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TOLERANCES, COVERING SYSTEMS, AND THE AXIOM OF CHOICE

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Dedicated to the memory of Milan Sekanina

Abstract. A *tolerance relation* of an algebra is a binary relation that is reflexive, symmetric, and has the Substitution Property. A number of authors (I. Chajda, G. Czédli, L. Klukovits, J. Niederle, I. Rosenberg, D. Schweigert, B. Zelinka, and the present authors) investigated how tolerances can be described by the system of *blocks* (maximal connected subsets). In this paper we show how to modify known results from idempotent algebras to arbitrary algebras. We prove the known characterization for lattices without the Axiom of Choice. For lattices with the Chain Condition, G. Czédli and L. Klukovits obtained a much better result. We generalize their result to arbitrary lattices, again avoiding the use of the Axiom of Choice. Finally, we show that for semilattices, the existence of a tolerance-block is equivalent to the Axiom of Choice.

Key words. Tolerance relation, covering system, Axiom of Choice, universal algebra, lattice.
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1. INTRODUCTION

It is well-known that there is a one-to-one correspondence between congruence relations of an algebra $\langle A; F\rangle$ and partitions of A with the Substitution Property; in fact, in informal discussions, the congruence relation Θ is often identified with the corresponding partition. There is a similar one-to-one correspondence between tolerance relations of an algebra $\langle A; F\rangle$ and a certain type of covering systems of the set A.

For a binary relation ϱ on the set A, a subset B of A is a *ϱ-block* iff B is a *ϱ-connected set* (i.e., $a\varrho b$ for all $a, b \in B$) that is *maximal* (i.e., if $B \subseteq C$ and C is also ϱ-connected, then $B = C$). I. Chajda [1] observed that there is a one-to-one correspondence between tolerance relations and covering systems of blocks. For lattices, G. Czédli [5] proved that one can define a lattice on the blocks of a tolerance relation, generalizing the concept of *quotient lattice.*

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Various papers deal with the covering systems one obtains from blocks of a tolerance relation. A recent one is G. Czédli and L. Klukovits [6] in which they obtain a characterization of the covering system of blocks of a tolerance relation for an *idempotent algebra* (an algebra $\langle A; F\rangle$ is *idempotent*, if it has no nullary operations and $f(x, \ldots, x) = x$ for all $x \in A$) sharpening earlier results of I. Chajda [1] and I. Chajda, J. Niederle, and B. Zelinka [2]; see Theorem 1 in § 2.

For lattices, the covering system of blocks of a tolerance relation was characterized in G. Czédli [5] and in I. Rosenberg and D. Schweigert [13]; see Theorem 4 in § 2. A much more useful characterization was obtained for lattices with the Chain Condition in G. Czédli [5]. This result was proved again in G. Czédli and L. Klukovits [6], applying the characterization for idempotent algebras; see Theorem 5 in § 2.

In this paper we offer two generalizations of the result of G. Czédli and L. Klukovits [5] for idempotent algebras: To algebras in which the tolerance-blocks are subalgebras (Theorem 2 in § 2) and to arbitrary algebras (Theorem 3 in § 2).

For lattices, we make two contributions. First, we show that, using some results of G. Grätzer and G. H. Wenzel [11], the older characterization theorem, Theorem 4, can be proved without the Axiom of Choice. Secondly, we generalize Czédli's result to arbitrary lattices: Theorem 6 in § 2.

However, the lattice proof cannot be extended to algebras, in general, or idempotent algebras, in particular. We prove that the existence of a tolerance-block in a semilattice is equivalent to the Axiom of Choice; see Theorem 7 in § 5.

Tolerances have been used recently in universal algebra in an attempt to describe the variety generated by the product of two varieties in G. Grätzer and G. H. Wenzel [11] and E. Fried and G. Grätzer [7] and [8], and in lattice theory to describe monotone functionally complete finite lattices in M. Kindermann [12].

For the basic concepts of universal algebra and lattice theory, we refer the reader to G. Grätzer [9] and [10].

Conditions will be denoted by mnemonic names. We shall refer to condition (XX) of Theorem n as (n. XX); "n." is dropped if the context is clear.

