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REALIZATIONS OF TOPOLOGIES AND CLOSURE OPERATORS BY SET SYSTEMS AND BY NEIGHBOURHOODS

HORST HERRLICH
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Dedicated to the memory of my friend Milan Sekanina

Abstract. Milan Sekanina and his collaborators have investigated the realizability of topologies and of closure operators by set systems. In particular they have shown that Top has precisely two [8] and Clos has no [3, 7, 2] realization by set systems. Moreover Top and Clos have precisely one realization by Conv [10]. In this paper it is shown that Top has a large (even illegitimate) collection of realizations by neighbourhoods, but Clos has only one. Moreover Clos has precisely two realization by uniform neighbourhoods.

Key words: realizations of constructs, topological space, closure space, (uniform) neighbourhood space.

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TERMINOLOGY

Constructs are pairs \((A, U)\) consisting of a category \(A\) and a faithful functor \(U: A \to \text{Set}\) [1]. A realization of a construct \((A, U)\) by a construct \((B, V)\) is a full embedding \(E: A \to B\) with \(U = V \circ E\) [6].

Top is the construct of topological spaces and continuous maps.

Clos is the construct of closure spaces (sets with a closure operation satisfying Kuratowski’s axioms except possibly the idempotency axiom) and continuous (= closure-preserving) maps.

Neigh has as objects all neighbourhood spaces, i.e. pairs \((X, N)\) where \(N: X \to \mathcal{P}X\) is a map, associating with any \(x \in X\) a collection \(N(x)\) of subsets \(U\) of \(X\) with \(x \in U\); and has as morphisms \(f: (X, N) \to (X', N')\) all maps \(f: X \to X'\) such that \(x \in X\) and \(U \in N'(f(x))\) imply \(f^{-1}(U) \in N(x)\).

UNeigh has as objects all uniform neighbourhood spaces, i.e., pairs \((X, <)\), where \(<\) is a binary relation on \(\mathcal{P}X\) satisfying the conditions (1) \(A < B \to A \subseteq B\) and (2) \(A \subset B < C \subseteq D \to A < D\).
and has as morphisms $f: (X, <) \to (X', <')$ all maps $f: X \to X'$ such that $A <' B$ implies $f^{-1}[A] < f^{-1}[B]$.

$\mathcal{S}et$ has as objects all pairs $(X, \mathcal{S})$ with $\mathcal{S} \subseteq \mathcal{P}X$ and as morphisms $f: (X, \mathcal{S}) \to (X', \mathcal{S}')$ all maps $f: X \to X'$ such that $A \in \mathcal{S}'$ implies $f^{-1}[A] \in \mathcal{S}$.

RESULTS

Proposition 1 [8]. Top has precisely two realizations by $\mathcal{S}et$.

Proposition 2 [3, 7, 2]. Clos has no realization by $\mathcal{S}et$.

Proof: Assume that $E: \text{Clos} \to \mathcal{S}et$ is a realization.

Notation: $E(X, cl) = (X, \mathcal{P}(cl))$. Then $E: \text{Clos} \to \mathcal{S}et$, defined by $E(X, cl) = (X, \mathcal{P}(cl) \cup \{\emptyset, X\})$, is a realization too. On a 3-element set $X$ there are precisely $4^3 = 64$ closure structures and precisely $2^{(2^3-2)} = 64$ subsets $\mathcal{S}$ of $\mathcal{P}X$ with $\{\emptyset, X\} \subseteq \mathcal{S}$. Hence $E$ induces an order-isomorphism between the ordered sets $F_1$ of all closure structures on $X$ and $F_2$ of all subsets $\mathcal{S}$ of $\mathcal{P}X$ with $\{\emptyset, X\} \subseteq \mathcal{S}$. Since $F_1$ has precisely 3 atoms and $F_2$ has 6, this cannot be.

Proposition 3. Top has a proper class (even an illegitimate collection) of realizations by Neigh.

