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Martin Čadek

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A NOTE TO POINTWISE TRANSFORMATIONS OF LINEAR QUASI-DIFFERENTIAL EQUATIONS

MARTIN ČADEK

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Abstract. In this paper the description of all pointwise transformations between linear quasi-differential equations of the n-th order is derived.

Key words. Linear quasi-differential equation, pointwise transformation.

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1. INTRODUCTION

In connection with the investigation of linear differential equations several kinds of pointwise transformations are often considered. In this paper we shall derive the general form of pointwise transformations of linear quasi-differential equations of the n-th order.

Let $A = (A^i_j)$ be $n \times n$-matrix with component A^i_j in the i-th row and in the j-th column and let $\mathbf{K}_n$ be the set of all $n \times n$-matrices A such that $A^i_{i+1} \neq 0$ and $A^i_j = 0$ for $1 \leqq i + 2 \leqq j \leqq n$. In accordance with [3] and [4] a linear quasi-differential equation of the n-th order is by the definition the scalar differential equation that can be obtained by elimination from the system

$$\mathbf{z}' = S(t)\,\mathbf{z}, \tag{1}$$

where $S(t) \in \mathbf{K}_n$. If $\mathbf{z}$ denotes a column vector in $\mathbf{R}^n$, we can compute the i-th component of $\mathbf{z}$ from the equation (1) through the previous components and their derivatives:

$$z^i = \frac{1}{S^{i-1}_i(t)}\Big[(z^{i-1})' - \sum_{j=1}^{i-1} S^{i-1}_j(t)\, z^j\Big], \qquad 2 \leqq i \leqq n.$$

By this procedure we can convert the equation (1) into a scalar linear differential equation with unknown function z^1. Then the i-th component of the vector solution of the equation (1) is called the $(i-1)$-th quasi-derivative of the function z^1.

In [5] Stäckal proved that the most general pointwise transformation converting any linear homogeneous differential equation of the n-th order ($n \geqq 2$) in independent variable x and dependent variable y into an equation of the same kind in variables t and z is of the form

$$t = k(x),$$
$$z = m(x)\, y,$$

where $k'(x) \neq 0$ and $m(x) \neq 0$, see also [1] and [6]. Our approach to description of all pointwise transformations of linear quasi-differential equations of the n-th order will be rather different. Instead of scalar linear differential or quasi-differential equations of the n-th order we shall take into account the linear systems (1) which correspond with scalar equations of the n-th order and pointwise transformations of scalar solutions will be replaced by pointwise transformations of fundamental matrix solutions of these systems. The main result of this approach is formulated in the next section. Its proof, based on Theorem 1 in [2], is given in Section 3.

2. NOTATION AND MAIN RESULT

The first part of this section is a list of symbols used below. Let $n \geqq 2$ be integer, let $r \geqq 0$ be an integer or ∞ and let I and J be open intervals. The set of all real $n \times n$-matrices will be denoted as $\mathbf{M}_n$. The symbol $\mathbf{K}_n$ has been already defined in Section 1 and $\mathbf{L}_n$ will be the set of all $n \times n$-matrices (A^i_j) such that $A^i_{i+1} = 1$ for $1 \leqq i \leqq n - 1$ and $A^i_j = 0$ for $1 \leqq i \leqq n - 1$, $i + 1 \neq j$. The symbol $\mathbf{GL}_n$ will stand for the set of all regular matrices from $\mathbf{M}_n$. $\mathbf{GL}_n^+$ will stand for the set of all $n \times n$-matrices with positive determinants. The symbols E, tr, det, ${}^{-1}$, * will stand for the unit matrix, the trace, the determinant, the inverse and the transpose, respectively. The elements of $\mathbf{R}^n$ will be considered as column vectors. The principal diagonal and the subprincipal diagonal of a matrix $A \in \mathbf{M}_n$ are formed by components A^i_i and A^i_{n+1-i} for $1 \leqq i \leqq n$, respectively.

