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ON OSCILLATORY SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS OF THE n-th ORDER VANISHING AT INFINITY

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Abstract. In the paper the oscillatory proper solutions of nonlinear differential equations are studied. There are given sufficient conditions for the existence of such solutions vanishing at infinity.

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Consider the differential equation

(1) $$y^{(n)} = f(t, y, \dots, y^{(n-1)}), \qquad n \geqq 4.$$

Let $R = (-\infty, \infty)$, $R_+ = [0, \infty)$, n_0 be the entire part of $n/2$, $D_n = R_+ \times R^n$, $N = \{1, 2, \dots\}$.

In all the paper we shall suppose that $f : D_n \to R$ fulfils the local Carathéodory conditions, there exists $\alpha \in \{0,1\}$ such that

(2) $$(-1)^\alpha f(t, x_1, \dots, x_n)\, x_1 \geqq 0 \qquad \text{in } D_n$$

holds and $n + \alpha$ is odd; n_0 is even.

Denote $C^k(I)$, $I \subset R_+$ the set of all continuous function which have continuous derivatives up to the order k on I. $L_{loc}(R_+)$ is the set of all functions, defined on R_+, integrable on every finite segment of R_+.

By a solution of (1) defined on $I \subset R_+$ we shall mean a function $y \in C^{n-1}(I)$ for which $y^{(n-1)}$ is absolutely continuous on each segment of the interval I and y satisfies (1) for almost all $t \in I$. Let y be a solution of (1) defined on $[\bar{t}, \infty)$, $0 \leqq \bar{t}$. y is called proper if for large t: $\sup\limits_{t \leqq s < \infty} |y(s)| > 0$ holds. y is called singular if there exists $b \in (\bar{t}, \infty)$ such that $y \equiv 0$ on $[b, \infty)$ and $\sup\limits_{t \leqq s < b} |y(s)| > 0$ for $t \in [\bar{t}, b)$. The proper (singular) solution y is called oscillatory if there exists an increasing sequence of its zeros tending to ∞ (to b). In the other case y is called nonoscillatory. The solution $y \equiv 0$ on R_+ is called trivial.

In the two last decades a great effort is devoted to the study of proper solutions of (1), (2). One of the important problems consists in the study of conditions under which the proper solution of (1), (2) tends to zero for $t \to \infty$ (see [3], [6], [7], [9]). Our goal lies in studying this problem foı oscillatory solutions.

The problem of vanishing of a solution y for $t \to \infty$ is very closely tied to the validity of the condition

$$I(y) = \int_0^\infty [y^{(n_0)}(t)]^2 \, dt < \infty \tag{3}$$

as it is shown in [3, 7]. Moreover, if $I(y) = \infty$ for an oscillatory proper solution y and for f being continuous on D_n, then $y^{(n_0-\alpha)}$ is unbounded, and under the validity of some assumptions on f, y is unbounded, too (see [3]).

First, let us state an inequality of Kolmogorov–Horny type:

Lemma 1. *Let* $-\infty < a < b < \infty$, $k \geqq 2$, $y \in C^{k-1}[a, b]$, $y^{(k-1)}$ *is absolute continuous,* $v_i = \max_{a \leq t \leq b} | y^{(i)}(t) |$ *for* $i = 0, 1, \ldots, k-1$, $v_k \geqq | y^{(k)}(t) |$ *for almost all* $t \in [a, b]$. *Let* $y^{(i)}$ *have a zero in* $[a, b]$ *for* $i = 1, 2, \ldots, k-1$. *Then*

$$v_i \leqq 2^{(k+i-1)(k-i)} v_0^{(k-i)/k} v_k^{i/k}, \qquad i = 0, 1, \ldots, k.$$

Kiguradze [6] proved this statement for $y \in C^k[a, b]$ when proving his lemma 9.1. In our case the proof is the same.

Now, we give two existence theorems for oscillatory proper solutions tending to zero for $t \to \infty$ which generalize some results in [4].