2. RESULTS

The main result of G. Czédli and L. Klukovits [6] is as follows:

Theorem 1. *Let $\langle A; F\rangle$ be an idempotent algebra. A family $\mathscr{C}$ of nonempty subsets of A is the set of all blocks of a tolerance relation iff the following conditions hold:*

(Cov) $\mathscr{C}$ *is a* covering system, *i.e.*, $\bigcup \mathscr{C} = A$.

(AnC) $\mathscr{C}$ *is an* antichain, *i.e.* $X \subseteq Y$ *implies that* $X = Y$, *for* $X, Y \in \mathscr{C}$.

(SP) $\mathscr{C}$ *has the* Substitution Property, *i.e., for any n-ary operation* $f \in \mathscr{F}$ *and* $X_1, \ldots, X_n \in \mathscr{C}$ *there exists an* $X \in \mathscr{C}$ *such that* $f(X_1, \ldots, X_n) = \{f(x_1, \ldots, x_n) \mid x_1 \in X_1, \ldots, x_n \in X_n\} \subseteq X$.

(2-SA) *For any* 2-covered subalgebra B *of* $\langle A; F\rangle$, *i.e., any subalgebra* B *of* $\langle A; F\rangle$ *such that for any two elements* $a, b \in B$ *there is an* $X \in \mathscr{C}$ *satisfying* $a, b \in X$, *there exists an* $E \in \mathscr{C}$ *with* $B \subseteq E$.

We have two generalizations of Theorem 1:

Theorem 2. *Let* $\langle A; F\rangle$ *be an algebra, and let* $\mathscr{C}$ *be a family of subalgebras of* $\langle A; F\rangle$. *Then* $\mathscr{C}$ *is the set of all blocks of a tolerance relation iff the conditions* (1.Cov), (1.AnC), (1.SP), *and* (1.2-SA) *hold.*

Theorem 2 is a direct generalization of Theorem 1 since in an idempotent algebra all tolerance-blocks are subalgebras (I. Chajda and B. Zelinka [4]). Indeed, if D is a tolerance-block of the tolerance relation τ, f is an n-ary operation, $d_1, \ldots, d_n \in D$, then for every $d \in D$, $d\tau d_i$ for $i = 1, \ldots, n$, hence by the Substitution Property for τ and the idempotency of f,

$$d = f(d, \ldots, d) \equiv f(d_1, \ldots, d_n).$$

By the maximality of D, $f(d_1, \ldots, d_n) \in D$, i.e., D is a subalgebra.

Theorem 3. *Let* $\langle A; F\rangle$ *be an* arbitrary algebra. *A family* $\mathscr{C}$ *of nonempty subsets of* A *is the set of all blocks of a tolerance relation iff the conditions* (1.Cov), (1.AnC), (1.SP), *and the following condition hold:*

(2-SS) *For any* 2-covered subset B *of* A, *i.e., any subset* B *of* A *such that for any two elements* $a, b \in B$ *there is an* $X \in \mathscr{C}$ *satisfying* $a, b \in X$, *there exists an* $E \in \mathscr{C}$ *with* $B \subseteq E$.

The characterization for lattices is as follows (I. Rosenberg and D. Schweigert [13]; the result in G. Czédli [5] is somewhat different):

Theorem 4. *Let* L *be a lattice. A family* $\mathscr{C}$ *of nonempty subsets of* L *is the set of all blocks of a tolerance relation iff the following conditions hold:*

(Cov) $\mathscr{C}$ *is* a covering system, *i.e.,* $\bigcup \mathscr{C} = L$.

(AnC) $\mathscr{C}$ *is an* antichain, *i.e.* $X \subseteq Y$ *implies that* $X = Y$, *for* $X, Y \in \mathscr{C}$.

(SP) $\mathscr{C}$ *has the* Substitution Property, *i.e., for all* $X, Y \in \mathscr{C}$ *there exist* $U, V \in \mathscr{C}$ *such that* $X \mathbin{\bigvee\!\!\!\!\bigvee} Y \subseteq U$ *and* $X \mathbin{\bigwedge\!\!\!\!\bigwedge} Y \subseteq V$.

