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ON ITERATION GROUPS OF CERTAIN FUNCTIONS

FRANTIŠEK NEUMAN

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In honour of the 60th birthday anniversary of Prof. M. Ráb

Abstract. This paper contains a characterization of iteration groups formed, up to conjugacy, by certain functions of the form

\[ \frac{a \tan x + b}{c \tan x + d}, \quad |ad - bc| = 1, \]

under the condition that graphs of different elements of such a group do not intersect each other.

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I. INTRODUCTION

For description of global transformations of linear differential equations, it is important to characterize all groups of those transformations that keep a given equation unchanged, see [5] and [6]. This characterization requires the following result concerning iteration groups of certain functions.

II. NOTATION, DEFINITIONS AND SOME BASIC FACTS

In accordance with O. Borůvka [2], the fundamental groups \( \mathcal{F}_1 \) is defined as the group of all functions \( f: \mathbb{R} \to \mathbb{R} \) given by the formula

\[ f(t) = \arctan \frac{a \tan t + b}{c \tan t + d}, \]

\( a, b, c, d \in \mathbb{R}, \quad |ad - bc| = 1, \) where \( \arctan \) denotes this branch of \( \arctan x + k\pi \) that makes function \( f \) continuous on \( \mathbb{R} \). Then the elements of the fundamental
group $\mathcal{F}_1$ are real analytic bijections of $\mathbb{R}$ onto $\mathbb{R}$, they are increasing exactly when $ad - bc = 1$. The group operation "$\circ$" is the composition of functions; for brevity the symbol $\circ$ is sometimes omitted.

Consider the following groups, whose elements are some functions of the fundamental group $\mathcal{F}_1$, restricted to an open interval $I \subset \mathbb{R}$.

$\mathcal{F}_2$: $f: (0, \infty) \to (0, \infty),$

$$f(t) = \arctan \frac{a \tan t}{b \tan t + 1/a}, \quad a \in (0, \infty), b \in \mathbb{R}.$$ 

$\mathcal{F}_3$: for each positive integer $m$

$$f: (0, mn) \to (0, mn),$$

$$f(t) = \arctan \frac{a \tan t}{b \tan t + 1/a}, \quad a \in (0, \infty), b \in \mathbb{R}.$$ 

$\mathcal{F}_4$: for each positive integer $m$

$$f: (0, m\pi - \pi/2) \to (0, m\pi - \pi/2),$$

$$f(t) = \arctan (a \tan t), \quad a \in (0, \infty).$$

Let the topology on $\mathcal{F}_1$ be the relative topology on

$$\{(a, b, c, d) \in \mathbb{R}^4; \mid ad - bc \mid = 1\},$$

where $\mathbb{R}^4$ is considered with the usual topology.

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two groups whose elements are (some) bijections of intervals $I_1$ and $I_2$ onto themselves, respectively. We say that the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ are $C^k$-conjugate (with respect to $\varphi$) for some positive integer $k$ if there is a $C^k$-diffeomorphism $\varphi$ of interval $I_1$ onto interval $I_2$, i.e. $\varphi(I_1) = I_2$, $\varphi \in C^k(I_1)$, $d\varphi(x)/dx \neq 0$ on $I_1$,

such that

$$\mathcal{G}_2 = \varphi \circ \mathcal{G}_1 \circ \varphi^{-1} := \{\varphi \circ f \circ \varphi^{-1}; f \in \mathcal{G}_1\}.$$ 

If $\mathcal{G}_1$ is a topological group the topology on $\mathcal{G}_2$ is induced by the conjugacy.

Let $\alpha$ be an element of a group. For any integer $k$ define the element $\alpha^{[k]}$ as follows:

$\alpha^{[0]}$ is the unit element of the group,

$\alpha^{[k]} = \alpha^{[k-1]} \circ \alpha$ for positive $k$,

$\alpha^{[k]} = (\alpha^{-1})^{[-k]}$ for negative $k$,

$\alpha^{-1}$ being the inverse to $\alpha$. Element $\alpha^{[k]}$ is called the $k$th iterate of $\alpha$.

A group is said to be partially (linearly) ordered if the set of its elements is partially (linearly) ordered and, for each its elements $\alpha$, $\beta$ and $\gamma$, the relation $\alpha \leq \beta$ implies both $\alpha \circ \gamma \leq \beta \circ \gamma$ and $\gamma \circ \alpha \leq \gamma \circ \beta$.