Theorem 1. *Suppose that there exist positive numbers* ε, M *and functions* $a : R_+ \to R_+$, $g : [0, \varepsilon] \to R_+$ *such that* $a \in L_{loc}(R_+)$, $g \in C^0[0, \varepsilon]$, $g(0) = 0$, $g(x) > 0$ *for* $x > 0$ *and*

$$a(t)\, g(|x_1|) \leqq | f(t, x_1, \ldots, x_n) | \leqq M \sum_{i=1}^n |x_i| \quad \text{on} \quad R_+ \times [-\varepsilon, \varepsilon]^n \tag{4}$$

holds and let one of the following assumptions be valid

1° $$\alpha = 1, \qquad a(t) = \frac{1}{1+t};$$

2° $$\lambda \in [1, \infty), \qquad g(x) = x^\lambda, \qquad \lim_{t\to\infty} t^{n - \frac{\lambda-1}{2}(1+\varepsilon)} a(t) = \infty;$$

3° $$\alpha = 0, \qquad a(t) = M_1 > 0 \quad \textit{for } t \in R_+,$$

Then there exists an oscillatory proper solution y of (1), *defined in some neighbourhood of ∞ and y fulfils the conditions*

$$\lim_{t\to\infty} y^{(i)}(t) = 0, \qquad i = 0, 1, \dots, n-1. \tag{5}$$

Proof. Let $\alpha = 1$ and $\lambda = 1$ in the case 1°. We can suppose, that g is nondecreasing for 1° without loss of generality. Let us define functions $\bar{f}: D_n \to R$ and $\bar{g}: R_+ \to R_+$

$$\bar{x}_i = x_i \qquad \text{for } |x_i| \leqq \varepsilon,$$

$$\bar{x}_i = \varepsilon \operatorname{sign} x_i \qquad \text{for } |x_i| > \varepsilon, \qquad i = 1, 2, \dots, n,$$

$$\begin{aligned} \bar{f}(t, x_1, \dots, x_n) &= f(t, \bar{x}_1, \dots, \bar{x}_n) \qquad \text{for } |x_1| \leqq \varepsilon, \\ &= a(t)\,|x_1|^{\lambda} \operatorname{sign} x_1 + f(t, \bar{x}_1, \dots, \bar{x}_n) \qquad \text{for } |x_1| > \varepsilon, \end{aligned} \tag{6}$$

$$\bar{g}(x) = g(\bar{x}) \qquad \text{for } 1°, \qquad \bar{g}(x) = x^{\lambda} \qquad \text{for } 2°.$$

From these definitions and from (4) we have in both cases 1° and 2° in D_n

$$a(t)\,\bar{g}(|x_1|) \leqq |\bar{f}(t, x_1, \dots, x_n)| \leqq a(t)\,|x_1|^{\lambda} + Mn\varepsilon \leqq M_1\,|x_1|^{\lambda} + Mn\varepsilon, \tag{7}$$

where M_1 is a suitable constant. Thus, with respect to [8, Th. 1.1, 1.3] and [7, Lemma 1.5] there exists a non-trivial solution y of the differential equation

$$y^{(n)} = \bar{f}(t, y, \dots, y^{(n-1)}) \tag{8}$$

defined on R_+ and satisfying (3); such solution fulfils

$$\lim_{t\to\infty} y(t) = 0 \tag{9}$$

in case 2°. As according to (4) singular solutions do not exist, y is proper. Further, with respect to (7) the equation (8) has only oscillatory proper solutions (see so called Property A, [6, Consequence of Th. 14.1]) and the relation (9) follows the case 1° from Theorem 4 of [3] (this theorem is proved only for $f \in C^0(D_n)$, but if f fulfils the local Carathéodory conditions, the proof is similar). Thus (9) is valid in both cases and y is oscillatory proper. It is clear from (7), (8), (9), that $y^{(n)}$ is bounded for almost all t on R_+ and according to Lemma 1 (5) holds. It is clear that with respect to (6) y is a solution of (1), too. For $\alpha = 0$ the proof is similar. We must use Property B (see [6]) instead of Property A and [2] instead of [3]. The theorem is proved.

