\varrho} \times Q_{\bar{y}}^{\sigma} \to D$ as $H(x, y)(t) := F(t, x(t), y(t))$. Taking into account the hypotheses of Lemma 3, it follows that $H(\bar{x}, \bar{y}) = 0$ and $\frac{\partial H}{\partial y}(\bar{x}, \bar{y})$ has a bounded inverse. Then, by the implicit function theorem on Banach spaces, there is a unique solution z_0 of (16_0) for x close to $\bar{x}$. Moreover, we can choose the pair (x, z_0) in such a way that the matrix $\frac{\partial F}{\partial y}(t, x(t), z_0(t))$ has a bounded inverse, for every $t \in \boldsymbol{R}$. Hence, from $(16_0)-(16_k)$ we obtain $z_0, z_1, \dots, z_k$ successively.

Let us set $Z_k(t, \varepsilon) := \sum_{i=0}^{k} \varepsilon^i z_i(t)$. In what follows we shall assume that the asumptions of Lemma 3 hold, unless the contrary is explicitly stated.

Lemma 5. *Suppose that the functions F and x have continuous and bounded derivatives up to the $(k + 1)$-st order inclusive in all its variables. Then the following asymptotic estimate holds:*

$$\|\Pi_\varepsilon x - Z_k(\cdot, \varepsilon)\| \leqq C\varepsilon^{k+1},$$

where C is independent of ε for sufficiently small ε.

Proof. Analogously to the proof of Lemma 3 in [4], we obtain that $u_k = \Pi_\varepsilon x - Z_k(t, \varepsilon)$ satisfies the following rough estimate

$$\| u_k \| \leqq \sigma + C_1\varepsilon + C_2\varrho, \tag{17}$$

for some positive constants C_1 and C_2.

Since $Z_k(\cdot, \varepsilon)$ satisfies the system

$$\varepsilon Z_k'(t, \varepsilon) = F(t, x(t - \varepsilon), Z_k(t, \varepsilon)) + P_k(t, \varepsilon),$$

with $\| P_k \| \leqq C_3\varepsilon^{k+1}$, it follows that $u_k \in D$ and is solution of the system

$$\varepsilon u_k'(t) = A(t)\, u_k + R(t, u_k, \varepsilon), \tag{18}$$

where $R(t, u_k, \varepsilon) = F(t, x(t - \varepsilon), u_k + Z_k) - F(t, x(t - \varepsilon), Z_k) + P_k - A(t)\, u_k$.

A similar argument to the one applied in the proof of (13), and (17) yield

$$\| R(\cdot, u_k(.), \varepsilon) \| \leqq C_4(\varepsilon^{k+1} + \eta^*(\varepsilon, \varrho, \sigma) \| u_k \|),$$

where η^* is a continuous function, $\eta^*(\varepsilon, \varrho, \sigma) \geqq 0$ for each $\varepsilon \geqq 0$, $\varrho \geqq 0$, $\sigma \geqq 0$; $\eta^*(\varepsilon, \varrho, \sigma) \to 0$ as $\varepsilon, \varrho, \sigma \to 0$.

Let us define the operator T_ε as in the proof of Lemma 3. Since $u_k \in D$ is solution of (18), by virtue of Lemma 2, we get $\| u_k \| = \| T_\varepsilon u_k \| = \| K_\varepsilon R(\cdot, u_k(\cdot), \varepsilon) \| \leqq \leqq KC_4(\varepsilon^{k+1} + \eta^*(\varepsilon, \varrho, \sigma) \| u_k \|)$.

This implies our assertion.

Of course, the application of Lemma 5 is only meaningful, if the solution $x(\cdot, \varepsilon)$ in D of system (1) verifies the required hypotheses. We can justify this procedure using the following:

Lemma 6. *Suppose that the system (1) has a solution $(x(\cdot, \varepsilon), y(\cdot, \varepsilon))$ in D asymptotically close to $(\bar{x}, \bar{y})$. If G and F have continuous and bounded derivative in all its variables up to the $(k + 1)$-st order inclusive, then $(x(\cdot, \varepsilon), y(\cdot, \varepsilon))$ has bounded and continuous derivatives up to the $(k + 1)$-st order inclusive uniformly bounded in ε on $[0, \varepsilon_0]$, for each $t \geqq t_0 + \varepsilon(k - 1)$.*

Proof. Let us set $x(t) := x(t, \varepsilon)$, $y(t) := y(t, \varepsilon)$. For $t \geqq t_0 + \varepsilon(k - 1)$ the existence of the derivatives of $(x(t), y(t))$ follows from the fact that the solutions of system with delay become smoother with increasing values of t.