(2-SL) *For any* 2-covered sublattice B of L, *i.e., any sublattice* B *of* L *such that for any two elements* $a, b \in B$ *there is an* $X \in \mathscr{C}$ *satisfying* $a, b \in X$, *there exists an* $E \in \mathscr{C}$ *with* $B \subseteq E$.

In the condition (SP) above we use the notation:

$$X \mathbin{\bigvee\!\!\!\!\bigvee} Y = \{x \vee y \mid x \in X, y \in Y\},$$
$$X \mathbin{\bigwedge\!\!\!\!\bigwedge} Y = \{x \wedge y \mid x \in X, y \in Y\}.$$

Theorem 1 is applied by G. Czédli and L. Klukovits to lattices with the Chain Condition to obtain a much sharper form of Theorem 4, first proved in G. Czédli [5]:

Theorem 5. *Let L be a lattice satisfying the* Chain Condition, *i.e., all chains in L are finite. A family $\mathscr{C}$ of nonempty subsets of L is the set of all blocks of a tolerance relation iff $\mathscr{C}$ is a system of intervals of L and the following conditions hold:*

(Cov) $\mathscr{C}$ *is* a covering system, *i.e.*, $\bigcup \mathscr{C} = L$.

(SP) $\mathscr{C}$ *has the* Substitution Property, *i.e., for all* $[a_1, a_2]$, $[b_1, b_2] \in \mathscr{C}$ *there exist* $[u_1, u_2]$, $[v_1, v_2] \in \mathscr{C}$ *such that* $u_1 = a_1 \vee b_1$, $u_2 \geqq a_2 \vee b_2$ *and* $v_1 \leqq a_1 \wedge b_1$, $v_2 = a_2 \wedge b_2$.

(UE) *The intervals in $\mathscr{C}$ have* unique endpoints, *i.e.*, $a_1 = b_1$ *iff* $a_2 = b_2$, *for* $[a_1, a_2], [b_1, b_2] \in \mathscr{C}$.

The following theorem generalizes Theorem 5 to arbitrary lattices:

Theorem 6. *Let L be a lattice. A family $\mathscr{C}$ of nonempty subsets of L is the set of all blocks of a tolerance relation iff all $X \in \mathscr{C}$ are convex sublattices of L and the following conditions hold:*

(Cov) $\mathscr{C}$ *is a* covering system, *i.e.*, $\bigcup \mathscr{C} = L$.

[SP] $\mathscr{C}$ *has the* Substitution Property, *i.e., for all* $X, Y \in \mathscr{C}$ *there exist* $U, V \in \mathscr{C}$ *such that* $[U) = [X) \vee_d [Y)$, $(U] \supseteq (X] \vee_i (Y]$ *and* $(V] = (X] \wedge_i (Y]$, $[V) \supseteq [X) \wedge_d [Y)$.

(UE) *The convex sublattices in $\mathscr{C}$ have* unique endpoints, *i.e.*, $(X] = (Y]$ *iff* $[X) = [Y)$, *for* $X, Y \in \mathscr{C}$.

(2-CS) *For any* 2-covered convex sublattice *B of L, i.e., any convex sublattice B of L such that for any two elements $a, b \in B$ there is an $X \in \mathscr{C}$ satisfying $a, b \in X$, there exists an $E \in \mathscr{C}$ with $B \subseteq E$.*

Notation. For a nonempty subset X of a lattice L, the ideal and dual ideal generated by X is denoted $(X]$ and $[X)$, respectively. (Note that in [11], we used $(X]$ and $[X)$ for the order ideals generated by X.) We form three lattices from L: the lattice of ideals (ordered under $\subseteq$), the lattice of dual ideals (ordered under $\supseteq$), and the lattice L/τ of all τ-blocks (ordered by $\leqq$, see Lemma 2 below). To avoid confusion, the lattice operations will be denoted by $\vee_i$ and $\wedge_i$ in the ideal lattice, by $\vee_d$ and $\wedge_d$ in the dual ideal lattice, and by $\vee_b$ and $\wedge_b$ in L/τ (the lattice of τ-blocks). Recall that $\vee_d$ is intersection and $\wedge_d$ is the dual ideal generated by the union.

