Archivum Mathematicum

Alexander Haščák

On the relationship between the initial and the multipoint boundary value problems for n-th order linear differential equations with delay

Archivum Mathematicum, Vol. 26 (1990), No. 4, 207--213

Persistent URL: http://dml.cz/dmlcz/107390

ARCHIVUM MATHEMATICUM (BRNO)
Vol. 26, No. 4 (1990), 207–214

ON THE RELATIONSHIP BETWEEN THE INITIAL AND THE MULTIPOINT BOUNDARY VALUE PROBLEMS FOR n-TH ORDER LINEAR DIFFERENTIAL EQUATIONS WITH DELAY

ALEXANDER HAŠČÁK

(Received January 5, 1987)

Abstract. In the paper it is shown that for each solution $\varphi(t; \tau_0, \Phi)$ of initial value problem for linear differential equation with delay there are the solutions $\varphi_v = \varphi(t; \bar{\tau}_v, \bar{\beta}_v)$, $v = 1, 2, \ldots$ of boundary value problems such that

$$\lim_{v \to \infty} \varphi^{(k)}(t; \bar{\tau}_v, \bar{\beta}_v) = \varphi^{(k)}(t; \tau_0, \Phi), \qquad k = 0, 1, \ldots, n-1$$

uniformly in the considered interval.

Key words. Linear differential equation with delay, initial value problem for differential equation with delay, multipoint boundary value problem for linear differential equation with delay.

MS Classification. 34 K 10.

Let us consider the following n-th order linear differential equation with delay

$$(E_n) \qquad x^{(n)}(t) + \sum_{i=1}^{n} a_i(t)\, x^{(n-i)}(t) + \sum_{i=1}^{n} b_i(t)\, x^{(n-i)}(t - \Delta(t)) = 0$$

having continuous coefficients $a_i(t)$, $b_i(t)$, $i = 1, \ldots, n$ and a continuous delay $\Delta(t) \geqq 0$ on an interval $I = \langle a, b \rangle$. The underlying initial value problem for the equation (E_n) is defined as follows:

Let $\tau_0 \in \langle a, b)$ and let on the initial set

$$E_{\tau_0} = \{t - \Delta(t) : t - \Delta(t) < \tau_0,\ t \in \langle \tau_0, b \rangle\} \cup \{\tau_0\}$$

bounded and continuous vector function

$$\Phi(t) = (\Phi_0(t), \Phi_1(t), \ldots, \Phi_{n-1}(t))$$

be given. The problem is to find the solution of the equation (E_n) which satisfies the conditions

$$(IV)\qquad \begin{aligned} x^{(k)}(\tau_0) &= \Phi_k(\tau_0), \qquad k = 0, 1, \dots, n-1, \\ x^{(k)}(t - \Delta(t)) &= \Phi_k(t - \Delta(t)), \qquad \text{if} \qquad t - \Delta(t) < \tau_0. \end{aligned}$$

Theorem 1. *Under the above assumptions the initial value problem* (E_n), (IV) *has exactly one solution* $\varphi(t; \tau_0, \Phi)$ *which is defined on the interval* $\langle \tau_0, b \rangle$.

Definition 1. A vector function Φ is called admissible if it is continuous and bounded on its domain of definition.

In [1] is considered the following multipoint boundary value problem for (E_n):

Let $\tau_0 \in \langle a, b)$,

$$(\tau)\qquad \begin{aligned} &\tau_1, \tau_2, \dots, \tau_m \in \langle \tau_0, b), \qquad \tau_1 \leqq \tau_2 \leqq \dots \leqq \tau_m (m \leqq n), \\ &r_1, r_2, \dots, r_m \in N, \qquad r_1 + r_2 + \dots + r_m = n \end{aligned}$$

and let

$$\bar{\beta})\qquad \beta_1^{(1)}, \dots, \beta_1^{(r_1)}, \beta_2^{(1)}, \dots, \beta_m^{(r_m)} \in R.$$