Theorem 2. *Let there exist a number $\varepsilon > 0$ and functions $a: R_+ \to R_+$, $d: R_+ \to R_+$, $g: R_+ \to R_+$, $h: R_+^{n_0+1} \to R_+$, such that $a, d \in L_{\mathrm{loc}}(R_+)$, $g \in C^0(R_+)$, g is nondecreasing, $g(x) > 0$ for $x > 0$, $h(., x_1, \dots, x_{n_0}) \in L_{\mathrm{loc}}(R_+)$ for arbitrary $x_1, \dots, x_{n_0} \in R_+$, h is nondecreasing with respect to the last n_0 variables. Further, let at least one of the cases 1°, 2° from Theorem 1 be valid and*

$$(10)\qquad |f(t, x_1, \ldots, x_n)| \leqq \mathrm{d}(t) \sum_{i=1}^{n} |x_i| \qquad on\ R_+ \times [-\varepsilon, \varepsilon]^n,$$

$$a(t)\, g(|x_1|) \leqq |f(t, x_1, \ldots, x_n)| \leqq h(t, |x_1|, \ldots, |x_{n_0}|) \times$$

$$(11)\qquad \times [1 + \sum_{i=n_0+1}^{n} |x_i|^{\gamma_i}] \qquad on\ J$$

hold where $J = R_+ \times [-\varepsilon, \varepsilon]^{s+1} \times R^{n-s-1}$, $\gamma_i = \dfrac{2n - 2n_0 - 1}{2i - 2n_0 - 1}$ *and* $s = n_0 - 2$ ($s = n_0 - 1$) *in the case* 1° (2°). *Then there exists an oscillatory proper solution* y *of* (1) *defined in some neighbourhood of* ∞ *such that*

$$(12)\qquad \lim_{t \to \infty} y^{(i)}(t) = 0, \qquad i = 0, 1, \ldots, s$$

holds. If $J = D_n$, *then this solution is defined on* R_+.

Proof is similar to that one of Theorem 1, only we use

$$\begin{aligned} \bar{f}(t, x_1, \ldots, x_n) &= f(t, \bar{x}_1, \ldots, \bar{x}_{s+1}, x_{s+2}, \ldots, x_n) \qquad \text{for } |x_1| \leqq \varepsilon \\ &= a(t)\, |x_1|^{\lambda} \operatorname{sign} x_1 + f(t, \bar{x}_1, \ldots, \bar{x}_{s+1}, x_{s+2}, \ldots, x_n) \qquad \text{for } |x_1| > \varepsilon \end{aligned}$$

instead of (6). If $J = D_n$, we use directly f instead of $\bar{f}$.

Remark. If the inequality (10) is omited from the assumptions of Theorem 2, then there exists a solution of (1) with the property (12) which is either oscillatory proper or oscillatory singular. This conclusion follows from the proof and from the fact that nonoscillatory singular solutions do not exist (see Kiguradze's lemma [6, lemmas 14.1, 14.2]).

In the rest of the paper we shall study the differential equations of the fourth order, $n = 4$, $\alpha = 1$.

Lemma 2. *Let* $n = 4$, $\alpha = 1$ *and let* y *be an oscillatory proper solution of* (1) *or which* (3) *holds. Then* $\lim_{t \to \infty} y^{(i)}(t) = 0$, $i = 1, 2$. *If, moreover, there exist* $\varepsilon > 0$ *and a continuous function* $g : R_+ \to R_+$ *such that* $g(x) > 0$ *for* $x > 0$ *and*

$$\frac{1}{t+1} \cdot g(|x_1|) \leqq |f(t, x_1, x_2, x_3, x_4)| \text{ on } R_+ \times [-\varepsilon, \varepsilon]^4$$

hold, then $\lim_{t \to \infty} y^{(j)}(t) = 0$, $j = 0, 1, 2$ *is valid.*

Proof. Define $z(t) = -y'(t)\, y(t) + 2 \int_0^t [y'(t)]^2\, \mathrm{d}t$, $t \in R_+$. Then (see (1), (2))

$$z'(t) = -y''(t)\, y(t) + [y'(t)]^2, \qquad z''(t) = -y'''(t)\, y(t) + y'(t)\, y''(t),$$

$$z'''(t) = -y^{(4)}(t)\, y(t) + [y''(t)]^2 \geqq 0 \qquad \text{for almost all } t \in R_+.$$