A partially ordered group is called archimedean if the following implication holds:
if $\alpha^{[m]} \leq \beta$ is satisfied for some elements $\alpha$ and $\beta$ and for all integers $n$, then $\alpha$ is the unit element of the group.

The following theorem is due to O. Hölder [3]: There exists an order preserving isomorphism of any linearly ordered archimedean group into a subgroup of the additive group of real numbers $\mathbb{R}$.

For proof see also for example A. I. Kokorin and V. M. Kopytov [4].

A group is said to be a cyclic group if there exists an element $\alpha$ of it such that all elements are iterates of $\alpha$. Element $\alpha$ of this property is called a generator of the cyclic group. If, in addition,

$$\alpha^{[m]} \neq \alpha^{[n]}$$

for generator $\alpha$ and different integers $m$ and $n$, then the group is an infinite cyclic group.

Now, consider an open interval $I \subset \mathbb{R}$. Let $n \geq 1$ be an integer and $\mathcal{G}$ denote a group of some $C^n$-diffeomorphisms of $I$ into $I$. Moreover, suppose that graphs of different elements of $\mathcal{G}$ do not intersect each other (on $I$).

III. THEOREM

If $\mathcal{G}$ is $C^n$-conjugate to a closed subgroup of increasing elements of the group $\mathcal{F}_1$, or $\mathcal{F}_2$, or $\mathcal{F}_3$, or $\mathcal{F}_4$,

then either $\mathcal{G}$ is trivial,

or $\mathcal{G}$ is an infinite cyclic group with a generator $h_\epsilon \in C^n(I)$, $dh_\epsilon(x)/dx > 0$ and $h_\epsilon(x) \neq x$ on $I$,

or $\mathcal{G}$ is $C^n$-conjugate to the group of all translations $\{h_c; c \in \mathbb{R}\}$,

$$h_c: \mathbb{R} \to \mathbb{R}, \quad h_c(x) = x + c.$$

Proof

Since different elements of the group $\mathcal{G}$ do not intersect each other on $I$, $\mathcal{G}$ can be linearly ordered in the following manner:

- for $h_1, h_2 \in \mathcal{G}$ we write $h_1 \leq h_2$,

if either $h_1(x_0) < h_2(x_0)$ for some (then any) number $x_0 \in I$, or $h_1 = h_2$.

Moreover, $\mathcal{G}$ is archimedean, because for $h \neq \text{id}_I$ there holds $h(x) \neq x$ on $I$ an the sequences

$$\{h^{[1]}(x_0)\}_{i=1}^{\infty} \quad \text{and} \quad \{h^{[1]}(x_0)\}_{i=-\infty}^{-1}$$

converge to both ends of interval $I$ for any $x_0 \in I$. Due to the Hölder Theorem there exists an order preserving isomorphism of $\mathcal{G}$ onto a subgroup $\tilde{\mathcal{G}}$ of the additive group $\mathbb{R}$.
If $\mathcal{G}$ is trivial then $\mathcal{G} = \{\text{id}\}$ and $\mathcal{\bar{G}} = \{0\}$.

Let $\mathcal{G}$ be not trivial and $\mathcal{G} = \{ie; i \in \mathbb{Z}, 0 \neq e \in \mathbb{R}\}$ be an infinite cyclic group generated by a nonzero number $e$. Denote by $h_e$ this element of group $\mathcal{G}$ that corresponds to the number $e$. Evidently $h_e \in C^n(I)$, $dh_e(x)/dx > 0$ and $h_e(x) \neq x$ on $I$. Moreover,

$$\mathcal{G} = \{h_e^i; i \in \mathbb{Z}\},$$

$h_e$ being a generator of the infinite cyclic group $\mathcal{G}$.

From now, let $\mathcal{G}$ be not trivial, neither it be an infinite cyclic group.

1. Consider first the case when $\mathcal{G}$ is $C^n$-conjugate to a closed subgroup of the fundamental group $\mathcal{F}$ with respect to a $C^n$-diffeomorphism $\varphi$ of $\mathbb{R}$ onto $I$. Let $h \in \mathcal{G}$, $h \neq \text{id}_I$. Then

$$\varphi^{-1}h\varphi(t) = \arctan \frac{a_{11} \tan x + a_{12}}{a_{21} \tan x + a_{22}} \in \mathcal{F},$$

and $a_{11}a_{22} - a_{12}a_{21} = 1$ because $dh(x)/dx > 0$ on $I$.