It remains to prove the uniform boundedness of the derivatives of (x, y) respect to $\varepsilon \in (0, \varepsilon_0]$. From (1) we have that x' is uniformly bounded respect to ε, for all $t \geqq t_0$; of course the derivative at t_0 represents the right-hand derivative. Moreover, for each $t > t_0 + \varepsilon$, we have

$$\varepsilon(y')' = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} x'(t - \varepsilon) + \frac{\partial F}{\partial y} y' =$$

$$= A(t)\, y' + \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} x'(t - \varepsilon) + \left(\frac{\partial F}{\partial y} - A(t)\right) y'(t)\right).$$

Taking into account this equation, the definition of T_ε and Lemma 2, we obtain

$$\| y' \| = \| T_\varepsilon y' \| = \left\| K_\varepsilon \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} x'(. - \varepsilon) + \left(\frac{\partial F}{\partial y} - A(.)\right) y'\right)\right\| \leqq$$

$$\leqq C_1 + C_2 \| y' \| \left(\sup_{t \geqq t_0} \left| \frac{\partial F}{\partial y}(t, x(t - \varepsilon), y(t)) - \frac{\partial F}{\partial y}(t, \bar{x}(t), y(t)) \right| + \right.$$

$$\left. + \sup_{t \geqq t_0} \left| \frac{\partial F}{\partial y}(t, \bar{x}(t), y(t)) - A(t) \right| \right),$$

$$\| y' \| \leqq C_1 + C_2(v_1(\varrho) + v_2(\sigma)) \| y' \|, \ .$$

where $v_i(r) \geqq 0$, $r \geqq 0$; $v_i(r) \to 0$ as $r \to 0$ $(i = 1, 2)$. Choosing σ and ϱ such that $v_1(\varrho) + v_2(\sigma) < 1/C_2$, we get

$$\| y' \| \leqq C_1(1 - C_2(v_1(\varrho) + v_2(\sigma)))^{-1}.$$

Since $x'' = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x} x'(t) + \frac{\partial G}{\partial x^*} x(t - \varepsilon) + \frac{\partial G}{\partial y} y'$, $\forall t > t_0 + \varepsilon$, then x'' is unifoımly bounded with respect to ε on $[t_0 + \varepsilon, +\infty)$.

An analogous argument prove the boundedness of others derivatives of (x, y).

Corollary. *If F and G are ω-periodic functions in t and (x, y) is an ω-periodic solution of (1), then the assertion of Lemma 6 holds for all $t \geqq t_0$.*

We point out that if we need that Lemma 6 holds for all $t > t_0$, then we must require more smoothness of initial data and additional boundary conditions.

5. THE MAIN RESULT

From Lemma 5, we have

$$\Pi_\varepsilon x(t) = z_0(t) + 0(\varepsilon), \qquad \forall t \geqq t_0, \tag{19}$$

where $z_0(t) + \varphi(t, x(t))$ the unique root of equation (16_0). Substituing (19) in (15), we obtain that x must verify the following equation

$$x'(t) = G_1(t, x(t), x(t - \varepsilon)) + r_1(t, x, \varepsilon), \qquad t \geqq t_0,$$

where $G_1(t, x(t), x(t - \varepsilon)) = G(t, x(t), x(t - \varepsilon), \varphi(t, x(t)))$ and r_1 satisfies the estimate $\| r_1 \| \leqq C\varepsilon$.

Repeating this procedure, we get that the required solution x of system (1) will have to satisfy the equation:

$$x'(t) = G_i(t, x(t), x(t - \varepsilon)) + r_i(t, x(t), \varepsilon) \tag{20}$$

for all $t \geqq t_0 + \varepsilon(i - 1)$, $i = 1, \dots, k + 1$; where $\| r_i \| = 0(\varepsilon^i)$.

These results are summarized in:

Theorem 7. *Under the hypotheses of Lemma 6, the required solution* $(x, y) \in D$ *of system* (1) *asymptotically close to* $(\bar{x}, \bar{y})$ *satisfies the chain of equations* (20).

Remark 1. Since the differences between the right-hand sides in (20) and the functions G_i are $0(\varepsilon^i)$, the methods developed for the case in which the right-hand sides are regular functions of ε are applicable to equation (20). Moreover, the study of the asymptotic properties of bounded (almost-periodic or periodic) solution of the original system can be carried out by means of equation (20).

Remark 2. All the foregoing reasoning clearly applies to systems of the form:

$$x'(t) = G(t, x(t), x(t - \varepsilon\tau_1), \dots, x(t - \varepsilon\tau_m), y(t), \dots, y(t - \varepsilon\tau_p^*),)$$
$$\varepsilon y'(t) = F(t, x(t), x(t - \varepsilon\tau_1), \dots, x(t - \varepsilon\tau_m), y(t), \varepsilon);$$

where τ_i, τ_j^* are positive constants, $i = 1, \dots, m$; $j = 1, \dots, p$.

Remark 3. We point out that systems of type (1) have been treated in [3] by using the method of the integral manifolds.

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