In [3] it was proved for $f \in C^0(D_4)$ that $|z^{(i)}|$ are nonincreasing

(13) $\quad \lim_{t\to\infty} z^{(i)}(t) = 0, \qquad i = 1, 2, \qquad \lim_{t\to\infty} y'(t) = 0, \qquad \int_0^\infty t\,|y^{(4)}(t)\,y(t)|\,dt < \infty$

hold. This statement can be proved for f fulfilling the local Carathéodory conditions similarly.

It was proved in [1] that there exist sequences $\{t_k^i\}$, $k \in N$, $i = 0, 1, 2, 3$ and $\tau \in \{-1, 1\}$ such that $0 \leqq t_k^0 < t_k^1 < t_k^2 < t_k^3 < t_{k+1}^0$, $\lim_{k\to\infty} t_0^k = \infty$, $y^{(i)}(t_k^i) = 0$, $(-1)^i \tau y^{(i)}(t) < 0$ on $[t_k^0, t_k^i)$, $i = 1, 2, 3$, $(-1)^j \tau y^{(j)}(t) > 0$ on (t_k^j, t_{k+1}^0), $j = 0, 1, 2$, $\tau y'''(t) \leqq 0$ on $[t_k^3, t_{k+1}^0]$ hold. Thus $|y''(t)|$ is over the tangent (under the tangent) on $[t_k^0, t_k^2]$ $([t_k^2, t_{k+1}^0])$,

(14) $\quad |y''(t)|\ (|y'''(t)|) \quad$ is nonincreasing on $[t_k^3, t_{k+1}^2]$ $([t_k^0, t_0^3])$, $\quad k \in N$.

Put

$$\Delta_k = t_k^2 - t_k^0, \qquad \Delta_{k1} = t_{k+1}^0 - t_k^2, \qquad \Delta_{k2} = t_k^2 - t_k^1, \qquad \Delta_{k3} = t_k^3 - t_k^2,$$
$$I_{k2} = [t_k^1, t_k^2], \qquad I_{k3} = [t_k^2, t_k^3], \qquad I_k = [t_k^0, t_k^2].$$

First, we prove that the sequence of the absolute values of local extremes of $y'' - \{|y''(_k^3)|\}_1^\infty$ is non-increasing. If $\Delta_{k2} > \Delta_{k3}$, then by use of (14) we have

(15) $\quad |y''(t_{k-1}^3)| \geqq |y''(t_k^1)| = \int_{I_{k2}} |y'''(t)|\,dt \geqq \int_{I_{k3}} |y'''(t)|\,dt = |y''(t_k^3)|.$

If $\Delta_{k2} \leqq \Delta_{k3}$, then by use of (14) we have

$$|y'(t_k^2)| = \int_{I_{k2}} |y''(t)|\,dt \leqq |y''(t_k^1)|\,\frac{\Delta_{k2}}{2},$$
$$|y'(t_k^2)| \geqq y'(t_k^2)| - |y'(t_k^3)| = \int_{I_{k3}} |y''(t)|\,dt \geqq |y''(t_k^3)|\,\frac{\Delta_{k3}}{2}$$

and thus

$$|y''(t_{k-1}^3)| \geqq |y''(t_k^1)| \geqq |y''(t_k^3)|\,\frac{\Delta_{k3}}{\Delta_{k2}} \geqq |y''(t_k^3)|.$$

From this and from (15), in both cases, the sequence $\{|y''(t_k^3)|\}_1^\infty$ is non-increasing. We prove by the indirect proof that $\lim_{t\to\infty} y''(t) = 0$. Thus, with respect to the last result we can suppose, that there exists a constant $M > 0$ such that