It is clear that Theorem 6 implies Theorem 5. Indeed, if the lattice L satisfies the Chain Condition, then convex sublattices are intervals, hence the first three conditions are equivalent. Condition (2-CS) trivially holds for L, since if B is a convex sublattice of L, then $B = [a, b]$ for some $a, b \in L$, $a \leqq b$, and the $X \in \mathscr{C}$ satisfying $a, b \in X$ also satisfies $B \subseteq X$. We shall show an example in Section 4 that (2-CS) cannot be dropped in general.

3. ALGEBRAS

For a tolerance relation τ, let $\mathscr{C}_\tau$ denote the set of all τ-blocks.

First, we prove Theorems 2 and 3.

Let τ be a tolerance relation, and $\mathscr{C} = \mathscr{C}_\tau$. Obviously, the conditions of Theorems 2 and 3 hold for $\mathscr{C}$; in particular, (2-SS) holds, since if B is 2-covered by $\mathscr{C}$, then B is τ-connected, hence (by the Axiom of Choice) it is contained in a maximal τ-connected set $E \in \mathscr{C}$.

Conversely, let the conditions of Theorem 2 or 3 hold for $\mathscr{C}$, and define the binary relation τ by $a\tau b$ iff $a, b \in B$ for some $B \in \mathscr{C}$. (1.Cov) and (1.SP) imply that τ is a tolerance relation. Let $\mathscr{C}_\tau$ denote the system of tolerance-blocks of τ. Since every $B \in \mathscr{C}$ is τ-connected by the definition of τ, there is a $B^* \in \mathscr{C}_\tau$ satisfying $B \subseteq B^*$.

We have to prove that $\mathscr{C} = \mathscr{C}_\tau$.

Let $B \in \mathscr{C}_\tau$. Then B is 2-covered by $\mathscr{C}$ by the definition of τ. In case of Theorem 2, B is a subalgebra, so condition (2-SA) can be applied; in case of Theorem 3, condition (2-SS) can be applied. In either case, $B \subseteq D$ for some $D \in \mathscr{C}$. Since D is τ-connected and B is a τ-block, therefore, $B = D$, proving that $B \in \mathscr{C}$.

Conversely, let $B \in \mathscr{C}$. Then $B \subseteq B^* \in \mathscr{C}_\tau \subseteq \mathscr{C}$ (the last containment by the previous paragraph), hence $B = B^*$ by (1.AnC). Thus $B \in \mathscr{C}_\tau$, proving Theorems 2 and 3.

4. LATTICES

We need some results from G. Grätzer and G. H. Wenzel [11]. These results are proved in [11] without the use of the Axiom of Choice.

Let L be a lattice and let τ be a tolerance relation on L. For a subset X of L, we define

$$X_\tau = \{y \mid y \in L, y \leqq x \textit{ for some } x \in X, \text{ and } y\tau x \textit{ for all } x \in X\}.$$

We define X^τ dually.

Lemma 1 (Lemma 5 of [11]). *If X is a τ-connected set, then $(X_\tau)^\tau$ is a τ-block.*

Let L/τ denote the lattice of tolerance-blocks. Lemma 1 shows that L/τ is non-empty. The lattice operations of L/τ can be described as follows:

Lemma 2 (Lemma 7 and Theorem 1 of [11]). *Let X and Y be τ-blocks. Then*

$$X \vee_b Y = (X \curlyvee Y)^\tau,$$

$$X \wedge_b Y = (X \curlywedge Y)_\tau,$$

and

$$X \leqq Y \textit{ iff for all } x \in X \textit{ there is a } y \in Y \textit{ satisfying } x \leqq y.$$

One should note the dual form of the last statement of Lemma 2:

$$X \leqq Y \text{ iff for all } y \in Y \text{ there is an } x \in X \text{ satisfying } x \leqq y.$$

First, we prove Theorem 4 without the Axiom of Choice. It is obvious that the conditions of Theorem 4 hold for the blocks of a tolerance relation. (To verify (4.Cov), use Lemma 1 with a singleton as the τ-connected set.)