The sets of all continuous and r-times continuously differentiable functions defined on M with values in N will be denoted as $C(M, N)$ and $C^r(M, N)$, respectively. We shall write $E(S, J)$ for the equation (1) with $S \in C(J, \mathbf{K}_n)$. This equation corresponds to the linear quasi-differential equation of the n-th order that will be denoted as $E_n(S, J)$. Analogously, the equation $E(R, I)$ with $R \in C(I, \mathbf{L}_n)$ of the form

$$\begin{pmatrix} 0, & 1, & 0, & \dots, & 0 \\ 0, & 0, & 1, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -p_0, & -p_1, & -p_2, & \dots, & -p_{n-1} \end{pmatrix}, \tag{2}$$

represents the n-th order linear homogeneous differential equation denoted as $E_n(p, I)$, where $p = (p_0, p_1, \ldots, p_{n-1})$.

As a pointwise transformation consider a mapping $T = (T_1, T_2)$: $I \times \mathbf{GL}_n^+ \to \to J \times \mathbf{M}_n$

$$t = T_1(x, Y), \qquad Z = T_2(x, Y),$$

satisfying the following assumptions:

(A) *$T = (T_1, T_2)$ is a homeomorphism of $I \times \mathbf{GL}_n^+$ into $J \times \mathbf{M}_n$.*

(B) *For every equation $E(R, I)$ with $R \in C^r(I, \mathbf{L}_n)$ there is an equation $E(S, J)$ with $S \in C(J, \mathbf{K}_n)$ such that for every solution $y \in C^1(I, \mathbf{GL}_n^+)$ of the equation $E(R, I)$ the couples (t, Z)*

$$t = T_1(x, Y(x)), \qquad Z = T_2(x, Y(x))$$

for $x \in I$ form a graph of a function $Z(t)$ representing a solution of $E(S, J)$.

Theorem. *Let $n \geqq 2$. Every mapping $T = (T_1, T_2)$ satisfying* (A) *and* (B) *is of the form*

$$T_1(x, Y) = k(x) \tag{3}$$

and either

$$T_2(x, Y) = M(x)(\det Y)^\lambda YC, \qquad \lambda \neq -\frac{1}{n}, \tag{4a}$$

or

$$T_2(x, Y) = M(x)(\det Y)^\lambda (Y^{-1})^* C, \qquad \lambda \neq \frac{1}{n}, \tag{4b}$$

where $\lambda \in \mathbf{R}$, $C \in \mathbf{GL}_n$, k is a C^1-diffeomorphism of I onto J, $M \in C^1(I, \mathbf{GL}_n)$, for every $x \in I$ the matrix $M(x)$ has only zeros above the principal diagonal in case (4a) *and $M^{-1}(x)$ has only zeros under the subprincipal diagonal in case* (4b).

Conversely, if a mapping T is of the form (3) *and* (4a) *or* (4b) *with λ, C, k and M satisfying the above conditions then* (A) *and* (B) *hold even if the condition $R \in C^r(I, \mathbf{L}_n)$ in* (B) *is replaced by the condition $R \in C(I, \mathbf{K}_n)$.*

Remark 1. Denote h the inverse function of k and put $N = M \circ h$. As it has been already noted in [2], the transformation (3), (4) converts the equation $E(R, I)$ either into the equation

$$Z' = [N'(t) N^{-1}(t) + \lambda h'(t) \operatorname{tr} R(h(t)) E + h'(t) N(t) R(h(t)) N^{-1}(t)] Z \tag{5a}$$

in case (4a), or into the equation

$$Z' = [N'(t) N^{-1}(t) + \lambda h'(t) \operatorname{tr} R(h(t)) E - h'(t) N(t) R^*(h(t)) N^{-1}(t)] Z \tag{5b}$$

in case (4b).

