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ON CORRECTNESS OF THE GENERALIZED BOUNDARY VALUE PROBLEM FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Dedicated to Academician Otakar Borůvka on the occasion of his ninetieth birthday

Abstract. By means of surjectivity theorems in $\mathbb{R}^n$ the correctness of the generalized boundary value problem for ordinary differential systems is investigated. A comparison theorem is proved which gives a necessary and sufficient condition for the correctness of the boundary value problem when its uniqueness is assured.

Key words. Generalized boundary value problem, surjective mapping, $\tau$-correctness, a subordinate functional, the orientation of a functional.

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In the sequel the following theorem on surjectivity in $\mathbb{R}^n$ from [2], [3] will be used. Here it will be given as

Lemma 1. Let $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous map. Then the following statements are true:

(a) If $g$ is injective, then $g$ is a homeomorphism of $\mathbb{R}^n$ onto itself if and only if it satisfies the condition

$$\lim_{|x| \to \infty} |g(x)| = \infty.$$  

(b) If $g$ satisfies (1) and one of the conditions:

Either

there is an $x_0 \in \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n, x \neq x_0$,

(2) $g(x) - x_0 = k(x - x_0)$ implies $k \geq 0$,

or

there is an $x_0 \in \mathbb{R}^n$ such that for each $x \in \mathbb{R}^n, x \neq x_0$,

(181)
(2') \[ g(x) - x_0 = k(x - x_0) \] \ implies k \leq 0,

then \( g \) is surjective.

Similarly as in [1], [2] under the generalized boundary value problem for the
differential system

\[ x' = f(t, x), \quad t \in I, x \in \mathbb{R}^n, \]

and for the given mapping \( F \) of the space \( C(i, \mathbb{R}^n) \) of all continuous vector functions \( x: i \to \mathbb{R}^n \) we understand the problem to find a solution \( x(i) \) of the
system (3) in the interval \( I \) for which \( F(x) \) is a given vector \( r \in \mathbb{R}^n \), that is

\[ F(x) = r. \]

Here and in what follows we suppose that the function \( f \) satisfies local Caratheodory conditions in \( i \times \mathbb{R}^n \) and if \( S \) is the set of all noncontinuable solutions
of the system (3), then

\[ S \cap C(i, \mathbb{R}^n) \neq \emptyset. \]

Let in the space \( C(i, \mathbb{R}^n) \) be a topology \( \tau \) given and let the functional \( F: C(i, \mathbb{R}^n) \to \mathbb{R}^n \) be continuous with respect to this topology.

Further we shall use the following definitions.

**Definition 1.** We shall say that the functional \( F \) is injective with respect to the
system (3) if it is injective on the set \( S \cap C(i, \mathbb{R}^n) \).

The functional \( F \) is surjective with respect to the system (3) if \( F(S \cap C(i, \mathbb{R}^n)) = \mathbb{R}^n. \)

**Definition 2.** The generalized boundary value problem (3), (4) is said to be
\( \tau \)-correct if \( F \) is injective and surjective with respect to the system (3) and the
inverse mapping \( (F|_{S \cap C(i, \mathbb{R}^n)})^{-1} \) of the mapping \( F|_{S \cap C(i, \mathbb{R}^n)} \) is continuous as
a mapping from \( \mathbb{R}^n \) to \( C(i, \mathbb{R}^n) \).

Denote \( x(t, r) \) the solution of the problem (3), (4) (if it exists). Hence \( F(x(t, r)) = r \)
and the \( \tau \)-correctness of the problem (3), (4) means that \( x(t, r) \) continuously depends
on \( r \) with respect to the topology \( \tau \).

Let the functional \( G: C(i, \mathbb{R}^n) \to \mathbb{R}^n \) be continuous with respect to the topology \( \tau \).

**Definition 3.** The functional \( F \) is said to be subordinate to the functional \( G \)
with respect to the differential system (3) if the following statement holds:

If the sequence \( \{G(x_k)\} \) is bounded in \( \mathbb{R}^n \), then the sequence \( \{F(x_k)\} \) is bounded,
too, for each sequence \( \{x_k\} \subset S \cap C(i, \mathbb{R}^n) \).

**Definition 4.** The functional \( G \) is said to have the same (the opposite) orientation
as the functional \( F \) with respect to the system (3) if the following implication
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holds: \( G(x(t)) = kF(x(t)) \) implies \( k \geq 0 \) \( (k \leq 0) \) for each solution \( x(t) \in S \cap C(i, R^n) \) such that \( F(x(t)) \neq 0 \).

The relation to have the same orientation is reflexive and symmetric.

By means of the notions given above we can state

**Theorem 1.** Let the boundary value problem (3), (4) be \( \tau \)-correct and let the functional \( G : C(i, R^n) \to R^n \) be continuous with respect to the topology \( \tau \).

Then the following statements are true:

1. If the functional \( G \) is injective with respect to the system (3), then the boundary value problem (3),

\[ G(x) = r \]

is \( \tau \)-correct if and only if the functional \( F \) is subordinate to the functional \( G \) with respect to the system (3).