Conversely, let $\mathscr{C}$ satisfy the four conditions of Theorem 4, and define the binary relation τ by

$$a\tau b \text{ iff } a, b \in B \text{ for some } B \in \mathscr{C}.$$

(4.Cov) and (4.SP) imply that τ is a tolerance relation. By the definition of τ, every $B \in \mathscr{C}$ is τ-connected. By Lemma 1, $B^* = (B_\tau)^\tau$ is a τ-block containing B.

Now let B be a τ-block. Then B is 2-covered by $\mathscr{C}$. By (2-SL), $B \subseteq D$ for some $D \in \mathscr{C}$. Thus $B \subseteq D$ and D is τ-connected. Since B is a τ-block, $B = D$, proving that $B \in \mathscr{C}$.

If $B \in \mathscr{C}$, then B^* is a τ-block with $B \subseteq B^*$, and by the previous paragraph, B^* is in $\mathscr{C}$. Again, by (AnC), $B = B^*$, so B is a τ-block. ∎

Now we prove Theorem 6. Let τ be a tolerance relation on the lattice L, let $\mathscr{C} = \mathscr{C}_\tau$. Conditions (6.Cov) and (2-CS) are obvious by Lemma 1. To verify (6.UE), let $X, Y \in \mathscr{C}$ and let $[X) = [Y)$. If $x \in X - Y$, then $x \in [X) = [Y)$, i.e., $x \geqq y$ for some $y \in Y$. Hence for every $x \in X$, $x \geqq y$ for some $y \in Y$. By Lemma 2, $X \geqq Y$ with respect to the ordering of the blocks. By symmetry, $Y \geqq X$, hence $X = Y$. Thus $(X] = (Y]$, verifying (6.UE).

Finally, we prove (6.SP). Let $X, Y \in \mathscr{C}$ and define $U = X \vee_b Y$. By Lemma 2 (and using that $X \vee\!\!\!\vee Y$ is downward directed),

$$\begin{aligned} U = X \vee_b Y &= (X \vee\!\!\!\vee Y)^\tau \\ &= \{z \mid z \geqq x \vee y \text{ for some } x \in X, y \in Y \\ &\qquad \text{and } z\tau x \vee y \text{ for all } x \in X, y \in Y\} \\ &= [X \vee\!\!\!\vee Y) = [X) \vee_d [Y), \end{aligned}$$

and so $[U) \subseteq [X) \vee_d [Y)$. Conversely, if $u \in [X) \vee_d [Y)$, then $u \geqq x$ and $u \geqq y$ for some $x \in X$ and $y \in Y$. Thus $u \geqq x \vee y \in U$, proving $[U) \supseteq [X) \vee_d [Y)$. Thus $[U) = [X) \vee_d [Y)$. We also have

$$(X] \vee_d (Y] = (X \vee\!\!\!\vee Y] \subseteq ((X \vee\!\!\!\vee Y)^\tau] = (U],$$

completing, by duality, the proof of (6.SP).

Conversely, let $\mathscr{C}$ satisfy the four conditions of Theorem 6. We verify that the four conditions of Theorem 4 hold; then by Theorem 4, we obtain $\mathscr{C} = \mathscr{C}_\tau$ for some tolerance relation τ, proving Theorem 6.

(4.Cov) is the same as (6.Cov). To verify (2-SL), let B be a 2-covered sublattice of L. Then $C = \bigcup([a, b] \mid a, b \in B)$ is the convex hull of B, and C is also 2-covered by $\mathscr{C}$. By (2-CS), $C \in D$ for some $D \in \mathscr{C}$, and so $B \subseteq D \in \mathscr{C}$, proving (2-SL).

To prove (4.AnC), take $X, Y \in \mathscr{C}$ with $X \subseteq Y$. By (6.SP), there exists a $U \in \mathscr{C}$ such that $[U) = [X) \vee_d [Y)$ and $(U] \supseteq (X] \vee_i (Y]$. But $[X) \cap [Y) = [X) \vee_d [Y) = [X)$ since $X \subseteq Y$ implies that $[X) \subseteq [Y)$. Thus $[X) = [U)$ which by (6.UE) implies that $(X] = (U]$, and so $X = U$. Therefore, $(X] \supseteq (X] \vee_i (Y] = (Y]$, yielding $(X] = (Y]$. By (6.UE) again, $[X) = [Y)$, and so $X = Y$.