Let the previous Theorem be illustrated by an example. Consider the transformation (3) and (4b), where k is an identity, $\lambda = 0$ and

$$M(x) = C^{-1} = \begin{pmatrix} 0, & 0, & \dots, & 0, & -1 \\ 0, & 0, & \dots, & 1, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & (-1)^{n-1}, & \dots, & 0, & 0 \\ (-1)^n, & 0, & \dots, & 0, & 0 \end{pmatrix}.$$

Similarly to [4] we can make sure that this transformation converts every equation (1) that corresponds to the equation

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_0(x)\, y = 0, \tag{6}$$

into the equation (1) that represents the adjoint equation to (6)

$$(\dots((z' - p_{n-1}(x)\, z)' + p_{n-2}(x)\, z)' - \dots + (-1)^{n-1} p_1(x)\, z)' + (-1)^n p_0(x)\, z = 0.$$

The next statement is a consequence of the main Theorem. It describes all pointwise transformations of equations (1) with coefficients in $\mathbf{L}_n$ into equations (1) with coefficients in $\mathbf{K}_n$, taking into account only vector solutions.

Consequence. *Let $n \geqq 2$ and let $\tau = (\tau_1, \tau_2)$ be a homeomorphism of $I \times \mathbf{R}^n$ into $J \times \mathbf{R}^n$ satisfying the assumption:*

For every n-th order linear differential equation $E_n(p, I)$ with $p_i \in C^r(I, \mathbf{R})$ there is a linear quasi-differential equation of the n-th order $E_n(S, J)$ with $S \in C(J, \mathbf{K}_n)$ such that for every solution $y \in C^n(I, \mathbf{R})$ of the former equation the couples $(t, z^{[i]})$, $0 \leqq i \leqq n-1$,

$$t = \tau_1(x, (y(x), y'(x), \dots, y^{(n-1)}(x))^*),$$
$$(z^{[1]}, z^{[2]}, \dots, z^{[n-1]})^* = \tau_2(x, (y(x), y'(x), \dots, y^{(n-1)}(x))^*),$$

for $x \in I$ form graphs of functions $z^{[i]}(t) \in C^1(J, \mathbf{R})$ and $z^{[0]}(t)$ is a solution of the equation $E_n(S, J)$ with the following quasi-derivatives $z^{[1]}(t)$, $z^{[2]}(t), \dots, \dots, z^{[n-1]}(t)$.

Then τ is of the form

$$\begin{aligned} \tau_1(x, \mathbf{y}) &= k(x), \\ \tau_2(x, \mathbf{y}) &= M(x)\, \mathbf{y}, \end{aligned} \tag{7}$$

where k is a C^1-diffeomorphism of I onto J, $M \in C^1(I, \mathbf{GL}_n)$ and $M(x)$ is a matrix having only zeros above the principal diagonal for every $x \in I$.

Remark 2. According to the statement of Consequence every solution $y \in C^n(I, \mathbf{R})$ of some equation $E_n(p, I)$ is converted into the function

$$z^{[0]}(t) = M_1^1(h(t))\, y(h(t)),$$

where h denotes the inverse function to k. That means, roughly speaking, that the pointwise transformations of the above type do not depend on derivatives.

3. PROOFS

First, we shall prove Theorem. Let us suppose that T is a mapping satisfying **(A)** and **(B)**. Then **(A1)** and **(B1)** of Theorem 1 in [2] hold for T as well. Therefore, due to this Theorem 1, we get that T is of the form (3) and (4a) or (4b), where $\lambda \in \mathbf{R}$, $C \in \mathbf{GL}_n$, k is a homeomorphism having the inverse function $h \in C^1(J, I)$ and $N = M \circ h \in C^1(J, \mathbf{GL}_n)$. It remains to show the claim on the form of the matrix $M(x)$ and $h'(t) \neq 0$ for every $t \in J$.

According to Remark 1 the transformation T converts every equation $E(R, I)$ into the equation $E(S, J)$ of the form (5a) or (5b). We shall request $S \in C(J, \mathbf{K}_n)$ for every $R \in C^r(I, \mathbf{L}_n)$. Let us distinguish cases (5a) and (5b).