Proof: Let $C$ be a strongly rigid proper class of Hausdorff spaces with more than one point. (Such a class exists by [5, 4]; cf. also [11]). For every subclass $\Gamma$ of $C$ define a realization $E_\Gamma: \text{Top} \to \text{Neigh}$ by $E_\Gamma(X, cl) = (X, N_\Gamma(cl))$ where $U \in N_\Gamma(cl)(x)$ provided $U$ is an open neighbourhood of $x$ in $(X, cl)$ or there exists $(X', cl')$ in $\Gamma$, a continuous map $f: (X, cl) \to (X', cl')$, and a neighbourhood $V$ of $f(x)$ in $(X', cl')$ with $U = f^{-1}[V]$.

The realizations $E_\Gamma$ are pairwise different, since, if $(X, cl)$ belongs to $\Gamma \setminus \Gamma'$, then for any $x \in X$, $N_\Gamma(cl)(x)$ consists of all neighbourhoods of $x$ in $(X, cl)$ and $N_{\Gamma'}(cl)(x)$ consists of all open neighbourhoods of $x$ in $(X, cl)$.

Proposition 4. Clos has precisely one realization by Neigh.

Proof: For every closure space $(X, cl)$ define a map $N_{cl}: X \to \mathcal{P}\mathcal{P}X$ by $N_{cl}(x) = \{U \subseteq X \mid x \notin cl(X \setminus U)\}$. Then $E: \text{Clos} \to \text{Neigh}$, defined by $E(X, cl) = (X, N_{cl})$ is a realization.

For uniqueness, consider an arbitrary realization $\tilde{E}: \text{Clos} \to \text{Neigh}$.

Notation: $\tilde{E}(X, cl) = (X, \tilde{N}_{cl})$. Let $(X, cl)$ be a closure space. Then the following hold:

(a) $\tilde{N}_{cl}(x) \neq \emptyset$ for every $x \in X$.

Proof: Assume $\tilde{N}_{cl}(x_0) = \emptyset$ for some $x_0 \in X$. Let $(X', cl')$ be an arbitrary closure space, let $x$ be an arbitrary element of $X'$, and let $f: X \to X'$ be the constant
map with value \( x \). Then continuity of \( f: (X, \text{cl}) \to (X', \text{cl'}) \) implies \( \bar{N}_{\text{cl}}(x) = \emptyset \). This in turn implies that every map between closure spaces is a morphism. Contradiction.

(b) \( X \in \bar{N}(x) \) for every \( x \in X \).
Proof: This follows from (a), since every constant map between closure spaces is continuous.

(c) \( X = \{1, 2\} \):

(c1) if \( \text{cl}\{1\} = \text{cl}\{2\} = X \), then \( \bar{N}_{\text{el}}(1) = \bar{N}_{\text{el}}(2) = \{X\} \),
(c2) if \( \text{cl}\{1\} = \{1\} \) and \( \text{cl}\{2\} = \{2\} \), then \( \bar{N}_{\text{el}}(1) = \{1\}, X \) and \( \bar{N}_{\text{el}}(2) = \{2\}, X \),
(c3) if \( \text{cl}\{1\} = X \) and \( \text{cl}\{2\} = \{2\} \), then one of the following two cases holds:
   Case A: \( \bar{N}_{\text{el}}(1) = \{1\}, X \) and \( \bar{N}_{\text{el}}(2) = \{X\} \),
   Case B: \( \bar{N}_{\text{el}}(1) = \{X\} \) and \( \bar{N}_{\text{el}}(2) = \{2\}, X \).

Proof: follows immediately from the fact that, there are only 4 neighbourhood structures on \( \{1, 2\} \), which satisfy (b).

(d) \( X = \{1, 2, 3\}\) if \( \text{cl}\{1\} = \text{cl}\{2\} = X \) and \( \text{cl}\{3\} = \{2, 3\} \), then one of the following two cases holds:
   Case A: \( \bar{N}_{\text{el}}(1) = \{1\}, X \) and \( \bar{N}_{\text{el}}(2) = \{X\} \),
   Case B: \( \bar{N}_{\text{el}}(1) = \{X\} \) and \( \bar{N}_{\text{el}}(2) = \{2\}, X \).