Finally, we prove (4.SP). Given $X, Y \in \mathscr{C}$, by (6.SP), there exist $U, V \in \mathscr{C}$ such that $[U) = [X) \vee_d [Y)$, $(U] \supseteq (X] \vee_i (Y]$ and $(V] = (X] \wedge_i (Y]$, $[V) \supseteq [X) \wedge_d Y)$. Thus $[X) \vee_d [Y) = [X \mathbin{\vee\!\!\!\vee} Y) = [U)$ and $(X] \vee_i (Y] = (X \mathbin{\vee\!\!\!\vee} Y] \subseteq (U]$. Now take $x \in X$ and $y \in Y$. Then $x \vee y \in (X] \vee_i (Y] \subseteq (U]$, thus $x \vee y \leqq u_1$, for some $u_1 \in U$. On the other hand, $x \vee y \in [X) \vee_d [Y) = [U)$, so $u_2 \leqq x \vee y$ for some $u_2 \in U$. Thus $u_1 \leqq x \vee y \leqq u_2$, and by the convexity of U, we conclude that $x \vee y \in U$, proving that $X \mathbin{\vee\!\!\!\vee} Y \subseteq U$. Similarly, $X \mathbin{\wedge\!\!\!\wedge} Y \subseteq V$, verifying (4.SP). ∎

The following example shows that condition (2-CS) cannot be dropped from Theorem 6. Let L be the lattice of real numbers with the usual partial ordering. Let $\mathscr{C}$ be the system of all sets of the form $[r, r+1]$ for a rational number r. Then $\mathscr{C}$ satisfies the first three conditions of Theorem 6. However, $\mathscr{C}$ fails (2-CS): the set $B = [\sqrt{2}, 1 + \sqrt{2})$ is a convex sublattice which is 2-covered by $\mathscr{C}$; however, B is not contained in any member of $\mathscr{C}$.

5. SEMILATTICES

Our final result shows that the Axiom of Choice is needed to prove the existence of tolerance-blocks.

Theorem 7. *The Axiom of Choice is equivalent to the following statement:*

(TB) *For a semilattice $\langle S; \vee \rangle$ and a tolerance relation τ on $\langle S; \vee \rangle$, there exists a τ-block.*

Proof. Let $\mathscr{Y}$ be a nonempty collection of nonempty pairwise disjoint sets. Define $S = (\bigcup \mathscr{Y}) \cup \{u\}$, where $u \notin \bigcup \mathscr{Y}$. We define the semilattice $\langle S; \vee \rangle$ by $x \vee y = u$ for all $x \neq y$. Finally, we define the binary relation τ on S as follows:

$$x \tau y \text{ iff } \begin{cases} x = y, \\ x \in X \quad \text{and} \quad y \in Y \quad \text{for some} \quad X, Y \in \mathscr{Y}, X \neq Y, \\ x = u, \\ y = u. \end{cases}$$

It is trivial to check that τ is a tolerance relation.

Let B be a τ-block. We claim that for every $X \in \mathscr{Y}$, $B \cap X$ is a singleton $\{x_B\}$. Proof: if $x, y \in B \cap X$, then $x, y \in B$, hence $x \tau y$. On the other hand, $x, y \in X$, hence by the definition of τ, we obtain $x = y$. Therefore, $B \cap X$ contains at most one element. If $B \cap X$ is empty, define $B^* = B \cup \{x\}$, where $x \in X$. Observe that $x \tau y$

for every $y \in B$; indeed, either $x = u$ or $y \in Y$ for some $Y \in \mathcal{Y}$, $X \neq Y$; in both cases, $x\tau y$ holds. Thus B^* is τ-connected and $B \subset B^*$, a contradiction.

Now we can define the choice function f on $\mathcal{Y}$ by $f(X) = x_B$ for $X \in \mathcal{C}$. ∎

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