Case 1a. Let

$$C^{-1} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} C = \begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix},$$

$b \in \mathbb{R}$, for a non-singular 2 by 2 matrix $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$. Without loss of generality, let $\det C = 1$. Denote by $\psi$ one of the continuous functions, element of the group $\mathcal{F}$, given by the formula

$$\psi(t) = \arctan \frac{c_{11} \tan t + c_{12}}{c_{21} \tan t + c_{22}}.$$

It can be verified that

$$\psi^{-1}\varphi^{-1}h\varphi\psi(t) = \arctan (b^2 \tan t) \in \mathcal{F}.$$

Since $h(x) \neq x$ on $I$, we have

$$\psi^{-1}\varphi^{-1}h\varphi\psi(0) = k \pi$$

for some integer $k \neq 0$.

Case 1b. Let

$$C^{-1} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} C = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix},$$

$\det C = 1$ and $\psi \in \mathcal{F}$ be defined as in case 1a. Then

$$\psi^{-1}\varphi^{-1}h\varphi\psi(t) = \arctan (\tan t \pm 1) \in \mathcal{F},$$

$$\psi^{-1}\varphi^{-1}h\varphi\psi(\pi/2) = \pi/2 + k \pi$$
for some \( k \in \mathbb{Z} \setminus \{0\} \), otherwise \( h \) intersects \( \text{id}_I \).

Case 1c. Let

\[
C^{-1} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} C = \begin{pmatrix} \cos \omega \pi & \sin \omega \pi \\ -\sin \omega \pi & \cos \omega \pi \end{pmatrix},
\]

\( \omega \in \mathbb{R} \setminus \mathbb{Z} \), \( \det C = 1 \) and \( \psi \) be defined as above. Then

\[
\psi^{-1} \varphi^{-1} h \varphi \psi(t) = t + \omega \pi \in \mathcal{F}_1.
\]

Now, let \( h \) and \( g \) be two different elements of the group \( \mathcal{G} \) that do not belong to the same infinite cyclic group. Denote

\[
h_1 := \varphi^{-1} h \varphi \in \mathcal{F}_1 \quad \text{and} \quad g_1 := \varphi^{-1} g \varphi \in \mathcal{F}_1.
\]

Suppose first that

\[
\psi_1^{-1} h_1 \psi_1(t) = \text{Arctan} \left( b_1^2 \tan t \right), \quad \text{case la for } h,
\]

and

\[
\psi_2^{-1} g_1 \psi_2(t) = \text{Arctan} \left( b_2^2 \tan t \right), \quad \text{case la for } g,
\]

hold for suitable elements \( \psi_1 \) and \( \psi_2 \) of the fundamental group \( \mathcal{F}_1 \). Due to the initial values of \( \psi_1^{-1} h_1 \psi_1 \) and \( \psi_2^{-1} g_1 \psi_2 \) at 0, and with respect to the fact that the relation

\[
\psi(t + n\pi) = \psi(t) + n\pi,
\]

holds for every increasing element \( \psi \) of \( \mathcal{F}_1 \), there exist integers \( n_1 \) and \( n_2 \) such that either \( h_1^{[n_1]} \) and \( g_1^{[n_2]} \) coincide and then \( h \) and \( g \) belong to the same infinite cyclic group, or \( h_1^{[n_1]} \) and \( g_1^{[n_2]} \) intersect each other, the same being true for \( h_1^{[n_1]} \) and \( g_1^{[n_2]} \). However both cases were excluded from our considerations.

The same argument shows that neither the situation when

\[
\psi_1^{-1} h_1 \psi_1(t) = \text{Arctan} \left( \tan t + 1 \right), \quad \text{case lb for } h,
\]

and

\[
\psi_2^{-1} g_1 \psi_2(t) = \text{Arctan} \left( \tan t + 1 \right), \quad \text{case lb for } g,
\]

nor the case when

\[
\psi_1^{-1} h_1 \psi_1(t) = \text{Arctan} \left( b^2 \tan t \right), \quad \text{case la for } h,
\]

and

\[
\psi_2^{-1} g_1 \psi_2(t) = \text{Arctan} \left( \tan t + 1 \right), \quad \text{case lb for } g,
\]

can occur.

