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ON THE ASYMPTOTIC PROPERTIES OF SOLUTIONS OF NON-LINEAR THIRD ORDER DIFFERENTIAL EQUATION

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Dedicated to Academician O. Borůvka on the occasion of his 90th birthday

Abstract. In this paper we shall study some asymptotic properties of solutions defined on (a, ∞) of the differential equation of the form $u''' + q(t)\,u' + p(t)\,u^\alpha = 0$, where $q'(t), p(t)$ are continuous functions of $t \in (a, \infty)$, a is a real number and α is an odd integer.

Key words. Nonlinear differential equation, asymptotic properties of solutions, oscillatory or nonoscillatory solutions.

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1. Professor O. Borůvka in 1950 attracted many young mathematicians from Brno and Bratislava to his seminar in Brno on the theory of linear ordinary differential equations. There is also pointed out to a number of unsolved problems in third order differential equation theory. These problems were then intensively studied not only in Brno and Bratislava but also in other countries. The methods of this theory can be applied as suitable tools for the study of properties of solutions of some nonlinear differential equations.

In this paper, dedicated to the 90-th birthday of professor Borůvka, we shall study asymptotic properties of solutions of the differential equation of the form

$$u''' + q(t)\,u' + p(t)\,u^\alpha = 0, \tag{1}$$

where $q'(t)$, $p(t)$ are continuous functions of $t \in (a, \infty)$, a is a real number and α is an odd integer. Some results can be generalized to the case, where α is a ratio of odd integers.

There is a lot of papers devoted to the study of properties of solutions of the differential equation of the form (1) or a generalized form ([1], [3], [4], [7] and other).

2. We restrict our considerations to those real solutions of (1) which exist on the interval (a, ∞) and which are nontrivial on (β, ∞) for every $\beta \geqq a$.

The solution of (1) is oscillatory on (a, ∞) if it has arbitrarily large zeros, otherwise we call it nonoscillatory.

The following integral identity is true for solutions of the differential equation (1)

$$uu'' - \frac{1}{2}u'^2 + \frac{1}{2}qu^2 + \int_{t_0}^{t}\left[pu^{\alpha-1} - \frac{1}{2}q'\right]u^2\,d\tau = \text{const}, \tag{2}$$

where $t_0 \in (a, \infty)$ is a fixed number, $t \in (a, \infty)$ is variable.

The integral identity (2) is obtained by multiplying the differential equation (1 by u and integrating termwise from t_0 to t.

Theorem 1. *Let $p(t) \geqq 0$, $q'(t) < 0$ and $\int_{t_0}^{\infty} p\,d\tau = \infty$.*

Then every nonoscillatory solution u of the differential equation (1) *with the property*

$$u(t)\,u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)\,u^2(t) > 0$$

for $t \geqq t_0 > a$ has the property

$$\liminf_{t\to\infty} |u(t)| = 0. \tag{3}$$

Proof. Let $u = u(t)$ be a nonoscillatory solution of (1) on (a, ∞). The integral identity (2) for u is

$$uu'' - \frac{1}{2}u'^2 + \frac{1}{2}qu^2 = k - \int_{t_0}^{t}\left[pu^{\alpha-1} - \frac{1}{2}q'\right]u^2\,d\tau,$$

where $k > 0$ and $a < t_0 < \infty$.

Let $u(t) \neq 0$ for $t \geqq t_0$ and let (3) be not true. Then the preceding identity and the assumption that $\int_{t_0}^{\infty} p\,d\tau = \infty$ imply a contradiction and the theorem is proved.

Theorem 2. *Let $q(t) \geqq 0$, $q'(t) \geqq k > 0$ and $p(t) < 0$ for $t \in (a, \infty)$.*

Let u be a solution of (1) *with the property*

$$u(t)\,u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)\,u^2(t) \leqq 0,$$

for $t \geqq t_0$, $t_0 \in (a, \infty)$. Then $u = u(t)$ belongs to the class L^2. The proof follows from the integral identity (2) too which has the following form for u

$$u(t)\,u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)\,u^2(t) = k - \int_{t_0}^{t}[p(\tau)\,u^{\alpha-1}(\tau) - \frac{1}{2}q'(\tau)]\,u^2(\tau)\,d\tau,$$

$$k \leqq 0.$$

The relation $p(t)\,u^{\alpha-1}(t) - q'(t) < 0$ is true for $t \geqq t_0$. If we suppose that the assertion of Theorem 2 is not true we obtain a contradiction.