(16) $\qquad |y''(t_k^3)| \geqq M, \qquad k \in N$

holds. By virtue of (3), (14) and (16) we have

$$\infty > \sum_{k=1}^\infty \int_{t_k^2}^{t_{k+1}^0} [y''(t)]^2\,dt \geqq \sum_{k=1}^\infty \left\{ \int_{I_{k3}} \left[y''(t_k^3)\,\frac{t - t_k^2}{\Delta_{k3}} \right]^2 dt + \right.$$
$$\left. + \int_{t_k^3}^{t_{k+1}^0} \left[y''(t_k^3)\,\frac{t_{k+1}^0 - t}{\Delta_{k1} - \Delta_{k3}} \right]^2 dt \right\} \geqq \sum_{k=1}^\infty \frac{1}{3} [y''(t_k^3)]^2 (t_{k+1}^0 - t_k^2),$$

$$\sum_{k=1}^{\infty} \Delta_{k1} < \infty. \tag{17}$$

As

$$M \leqq |y''(t_k^3)| = \int_{I_{k3}} |y'''(t)| \, \mathrm{d}t \leqq |y'''(t_k^2)| \Delta_{k3} \leqq |y'''(t_k^2)| \Delta_{k1}, \tag{18}$$

then according to (17)

$$\lim_{k \to \infty} |y'''(t_k^2)| = \infty. \tag{19}$$

Thus, by use of (14)

$$\begin{aligned} |y'''(t_k^2)| \frac{\Delta_k^2}{2} &= \int_{I_k} |y'''(t_k^2)| (t_k^2 - t) \, \mathrm{d}t \leqq \int_{I_k} |y''(t)| \, \mathrm{d}t = \\ &= |y'(t_k^0)| + |y'(t_k^2)| \underset{k \to \infty}{\to} 0, \end{aligned} \tag{20}$$

$\lim_{k \to \infty} \Delta_k = 0$. From this and from (17) $M = \frac{2}{\pi} \max_{k \in N} (t_{k+1}^0 - t_k^0) < \infty$. By use of Levin's lemma [6, Lemma 4.7] and (3) we have

$$\begin{aligned} \int_{t_1^0}^{\infty} (y'(t))^2 \, \mathrm{d}t &\leqq \sum_{k=1}^{\infty} \int_{t_k^0}^{t_{k+1}^0} (y'(t))^2 \, \mathrm{d}t \leqq \sum_{k=1}^{\infty} M^2 \int_{t_k^0}^{t_{k+1}^0} (y''(t))^2 \, \mathrm{d}t \leqq \\ &\leqq M^2 \int_0^{\infty} (y''(t)^2 \, \mathrm{d}t < \infty. \end{aligned} \tag{21}$$

As the function z is nondecreasing, it is clear that $z(\infty) < \infty$ and (see (13) and (2))

$$\int_0^{\infty} t^2 [y''(t)]^2 \, \mathrm{d}t = 2 \int_0^{\infty} \int_t^{\infty} \int_t^{\infty} [y''(t)]^2 \, \mathrm{d}t \, \mathrm{d}t \, \mathrm{d}t \leqq 2 \int_0^{\infty} \int_t^{\infty} \int_t^{\infty} z'''(t) \, \mathrm{d}t \, \mathrm{d}t \, \mathrm{d}t < \infty.$$

If we use this result and Carlson's inequality [5], we get

$$\int_0^{\infty} |y''(t)| \, \mathrm{d}t \leqq \pi \Big[\int_0^{\infty} [y''(t)]^2 \, \mathrm{d}t \int_0^{\infty} t^2 [y''(t)]^2 \, \mathrm{d}t\Big]^{1/4} < \infty.$$

Thus, by virtue of (18) and (20)

$$\infty > \sum_{k=1}^{\infty} \int_{I_k} |y''(t)| \, \mathrm{d}t \geqq \frac{M}{2} \sum_{k=1}^{\infty} \frac{\Delta_k^2}{\Delta_{k1}}. \tag{22}$$

Let $N_1 \subset N$ be the set of indexes k such that $\Delta_k \geqq \Delta_{k1}$. Then it follows from (22) that $\sum_{k \in N_1} \Delta_k < \infty$. If $\Delta_k < \Delta_{k1}$, by use of (17) we have

$$\sum_{k \in N - N_1} \Delta_k \leqq \sum_{k \in N - N_1} \Delta_{k1} < \infty.$$

Thus

$$\sum_{k \in N} \Delta_k < \infty, \qquad \sum_{k \in N} \Delta_k + \Delta_{k1} < \infty$$

which contradicts the fact that $\lim_{t\to\infty} t_k^0 = \infty$. This contradiction proves that $\lim_{t\to\infty} y''(t) = 0$.