If one of the functions, say \( h \), is of the form described in case 1c, i.e.

\[
\psi_1^{-1} h_1 \psi_1(t) = t + \omega \pi, \quad \omega \in \mathbb{R} \setminus \mathbb{Z},
\]
then $g$ cannot be of the form in case 1a

$$
\psi_2^{-1}g_1\psi_2(t) = \text{Arctan}(b^2 \tan t) \quad \text{for } k \neq 1,
$$

or of the form of the case 1b

$$
\psi_2^{-1}g_1\psi_2(t) = \text{Arctan}(\tan t + 1),
$$

because then there again exist integers $n_1$ and $n_2$ such that $h^{[n_1]}$ and $g^{[n_2]}$ intersect each other.

Hence in this case 1 when the group $\mathcal{G}$ is $C^n$-conjugate to a closed subgroup of the whole fundamental group $\mathcal{F}$, it remains to consider only the situation when

$$
\psi_1^{-1}h_1\psi_1(t) = t + \omega_1\pi, \quad \omega_1 \in \mathbb{R} \setminus \mathbb{Z}
$$

and either

$$
\psi_2^{-1}g_1\psi_2(t) = \text{Arctan}(\tan t),
$$

or

$$
\psi_2^{-1}g_1\psi_2(t) = t + \omega_2\pi, \quad \omega_2 \in \mathbb{R} \setminus \mathbb{Z}.
$$

In the first of these cases

$$
\psi_2^{-1}g_1\psi_2(t) = t + k_1\pi \quad \text{for some } k_1 \in \mathbb{Z} \setminus \{0\}
$$

due to the initial value of this function at 0. Since

$$
\psi_1^{-1}g_1\psi_1(t) = (\psi_2\psi_1)^{-1} \psi_2g_1\psi_2^{-1}(\psi_2\psi_1)(t)
$$

and $\psi_2\psi_1$ is again an increasing element of the fundamental group $\mathcal{F}_1$, i.e. $\psi_2\psi_1(t + k\pi) = \psi_2\psi_1(t) + k\pi$, we have

$$
\psi_1^{-1}g_1\psi_1(t) = (\psi_2\psi_1)^{-1} (\psi_2\psi_1(t) + k\pi) = t + k\pi, \quad k \in \mathbb{Z}.
$$

Hence $\omega_1$ is an irrational number, otherwise $h_1$ and $g_1$ belong to the same infinite cyclic group and the same is true for the functions $h$ and $g$, that was already excluded. However, when $\omega_1$ is irrational, then the union of graphs of functions $h_1^{[n_1]}$ and $g_1^{[n_2]}$ for all $n_1$ and $n_2$ from $\mathbb{Z}$ is a dense set in $\mathbb{R}^2$. Now we have

$$
h = \psi_1\varphi(\text{id} + \omega_1\pi) \varphi^{-1} \psi_1^{-1} \quad \text{and} \quad g = \psi_1\varphi(\text{id} + k\pi) \varphi^{-1} \psi_1^{-1},
$$

where $\psi_1\varphi$ is a $C^n$-diffeomorphism of $\mathbb{R}$ onto $I$. Since the group $\mathcal{G}$ is closed, we conclude that it is $C^n$-conjugate to the group of all translations

$$
t \mapsto t + c, \quad \text{for all } c \in \mathbb{R}.
$$

Now, let

$$
\psi_1^{-1}h_1\psi_1(t) = t + \omega_1\pi, \quad \omega_1 \in \mathbb{R} \setminus \mathbb{Z}, \quad \text{case 1c for } h,
$$

and

$$
\psi_2^{-1}g_1\psi_2(t) = t + \omega_2\pi, \quad \omega_2 \in \mathbb{R} \setminus \mathbb{Z}, \quad \text{case 1c for } g.
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Then

\[ h_1^{[m]}(t) = \psi_1(\psi_1^{-1}(t) + n_1 \omega_1 \pi), \]
\[ g_1^{[n]}(t) = \psi_2(\psi_2^{-1}(t) + n_2 \omega_2 \pi) \]

and the condition \( h_1^{[m]}(t) \neq g_1^{[n]}(t) \) on \( \mathbb{R} \) implies

\[ \psi_3(t + n_1 \omega_1 \pi) \neq \psi_3(t) + n_2 \omega_2 \pi \]

for \( \psi_3 : = \psi_2^{-1} \psi_1 \in \mathcal{F} \), otherwise \( h_1^{[m]} \) coincides with \( g_1^{[n]} \) that shows that \( h_1 \) and \( g_1 \) belong to the same infinite cyclic group, the case already excluded from our considerations. Since