Lemma 1. *Let $q'(t) \geqq 0$, $p(t) < 0$ for $t \in (a, \infty)$. Then every solution u of the differential equation* (1) *with the property*

$$u(t_0)\,u''(t_0) - \frac{1}{2}u'^2(t_0) + \frac{1}{2}q(t_0)\,u^2(t_0) \geqq 0, \qquad t_0 \in (a, \infty)$$

has no zero on the right of t_0.

The proof follows from the integral identity (2).

Theorem 3. *Let the suppositions of Lemma 1 be fulfiled and let moreover $p(t) < -k^2$, $k > 0$ for $t > t_0$. Then every solution u of the differential equation* (1) *with the property*

$$u(t)\,u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)\,u^2(t) < 0, \qquad t \geqq t_0, \tag{4}$$

belongs to the class $L^{\alpha-1}$.

The proof follows from the integral identity (2) as in the preceding cases.

Corollary 1. *Let the hypotheses of Theorem 3 be fulfiled and let u be an oscillatory solution of* (1).

Then it fulfils the condition (4).

Proof. The integral identity (2) for u has the form

$$uu'' - \frac{1}{2}u'^2 + \frac{1}{2}qu^2 = k - \int_{t_0}^{t} \left[pu^{\alpha-1} - \frac{1}{2}q'\right]u^2\,d\tau. \tag{5}$$

It follows from Lemma 1, that a solution of (1) with a double zero is not oscillatory. Therefore the zeros of u are single. Let t_0 be one of the zeros of u. Then the constant k is negative i.e. $k < 0$ and the left side of (5) is negative in every zero of u. But the right side of (5) is increasing and therefore the solution u fulfils the condition (4).

Lemma 2. *Let v_1, v_2 be a fundamental set of solutions of the differential equation*

$$v'' + \frac{1}{4}q(t)\,v = 0. \tag{6}$$

Then v_1^2, v_1v_2, v_2^2 form a fundamental set of solutions of the differential equation

$$u''' + q(t)\,u' + \frac{1}{2}q'(t)\,u = 0. \tag{7}$$

For the proof see [5].

Rewrite the differential equation (1) in the form

$$u''' + q(t)\,u' + \frac{1}{2}\,q'(t)\,u = -\left[p(t)\,u^{\alpha-1} - \frac{1}{2}\,q'(t)\right]u.$$

By the method of variation of constants as in Lemma 2.3 [2], it is easy to verify that

$$(8)\quad u(t) = c_1u_1(t) + c_2u_2(t) + c_3u_3(t) - \int_{t_0}^{t} \frac{p(\tau)\,u^{\alpha-1}(\tau) - \frac{1}{2}\,q'(\tau)}{W(\tau)}\,W(t,\tau)\,u(\tau)\,\mathrm{d}\tau,$$

$a < t_0 < \infty$, where u_1, u_2, u_3 is a fundamental set of solutions of the differential equation (7), $W(t)$ is the wronskian of the solution u_1, u_2, u_3 and

$$W(t,\tau) = \begin{vmatrix} u_1(t), & u_2(t), & u_3(t) \\ u_1(\tau), & u_2(\tau), & u_3(\tau) \\ u_1'(\tau), & u_2'(\tau), & u_3'(\tau) \end{vmatrix},$$

c_1, c_2, c_3 are constants chosen so that the solution u and the function $c_1u_1 + c_2u_2 + c_3u_3$ satisfy at t_0 the same initial conditions. Clearly, for a fixed τ the function $W(t, \tau)$ solves the differential equation (7) and has a double zero at τ. From Lemma 2 it follows that we can choose $u_1 = v_1^2$, $u_2 = v_1v_2$, $u_3 = v_2^2$, where v_1, v_2 is a fundamental set of solutions of (6). Let v_1, v_2 be choosen so that their wronskian is equal 1. Then we can calculate that $W(t) = 2$.