Now we prove by indirect proof that $\lim_{t\to\infty} y(t) = 0$. Thus suppose without loss of generality that $y(t_k^1) \geqq M_1 > 0$, $k \in N$. Let s be an arbitrary number, $s \leqq \varepsilon$, $s \leqq M_1$. Define numbers τ_k^1, τ_k^2 in the following way: $t_k^0 < \tau_k^1 < \tau_k^2 < t_k^1$, $|y(\tau_k^1)| = \frac{s}{2}$, $|y(\tau_k^2)| = s$, $k \in N$, $J_k = [\tau_k^1, \tau_k^2]$. Then there exists an index $k_0 \in N$ such that (see (13): $\lim_{t\to\infty} z''(t) = 0$)

$$(23) \qquad s/2 \leqq |y(t)| \leqq s, \quad |y^{(i)}(t)| \leqq s, \quad i = 1, 2, 3, \quad t \in J_k, \quad k \geqq k_0$$

holds. Moreover, $\tau_k^2 - \tau_k^1 \geqq 1/2$ because

$$s/2 = |y(\tau_k^2)| - |y(\tau_k^1)| = \int_{J_k} |y'(t)| \,\mathrm{d}t \leqq s(\tau_k^2 - \tau_k^1).$$

From this and from (23), (13) we have

$$\infty > \int_0^\infty t\,|y^{(4)}(t)\,y(t)| \,\mathrm{d}t \geqq \sum_{k=0}^{\infty} \int_{J_k} g(|y(t)|)\,|y(t)| \,\mathrm{d}t \geqq$$
$$\geqq \frac{cs}{2} \sum_{k=0}^{\infty} (\tau_k^2 - \tau_k^1) = \infty, \qquad c = \min_{s/2 \leqq x \leqq s} g(x).$$

This contradiction proves the theorem.

Remark. It follows from the proof of the lemma 2 that the sequences of the absolute values of local extremes of y' and y'' – $\{|y'(t_k^2)|\}$, $\{|y''(t_k^3)|\}$, $k \in N$ are nonincreasing.

By use of Lemma 2 the existence Theorem 2 can be generalized.

Theorem 3. *Let $n = 4$, $\alpha = 1$, there exist number $\varepsilon > 0$ and functions $g : R^2 \to R_+$, $d : R_+ \to R_+$, $h : R_+^3 \to R_+$ such that $g \in C_0(R_+^2)$, g is nondecreasing with respect to the first argument, $g(x_1, x_2) > 0$ for $x_1 > 0$, $d \in L_{\mathrm{loc}}(R_+)$, $h(., x_1, x_2) \in L_{\mathrm{loc}}(R_+)$ for arbitrary $x_1, x_2 \in R_+$ and h is nondecreasing with respect to the last two variables. Further, let* (10) *be valid and*

$$(24) \qquad \frac{1}{t+1} g(|x_1|, |x_2|) \leqq |f(t, x_1, x_2, x_3, x_4)| \quad \text{on } R_+ \times J \times [-\varepsilon, \varepsilon]^2,$$

$$|f(t, x_1, x_2, x_3, x_4)| \leqq h(t, |x_1|, |x_2|)(1 + |x_3|^3 + |x_4|) \quad \text{on } R_+ \times J_1 \times R$$

where $J = [\varepsilon, \varepsilon]^2$, $J_1 = [-\varepsilon, \varepsilon]^3$. Then there exists an oscillatory proper solution y

of (1), *defined in some neighbourhood of* ∞ *such that*

$$\lim_{t\to\infty} y^{(i)}(t) = 0, \qquad i = 0, 1, 2 \tag{25}$$

holds. If $J = R^2$ *and* $J_1 = R^3$, *then this solution is defined on* R_+.