(i) In case (5a) we seek all functions $N \in C^1(J, \mathbf{GL}_n)$ and $h \in C^1(J, I)$ so that for every $R \in C^r(I, \mathbf{L}_n)$ there exists a function $S \in C(J, \mathbf{K}_n)$ such that

$$N' + h'NR \circ h = \tilde{S}N, \tag{8}$$

where $\tilde{S} = S - \lambda h' \operatorname{tr} R \circ hE \in C(J, \mathbf{K}_n)$.

Write $R \circ h$ in the form (2) and compare the i-th rows in (8) for $1 \leqq i \leqq n - 1$.

$$\begin{aligned}
&(N_1^i)' && - h'N_n^i p_0 &&= \sum_{j=1}^{i+1} \tilde{S}_j^i N_1^j,\\
&(N_2^i)' + h'N_1^i && - h'N_n^i p_1 &&= \sum_{j=1}^{i+1} \tilde{S}_j^i N_2^j,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&(N_m^i)' + h'N_{m-1}^i && - h'N_n^i p_{m-1} &&= \sum_{j=1}^{i+1} \tilde{S}_j^i N_m^j, \qquad 2 \leqq m \leqq n,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&(N_n^i)' + h'N_{n-1}^i && - h'N_n^i p_{n-1} &&= \sum_{j=1}^{i+1} \tilde{S}_j^i N_n^j.
\end{aligned} \tag{9}$$

Supposing $h'(t)\, N_n^i(t) \neq 0$ for some $t \in J$ and $1 \leqq i \leqq n - 1$, we can choose $p_0(t), p_1(t), \dots, p_{n-1}(t)$ for the left hand side of (9) to vanish. Since $\tilde{S}_{i+1}^i(t) \neq 0$, we get from (9) that a nontrivial linear combination of $i + 1$ rows of the matrix $N(t)$ is zero, which is in contradiction to $N(t) \in \mathbf{GL}_n$. Hence, $h'(t)\, N_n^i(t) = 0$ for every $t \in J$ and all i, $1 \leqq i \leqq n - 1$. The function h is strictly increasing or decreasing, therefore the set $\{t \in J,\ h'(t) \neq 0\}$ is dense in J and, that is why the set $\{t \in J,\ N_n^i(t) = 0\}$ is dense in J as well. Continuity of the function N implies

$N_n^i(t) = 0$ for every $t \in J$. Consequently, from the last equation in (9) we obtain $h'(t)\, N_{n-1}^i(t) = 0$ for every $t \in J$ and $1 \leqq i \leqq n-2$. The same consideration as above yields $N_{n-1}^i(t) = 0$ for every $t \in J$ and $1 \leqq i \leqq n-2$. In the same way it can be proved by induction in (9) that the necessary condition for (8) is

$$N_j^i(t) = 0 \qquad \text{for } 1 \leqq i < j \leqq n \text{ and for every } t \in J. \tag{10}$$

Let us assume there is a $t \in J$ such that $h'(t) = 0$. Taking into account (10), from the $(i+1)$-th equation in (9) we get

$$\tilde{S}_{i+1}^i(t)\, N_{i+1}^{i+1}(t) = 0,$$

which contradicts to $\tilde{S}_{i+1}^i(t) \neq 0$, $\det N(t) \neq 0$ and (10). Hence,

$$h'(t) \neq 0 \qquad \text{for every } t \in J. \tag{11}$$

Let us prove the converse: If $h \in C^1(J, I)$, $N \in C^1(J, \mathbf{GL}_n)$ and (10) and (11) are fulfilled, then the equation (5a) corresponds to linear quasi-differential equation of the n-th order for every $R \in C(I, \mathbf{K}_n)$, i.e. for every $R \in C(I, \mathbf{K}_n)$ the equation

$$N' + \lambda h' \operatorname{tr} R \circ hN + h'NR \circ h = SN \tag{12}$$

has a solution $S \in C(J, \mathbf{K}_n)$. Comparing the i-th rows in (12) for $1 \leqq i \leqq n-1$, we get

$$(N_1^i)' + h'\Big(\sum_{j=1}^{i} N_j^i R_1^j \circ h + \lambda \operatorname{tr} R \circ h N_1^i\Big) = \sum_{j=1}^{n} S_j^i N_1^j,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(N_m^i)' + h'\Big(\sum_{j=m-1}^{i} N_j^i R_m^j \circ h + \lambda \operatorname{tr} R \circ h N_m^i\Big) = \sum_{j=m}^{n} S_j^i N_m^j, \qquad 2 \leqq m \leqq i,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$h' N_i^i R_{i+1}^i \circ h = \sum_{j=i+1}^{n} S_j^i N_{i+1}^j, \tag{13}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0 = \sum_{j=m}^{n} S_j^i N_m^j, \qquad i+2 \leqq m \leqq n.$$