Proof: Let \( (X', \text{cl'}) \) be the indiscrete closure space with underlying set \( X' = \{1, 2\} \). Then the maps \( f: (X', \text{cl'}) \to (X, \text{cl}) \), defined by \( f(x) = x \), and \( g: (X', \text{cl'}) \to (X, \text{cl}) \), defined by \( g(x) = x + 1 \), are continuous. Hence, by (c3), one of the following cases must hold:
   Case A: \( U \in \bar{N}_{\text{el}}(3) \to 1 \in U \),
   Case B: \( U \in \bar{N}_{\text{el}}(1) \to 3 \in U \).

Since \( (X, \text{cl}) \) is not indiscrete, \( \bar{N}_{\text{el}}(1) = \bar{N}_{\text{el}}(2) = \bar{N}_{\text{el}}(3) = \{X\} \) cannot hold. This implies (d).

(e) Case B cannot hold.
Proof: Assume that case B holds. Let \( (X, \text{cl}) \) be as in (d), let \( (X', \text{cl'}) \) be an arbitrary closure space, let \( x \) be an element of \( X' \), let \( U \) be a subset of \( X' \) with \( x \in U \), and let \( f: X' \to X \) be defined by
Then the following conditions are equivalent:
(1) \( U \in \tilde{N}_{cl}(x) \),
(2) \( f: (X', \tilde{N}_{cl}) \to (X, \tilde{N}_{el}) \) is a morphism in Neigh,
(3) \( f: (X', cl') \to (X, cl) \) is continuous,
(4) \( cl'\{x\} \subseteq U \).

Hence in particular, if \((X', cl')\) is a topological \(T_1\)-space, then \(\tilde{N}_{el}(x) = \{U \subseteq X \mid x \in U\}\) for every \(x \in X'\). Since there exist different \(T_1\)-topologies on an infinite set, \(\tilde{E}\) is not injective on objects. Contradiction.

(f) \(\tilde{E} = E\).

Proof: In view of (e), Case A must hold. Again, let \((X, cl)\) be as in (d) let \((X', cl')\) be an arbitrary closure space, let \(x\) be an element of \(X'\), let \(U\) be a subset of \(X'\) with \(x \in U\), and let \(f: X' \to Y\) be defined by

\[
f(y) = \begin{cases} 
1, & \text{if } y = x, \\
2, & \text{if } y \in U \setminus \{x\}, \\
3, & \text{if } y \in X \setminus U.
\end{cases}
\]

Then the following conditions are equivalent:
(1) \( U \in \tilde{N}_{el}(x) \),
(2) \( f: (X', \tilde{N}_{el}) \to (X, \tilde{N}_{el}) \) is a morphism in Neigh,
(3) \( f: (X', cl') \to (X, cl) \) is continuous,
(4) \( x \notin cl'(X \setminus U) \).

Thus \(\tilde{N}_{el} = N_{el}\), i.e., \(\tilde{E} = E\).

**Proposition 5.** Clos has precisely two realizations by UNeigh.

Proof. As in the proof of Proposition 4, two cases arise. Case A leads to the realization \(E_1: \text{Clos} \to \text{UNeigh}\), defined by \(E_1(X, cl) = (X, <_1 (cl))\), where \(A < B\) iff \(A \setminus \text{cl} (X \setminus B) = \emptyset\), i.e., iff \(B\) is a neighbourhood of \(A\) in the familiar sense. Case B does not lead to a contradiction but to the realization \(E_2: \text{Clos} \to \text{UNeigh}\), defined by \(E_2(X, cl) = (X, <_2 (cl'))\) where \(A < B\) iff \((X \setminus B) \cap \text{cl} A = \emptyset\), i.e., iff \(X \setminus A\) is a neighbourhood of \(X \setminus B\) in the familiar sense.

Remark. Since the construct Rere of reflexive relations has a realization \(E: \text{Rere} \to \text{Clos}\), given by

\[
x \in \text{cl} A \iff \exists a \in A a \text{ q x},
\]
since the restriction of $E$ to objects with finite underlying sets is an isomorphism, and since the proof of Proposition 5 depends only on finite closure space, Rere has precisely two realizations in $\text{UNeigh}$ (resp. in $\text{Neigh}$).

REFERENCES


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