Proof. The statement of the theorem can be proved similarly to Theorem 2, where we put $\bar{f}(t, x_1, x_2, x_3, x_4) = f(t, \bar{x}_1, \bar{x}_2, \bar{x}_3, x_4)$, $\bar{g}(x_1, x_2) = g(\bar{x}_1, \bar{x}_2)$; $\bar{x}_i$ are given by (6). Thus $\frac{1}{t+1}\bar{g}(|x_1|, |x_2|) \leqq |\bar{f}(t, x_1, x_2, x_3, x_4)|$ on $R_+ \times R^3 \times \times [-\varepsilon, \varepsilon]$. Then the existence of the solution y of (8), defined on R_+, is guaranteed. We must consider two facts, only. First, we can see that the assumptions of Lemma 2 are fulfilled and thus (25) is valid if y is oscillatory proper. The second fact consists in the conclusion that the proper solution y must be oscillatory. Let us prove this fact. According to Kiguradze's lemma every nonoscillatory proper solution with the property (3) must fulfil the conditions

$$y^{(i)}(t) \neq 0 \quad i = 0, 1, 2, 3, \quad |y(t)| \text{ is increasing,} \tag{26}$$

$$|y'(t)| \text{ is decreasing in some neighbourhood of } \infty,$$

$$\lim_{t\to\infty} y^{(j)}(t) = 0, \qquad j = 2, 3.$$

Suppose, that for y (26) holds. Then we have for a sufficiently large τ $\infty > |y'''(\tau)| = \int_\tau^\infty |y^{(4)}(t)|\,\mathrm{d}t \geqq \int_\tau^\infty \frac{1}{t+1}\bar{g}(|y(\tau)|, |y'(t)|)\,\mathrm{d}t \geqq K\int_\tau^\infty \frac{1}{t+1}\,\mathrm{d}t = \infty$, $K = \min_{0\leqq s\leqq \varepsilon_1} \bar{g}(\tau, s) > 0$, where $\varepsilon_1 = \max_{\tau\leqq t<\infty} |y'(t)|$. This contradiction shows that every proper solution is oscillatory. The theorem is proved.

REFERENCES

[1] M. Bartušek, *The Asymptotic Behaviour of Oscillatory Solutions of the Equation of the Fourth Order*, Arch. Math. (Brno), 21, No. 2 (1985), 93–104.

[2] M. Bartušek, *On Properties of Oscillatory Solutions of Non-linear Differential Equations of the n-th Order*, Differential Equations and Their Applications, Equadiff 6, Proc. 6th Int. Conf. Brno, Lecture Notes Math., 1192, 107–113.

[3] M. Bartušek, *On Oscillatory Solution of the Differential Equation of the n-th Order*, Arch. Math. (Brno), 22, No. 3 (1986), 145–156.

[4] M. Bartušek, *On Proper Oscillatory Solutions of the Non-linear Differential Equations of the n-th Order*, Arch. Math. (Brno), 24, No. 2 (1988), 89–98.

[5] F. Beckenbach, R. Bellman, *Inequalities*, Springer Verlag, Berlin, 1961.

[6] И. Т. Кигурадзе, *Некоторые сингулярные краевые задачи для обыкновенных дифференциальных уравнений*, Из. Тбилис. унив., Тбилиси, 1975.

[7] I. T. Kiguradze, *On Vanishing at Infinity of Solutions of Ordinary Differential Equations*, Czech. Math. J., 33 (108) 1983, 613–646.

[8] И. Т. Кигурадзе, *Об одной краевой задаче с условием на бесконечности для обыкновенных дифференциальных уравнений высших порядков*, Тр. Всесоюзного симпозиума в Тбилиси 21—23 ап. 1982 г. Изд. Тбилис. ун-та, Тбилиси 1986, 91—105.

[9] J. S. E. Wong, *On the Generalized Emden-Fowler Equation*, SIAM Rev., 17, No. 2, 1975, 339–360.

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