Another short calculation based on (8) yields

$$(9)\quad u(t) = c_1v_1^2(t) + c_2v_1(t)\,v_2(t) + c_3v_2^2(t) - \\ -\frac{1}{2}\int_{t_0}^{t}\left[p(\tau)\,u^{\alpha-1}(\tau) - \frac{1}{2}\,q'(\tau)\right]\begin{vmatrix} v_1(t), & v_2(t) \\ v_1(\tau), & v_2(\tau) \end{vmatrix}^2 u(\tau)\,\mathrm{d}\tau,$$

where $W(t,\tau) = \begin{vmatrix} v_1(t), & v_2(t) \\ v_1(\tau), & v_2(\tau) \end{vmatrix}^2$.

Using (9) we can derive some asymptotic properties of certain solutions of the differential equation (1).

Theorem 4. *Assume that every solution of the differential equation of the second order* (6) *is bounded on* (a, ∞) *and that* $q'(t) \geqq 0$, $p(t) > 0$ *for* $t \in (a, \infty)$ *and* $\int_{t_0}^{\infty} q'(t)\,\mathrm{d}t$ *exist for some* $t_0 > a$. *Then every solution* $u(t)$ *of the differential equation* (1) *with the property* $u(t) \neq 0$ *for* $t \geqq t_0$, *is bounded on* (a, ∞).

Proof. Let $u(t)$ be a solution of the differential equation (1), defined on (a, ∞) with the property $u(t) \neq 0$ for $t \in \langle t_0, \infty)$. Without loss of generality we can suppos $u(t) > 0$ for $t \in \langle t_0, \infty)$. Suppose that the fundamental set of solutions v_1, v_2

of the differential equation (6) is bounded on $\langle t_0, \infty)$ with the constant $k > 0$, i.e. $|v_1(t)| \leqq k$, $|v_2(t)| \leqq k$ for $t \in \langle t_0, \infty)$. Then from the relation (9) it follows for $t_1 \geqq t_0$

$$u(t) + \frac{1}{2} \int_{t_1}^{t} p(\tau) u^\alpha(\tau) \begin{vmatrix} v_1(t), & v_2(t) \\ v_1(\tau), & v_2(\tau) \end{vmatrix}^2 d\tau \leqq k^2(|c_1| + |c_2| + |c_3|) + \\ + \frac{1}{4} \int_{t_1}^{t} q'(\tau) \begin{vmatrix} v_1(t), & v_2(t) \\ v_1(\tau), & v_2(\tau) \end{vmatrix}^2 u(\tau) d\tau.$$

Let $\mu(t) = \max u(\tau)$ for $\tau \in \langle t_1, t\rangle$ and let $t_1 \geqq t_0$ and $M > 0$ be such that $\int_{t_1}^{\infty} q'(\tau) dt < M$ and $4Mk^4 \leqq 1$. Then there is

$$\mu(t) + \frac{1}{2} \int_{t_1}^{t} p(\tau) u^\alpha(\tau) \begin{vmatrix} v_1(t), & v_2(t) \\ v_1(\tau), & v_2(\tau) \end{vmatrix}^2 d\tau \leqq k^2(|c_1| + |c_2| + |c_3|) + \frac{1}{4}\mu(t)$$

and then

$$\frac{3}{4}\mu(t) + \frac{1}{2} \int_{t_1}^{t} p(\tau) u^\alpha(\tau) \begin{vmatrix} v_1(t), & v_2(t) \\ v_1(\tau), & v_2(\tau) \end{vmatrix}^2 d\tau \leqq k^2(|c_1| + |c_2| + |c_3|).$$

Thus the assertion of Theorem 4 follows from the last relation.

Remark 1. If we suppose e.g. $q(t) > 0$, $q'(t) \geqq 0$ and $\int_{t_0}^{\infty} q'(t) dt < M$, $M > 0$, then the solutions of (6) are oscillatory and bounded on $\langle t_0, \infty)$, [6].

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