\[ \psi_3(t + \pi) = \psi_3(t) + \pi, \]

we have

\[ \psi_3(t) = t + p(t), \]

where \( p \) is a \( \pi \)-periodic function: \( p(t + \pi) = p(t) \in C^3(\mathbb{R}) \). Hence

\[ t + n_1 \omega_1 \pi + p(t + n_1 \omega_1 \pi) \neq t + p(t) + n_2 \omega_2 \pi, \]

or

\[ p(t + n_1 \omega_1 \pi) - p(t) \neq (n_2 \omega_2 - n_1 \omega_1), \]

for all \( t \in \mathbb{R} \) and all \( n_1, n_2 \in \mathbb{Z}, n_1^2 + n_2^2 \neq 0. \)

If \( n_2 \omega_2 - n_1 \omega_1 = 0 \) for some \( n_1 \) and \( n_2 \) then either

\[ p(t + n_1 \omega_1 \pi) > p(t) \quad \text{on} \quad \mathbb{R}, \]

or

\[ p(t + n_1 \omega_1 \pi) < p(t) \quad \text{on} \quad \mathbb{R}. \]

Neither of these cases is possible for any continuous periodic function \( p \).

Hence \( n_2 \omega_2 - n_1 \omega_1 \neq 0 \) for all integers \( n_1 \) and \( n_2 \), \( n_1^2 + n_2^2 \neq 0 \), that means that \( \omega_1 \) and \( \omega_2 \) are rationally independent. Then for each number \( t_0 \in \mathbb{R} \) the set

\[ \{ g_1^{[m]} \circ h_1^{[n]}(t_0); n_1, n_2 \in \mathbb{Z} \} \]

is dense in \( \mathbb{R} \), because for different couples \((n_1, n_2)\) and \((n_1^*, n_2^*)\) the values,

\[ g_1^{[m]} \circ h_1^{[n]}(t_0) \text{ and } g_1^{[m]} \circ h_1^{[n']} (t_0) \]

are different, there are infinite number of couples \((n_1, n_2)\) satisfying \( |n_1 \omega_1 + n_2 \omega_2| < \varepsilon \) for any given \( \varepsilon > 0 \) and, moreover, \( \psi_1 \) and \( \psi_2 \) are \( C^\infty \)-diffeomorphisms of \( \mathbb{R} \) onto \( \mathbb{R} \) for any \( n \in \mathbb{N} \) satisfying

\[ \psi_1(t) = t + p_1(t), \quad \psi_2(t) = t + p_2(t), \]

with \( \pi \)-periodic functions \( p_1 \) and \( p_2 \).

Since \( \varphi \) is a \( C^\infty \)-diffeomorphism of \( \mathbb{R} \) onto \( I \), and the group \( \mathcal{G} \) is archimedean and closed, the union of graphs of all its elements is the whole square \( I^2 \). In such a situation we may apply Theorem 1 of G. Blanton and J. A. Baker [1] which
states: "Each group whose elements are $C^n$-diffeomorphisms of an interval $I$ onto $I$ and such that to each point $(x_0, y_0) \in I \times I$ there exists just one element $h$ of the group satisfying $h(x_0) = y_0$, is formed by functions
\[ \chi(\chi^{-1}(x) + c), \]
where $\chi$ is a $C^n$-diffeomorphism of $\mathbb{R}$ onto $\mathbb{R}$ and $c$ ranges through the real numbers". In our case we may write
\[ G = \chi \circ h_c \circ \chi^{-1}, \]
where $h_c : \mathbb{R} \to \mathbb{Z}, h_c(t) = t + c, c \in \mathbb{R}$.

2. Now, suppose that
\[ \varphi^{-1} h \varphi(t) = \text{Arctan} \left( \frac{a \tan t}{b \tan t + 1/a} \right), \quad t \in \mathbb{R}_+, \]
$a \in \mathbb{R}_+, b \in \mathbb{R}$, is an element of the two-parametric group $\mathcal{F}_2$ of increasing functions. Since $\lim_{t \to 0^+} \varphi^{-1} h \varphi(t) = 0$, we have
\[ \varphi^{-1} h \varphi(\pi) = \pi, \]
hence $\varphi^{-1} h \varphi = \text{id}_{\mathbb{R}_+}$ that is excluded from our considerations.