Then the fact that $\det(N_m^j) \neq 0$, $j, m \in \{i+2, i+3, \ldots, n\}$, for $1 \leqq i \leqq n-2$ implies $S_j^i = 0$ for $j \geqq i+2$. Consequently, from the $(i+1)$-th equation in (13) we obtain

$$h'(t)\, N_i^i(t)\, R_{i+1}^i(h(t)) = S_{i+1}^i(t)\, N_{i+1}^{i+1}(t).$$

Therefore according to (10) and (11), we have $S_{i+1}^i(t) \neq 0$ for all $t \in J$.

(ii) Now we are going to examine case (5b). First, we shall find a necessary condition on $N \in (J, \mathbf{GL}_n)$ and $h \in C^1(J, I)$ so that for every $R \in C'(I, \mathbf{L}_n)$ there exists a function $S \in C(J, \mathbf{K}_n)$ satisfying the equation

$$N'N^{-1} + \lambda h' \operatorname{tr} R \circ hE - h'NR^* \circ hN^{-1} = S.$$

Since $N'N^{-1} = -N(N^{-1})'$, we can rewrite the previous formula in the following way

$$(N^{-1})' + h'R^* \circ nN^{-1} = -N^{-1}\tilde{S}, \tag{14}$$

where $\tilde{S} = S - \lambda h' \operatorname{tr} R \circ hE \in C(J, \mathbf{K}_n)$.

Further, let us use V instead of N^{-1} and let $R \circ h$ be written in the form (2). For the j-th column, $2 \leqq j \leqq n$, the formula (14) reads as follows

$$\begin{aligned}
&(V_j^1)' && - h'p_0V_j^n &&= -\sum_{i=j-1}^{n} V_i^1\tilde{S}_j^i,\\
&(V_j^2)' + h'V_j^1 && - h'p_1V_j^n &&= -\sum_{i=j-1}^{n} V_i^2\tilde{S}_j^i,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&(V_j^m)' + h'V_j^{m-1} && - h'p_{m-1}V_j^n &&= -\sum_{i=j-1}^{n} V^m\tilde{S}_j^i, \qquad 2 \leqq m \leqq n,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&(V_j^n)' + h'V_j^{n-1} && - h'p_{n-1}V_j^n &&= -\sum_{i=j-1}^{n} V_i^n\tilde{S}_j^i.
\end{aligned} \tag{15}$$

Next we proceed similarly as in (i). Assuming $h'(t)\, V_j^n(t) \neq 0$ for some $t \in J$ and some j, $2 \leqq j \leqq n$, we can choose $p_0(t), p_1(t), \ldots, p_{n-1}(t)$ for the left hand side of (15) to vanish. Therefore a nontrivial linear combination of $n - j + 2$ columns of the matrix V is zero, which is a contradiction. That is why $h'(t)\, V_j^n(t) = 0$ for all $t \in J$ and all j, $2 \leqq j \leqq n$. Similarly as in (i), we derive $V_j^n(t) = 0$ for all $t \in J$ and $2 \leqq j \leqq n$. From the last equation in (15) we get $h'(t)\, V_j^{n-1}(t) = 0$ for every $t \in J$ and $3 \leqq j \leqq n$. As in (i), we can show $V_j^{n-1}(t) = 0$ for every $t \in J$ and $3 \leqq j \leqq n$. The induction analogously yields

$$V_j^i(t) = 0 \qquad \text{for every } t \in J \text{ and } i + j \geqq n + 2. \tag{16}$$

Considering (16) and supposing $h'(t) = 0$ for some $t \in J$ we get from the $(n - j + 2)$-th equation in (15)

$$V_{j-1}^{n+2-j}(t)\, \tilde{S}_j^{j-1}(t) = 0,$$

which is in contradiction to $\tilde{S}_j^{j-1}(t) \neq 0$, $\det V \neq 0$ and (16). Hence (11) is also true in this case.