2. If the functional \( F \) is subordinate to the functional \( G \) with respect to the system (3) and the functional \( G \) has the same (the opposite) orientation as the functional \( F \), then the functional \( G \) is surjective with respect to the system (3).

**Proof.** Define the mapping \( H : R^n \to C(i, R^n) \) by the relation

\[ H(r) = x(t, r) \]

for each \( r \in R^n \).

Since the boundary value problem (3), (4) is \( \tau \)-correct, the mapping \( H : R^n \to \to C(i, R^n) \) is a homeomorphism of \( R^n \) onto \( S \cap C(i, R^n) \). Hence the mapping

\[ g = GH \]

from \( R^n \) into \( R^n \) is continuous and if the functional \( G \) is injective with respect to the system (3), then \( g \) is injective, too. We apply Lemma 1. The condition (1) means that the inverse image of each bounded subset in \( R^n \) under the mapping \( g \) is bounded in \( R^n \).

1. Suppose that the functional \( G \) is injective with respect to the system (3) and that the functional \( F \) is subordinated to the functional \( G \) with respect to the system (3). Let \( \{r_k\} \) be an arbitrary sequence of points in \( R^n \) and \( x_k = x(t, r_k) \) the corresponding sequence of solutions of the system (3) in the interval \( i \), i.e. \( F(x_k) = r_k \), \( k = 1, 2, ... \) If the sequence \( g(r_k) = G(x_k) \) is bounded, then the sequence \( \{F(x_k)\} = \{r_k\} \) is bounded, too. But this means that the condition (1) is fulfilled and thus, by Lemma 1, \( g \) is a homeomorphism of the space \( R^n \) onto itself. Then \( G = gH^{-1} \) is a homeomorphism of the space \( S \cap C(i, R^n) \) onto \( R^n \) and hence the problem (3), (6) is \( \tau \)-correct.

If, on the other hand, the problem (3), (6) is \( \tau \) correct, then \( G \) is a homeomorphic mapping of the space \( S \cap C(i, R^n) \) onto \( R^n \) and \( g \), determined by (8), is a homeomorphism of \( R^n \) onto itself. By Lemma 1 the condition (1) is satisfied. Let \( \{G(x_k)\} \) be a bounded sequence. In view of the relation \( G(x_k) = g(r_k) \) and (1) we get that
the sequence \( \{r_k\} = \{F(x_k)\} \) is also bounded. Hence the functional \( F \) is subordinate to the functional \( G \) with respect to (3).

2. If the functional \( F \) is subordinate to the functional \( G \) with respect to the system (3) and the sequence \( \{g(r_k)\} = \{G(x_k)\} \) is bounded, then \( \{F(x_k)\} = \{r_k\} \) is bounded, too, which means that the condition (1) is fulfilled. The mapping \( g \) satisfies the condition (2) with the point \( x_0 = 0 \) if the equality \( G(x(t, r)) = kr = kF(x(t, r)) \) implies \( k \geq 0 \) for each \( r \neq 0, r \in \mathbb{R}^n \). But this means that the functionals \( G \) and \( F \) have the same orientation. Similarly the condition (2') with \( x_0 = 0 \) is fulfilled if \( G \) and \( F \) have the opposite orientation.

In applications of Theorem 1 the initial value problem is often compared with the given boundary value problem. As the existence and the uniqueness of the solution to the initial value problem implies the \( \tau_0 \)-correctness of this problem where the topology \( \tau_0 \) is the topology of uniform convergence (of locally uniform convergence) on \( i \) when \( i \) is a compact (a noncompact) interval we get the following

**Corollary 1.** (Compare with [2], p. 169). Let there exist a point \( t_0 \in i \) such that for each vector \( x_0 \in \mathbb{R}^n \) there exists a unique solution \( x(t) \) on \( i \) to the initial value problem (3),

\[
(9) \quad x(t_0) = x_0
\]

and let the functional \( G : C(i, \mathbb{R}^n) \to \mathbb{R}^n \) be continuous with respect to the topology \( \tau_0 \).

Then the following statements hold:

1. If the boundary value problem (3), (6) has at most one solution for each vector \( r \in \mathbb{R}^n \), then this problem is \( \tau_0 \)-correct if and only if the following implication holds:

\[
(10) \quad \text{If } \{x_k\} \text{ is a sequence of solutions of (3) on the interval } i \text{ such that } \{G(x_k)\} \text{ is bounded, then } \{x_k(t_0)\} \text{ is bounded.}
\]

2. If the implication (10) as well as the implication:

\[
(11) \quad \text{If } G(x) = kx(t_0), \text{ then } k \geq 0 \ (k \leq 0) \text{ for each solution } x(t) \text{ of (3) on } i \text{ such that } x(t_0) \neq 0,
\]

hold, then the boundary value problem (3), (6) has a solution for each \( r \in \mathbb{R}^n \).

In the paper [3] two boundary value problems have been compared.

**REFERENCES**


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