3m. If
\[ \varphi^{-1} h \varphi(t) = \text{Arctan} \left( \frac{a \tan t}{b \tan t + 1/a} \right), \quad \varphi^{-1} h \varphi; (0, m\pi) \to (0, m\pi), \]
$a \in \mathbb{R}_+, b \in \mathbb{R}$, then $\lim_{t \to 0^+} \varphi^{-1} h \varphi(t) = 0$ and $\lim_{t \to \pi} \varphi^{-1} h \varphi(t) = \pi$,
because $h$ as well as $\varphi^{-1} h \varphi$ are increasing functions. Hence $m = 1$, otherwise $h = \text{id}_{\mathbb{R}}$ that contradicts to our assumptions. However, if $a \neq 1$ and $b \neq 0$ then the equation
\[ \arctan \frac{a \tan t}{b \tan t + 1/a} = t, \]
i.e.
\[ a \tan t = (b \tan t + 1/a) \tan t \]
is satisfied for $t_1 \in (0, \pi)$ where
\[ \tan t_1 = \frac{a^2 - 1}{ab}. \]
This case is excluded from our considerations. Even the case $b = 0$ impossible since then
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\[ \varphi^{-1}h\varphi(t) = \arctan(a^2 \tan t) \]

intersects \( \text{id}_{(0, \pi)} \) at \( \pi/2 \).

If \( a = 1 \) then

\[
\varphi^{-1}h\varphi(t) = \arctan \frac{\tan t}{b \tan t + 1} = \arccot \frac{1 + b \tan t}{\tan t} = \arccot (\cot t + b), \quad t \in (0, \pi),
\]

hence \( h \) is conjugate to \( x \mapsto x + b, x \in \mathbb{R} \) for a fixed \( b \in \mathbb{R} \) by means of the function \( \varphi \circ \arccot : \mathbb{R} \rightarrow I \).

Now, let \( h \) and \( g \) be two different elements of the stationary group \( G \) that do not belong to the same infinite cyclic group. Then

\[
\psi^{-1}h\psi(x) + x + b_1 \quad \text{and} \quad \psi^{-1}g\psi(x) = x + b_2
\]
on \( \mathbb{R} \) where \( \psi = \varphi \circ \arccot \in C^\alpha(\mathbb{R}) \), and \( b_1/b_2 \) is irrational. Since the union of the graphs of functions

\[ x \mapsto x + n_1 b_1 + n_2 b_2 \quad \text{for all} \quad n_1, n_2 \in \mathbb{Z} \]
is dense in \( \mathbb{R}^2 \), and the group \( G \) is closed, it is \( C^\alpha \)-conjugate to the group of all translations:

\[ \{x \mapsto x + c, c \in \mathbb{R}\}. \]

4m. Finally, if

\[
\varphi^{-1}h\varphi(t) = \text{Arctan} \ (a \tan t), \quad a > 0,
\]

\[ \varphi^{-1}h\varphi: (0, m\pi - \pi/2) \rightarrow (0, m\pi - \pi(2)), \]

then

\[
\lim_{t \rightarrow 0^+} \varphi^{-1}h\varphi(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \pi/2^-} \varphi^{-1}h\varphi(t) = \pi/2,
\]

and hence \( m = 1 \). In this case \( h \) is conjugate to the function \( x \mapsto x + \ln a, x \in \mathbb{R} \) by means of the \( C^\alpha \)-diffeomorphism \( \varphi \circ \arctan \circ \exp : \mathbb{R} \rightarrow I \).

Now, analogously to case 3m, if \( h \) and \( g \) are two different elements of \( G \) that do not belong to the same infinite cyclic group, they are \( C^\alpha \)-conjugate to \( x + b_1 \) and \( x + b_2 \), respectively, with respect to the some \( C^\alpha \)-diffeomorphism, the quotient \( b_1/b_2 \) being irrational. Hence the group \( G \) is \( C^\alpha \)-conjugate to the group

\[ \{x \mapsto x + c; c \in \mathbb{R}\}, \]

that finishes the proof of the theorem.
IV. REMARK

The present paper gives technical details of the proof of Theorem 6.3.5 in the monograph [6], where main steps of the proof were outlined.

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Franťšek Neuman
Mathematical Institute of
the Czechoslovak Academy of Sciences
branch Brno
Mendlovo nám. 1
603 00 Brno
Czechoslovakia