Proof of the converse: If $h \in C^1(J, I)$ and $N \in C^1(J, \mathbf{GL}_n)$ satisfy (11) and (16), where $V = N^{-1}$, the equation (5b) corresponds to linear quasi-differential equation of the n-th order for every $R \in C(I, \mathbf{K}_n)$, i.e. for every $R \in C(I, \mathbf{K}_n)$ the equation

$$V' - \lambda h' \operatorname{tr} R \circ hV + h'R^* \circ hV = -VS \tag{17}$$

has a solution $S \in C(J, \mathbf{K}_n)$. Writing (17) for the j-th column, $2 \leqq j \leqq n$, we have

$$(V_j^1)' + h'\Big(\sum_{i=1}^{n+1-j} R_1^i \circ h\ _j^i - \lambda \operatorname{tr} R \circ hV_j^1\Big) = -\sum_{i=1}^{n} V_i^1 S_j^i,$$

....

$$(V_j^m)' + h'\Big(\sum_{i=m-1}^{n+1-j} R_m^i \circ hV_j^i - \lambda \operatorname{tr} R \circ hV_j^m\Big) = -\sum_{i=1}^{n+1-m} V_i^m S_j^i, \qquad 2 \leqq m \leqq n+1-j$$

....

$$h' R_{n+2-j}^{n+1-j} \circ hV_j^{n+1-j} = -\sum_{i-1}^{j-1} V_i^{n+2-j} S_j^i, \tag{18}$$

....

$$0 = -\sum_{i=1}^{n+1-m} V_i^m S_j^i, \qquad n+3-j \leqq m \leqq n.$$

Since $\det (V_i^m)_{i=1,2,\dots,j-2}^{m=n+3-j,\dots,n} \neq 0$ for $3 \leqq j \leqq n$, we get $S_j^i = 0$ for $j \geqq i+2$. Consequently, the $(n+2-j)$-th equation in (18) is

$$h'(t)\, R_{n+2-j}^{n+1-j}(h(t))\, V_j^{n+1-j}(t) = -V_{j-1}^{n+2-j}(t)\, S_j^{j-1}(t).$$

Therefore (11) and (16) yield $S_j^{j-1}(t) \neq 0$ for every $t \in J$ and $2 \leqq j \leqq n$. This completes the proof of Theorem.

The proof of Consequence proceeds in the similar way as the proof of Theorem 2 in [2], so we sketch it only. First, we prove that τ_1 depends on x only, i.e.

$$\tau_1(x, \mathbf{y}) = k(x),$$

where k is a homeomorphism of I onto J. Then we define the mapping $\boldsymbol{T} = (T_1, T_2) : I \times \mathbf{GL}_n^+ \to J \times \mathbf{M}_n$ by the formula

$$\begin{aligned} &T_1(x, Y) = k(x), \\ &T_2(x, Y) = (\tau_2(x, Y_1), \tau_2(x, Y_2), \dots, \tau_2(x, Y_n)), \end{aligned} \tag{19}$$

where $Y_i \in \mathbf{R}^n$, $1 \leqq i \leqq n$. The mapping T satisfies the assumptions (A) and (B), hence the form of this transformation is described by (3) and (4). Because of $\boldsymbol{T}$ being defined with the aid of (19), we get only

$$\boldsymbol{T}_2(x, Y) = M(x)\, Y,$$

where $M(x)$ has zeros above the principal diagonal. The remaining part of the proof follows from (19) and Theorem.

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Martin Čadek
Mathematical Institute
of the Czechoslovac Academy of Sciences, branch Brno
Mendlovo nám. 1
603 00 Brno
Czechoslovakia