The problem is to find the solution of the equation (E_n) which satisfies the conditions

$$(BV)\qquad x^{(v_i - 1)}(\tau_i) = \beta_i^{(v_i)}, \qquad v_i = 1, \dots, r_i; \quad i = 1, \dots, m.$$

Further, let an admissible vector function $\Phi(t) = (\Phi_0(t), \Phi_1(t), \dots, \Phi_{n-1}(t))$ defined on E_{τ_0} be given. By $H_\Phi^{\tau_0}$ we shall denote the following set of functions defined on E_{τ_0}

$$H_\Phi^{\tau_0} = \{(\Phi_0(t) + c_0, \Phi_1(t) + c_1, \dots, \Phi_{n-1}(t) + c_{n-1}) : c_i \in R, i = 0, 1, \dots, n-1\}.$$

For the formulation of the existence and uniqueness theorem for boundary value problem (E_n), (BV) the function $\chi(\varrho)$ is useful: let for the coefficients $a_i(t)$, $b_i(t)$, $i = 1, \dots, n$ the inequalities

$$(1)\qquad |a_i(t)| \leqq A_i, \qquad |b_i(t)| \leqq B_i, \qquad t \in \langle a, b \rangle, \qquad i = 1, \dots, n$$

hold. Then $\chi(\varrho)$ is defined by the formula

$$\chi(\varrho) = \sum_{i=1}^{n} \frac{A_i + B_i}{i \left[\frac{i-1}{2}\right]! \left[\frac{i}{2}\right]!} \varrho^i.$$

Theorem 2. (A. Haščák [2].) *Let* $\chi(b - a) < 1$ *and* $\tau_0 \in \langle a, b)$. *Then for each admissible function* $\Phi(t)$ *defined on* E_{τ_0} *there is a unique* $\psi \in H_\Phi^{\tau_0}$ *such that the solution* $\varphi(t; \tau_0, \psi)$ *satisfies the boundary condition* (E_n), (BV). *This solution will be denoted as* $\varphi(t; \tau, \bar{\beta})$.

The purpose of this note is to show a relation between the initial value problem (E_n), (IV) and boundary value problem (E_n), (BV).

Now we shall introduce notations, notions and preliminary remarks which will be needed in the sequel.

Let a function $f(t)$ in the interval $\langle a, b\rangle$ be given. Consider the points

$$a < \tau_1 < \tau_2 < \ldots < \tau_n < b.$$

Denote

$$\beta_i = f(\tau_i), \qquad i = 1, \ldots, n.$$

By difference quotient of the n-th order we shall understand

$$D^n(\tau_1, \ldots, \tau_n; \beta_1, \ldots, \beta_n) = [\tau_1, \ldots, \tau_n] =$$
$$= \frac{\beta_1}{(\tau_1 - \tau_2)(\tau_1 - \tau_3)\ldots(\tau_1 - \tau_n)} + \frac{\beta_2}{(\tau_2 - \tau_1)(\tau_2 - \tau_3)\ldots(\tau_2 - \tau_n)} +$$
$$+ \ldots + \frac{\beta_n}{(\tau_n - \tau_1)(\tau_n - \tau_2)\ldots(\tau_n - \tau_{n-1})}$$

and specially for $n = 1$

$$D^1(\tau_1; \beta_1) = [\tau_1] = \beta_1 \qquad \text{(see [3] p. 17).}$$

If the function f has continuous derivatives to the n-th order (including the n-th order) in $\langle a, b\rangle$, then there are numbers ξ_k, $k = 0, \ldots, n-1$, such that

$$\tau_1 < \xi_k < \tau_{k+1}$$

and

$$D^{k+1}(\tau_1, \ldots, \tau_{k+1}; \beta_1, \ldots, \beta_{k+1}) = \frac{f^{(k)}(\xi_k)}{k!}, \qquad k = 0, \ldots, n-1 \tag{2}$$

holds.

It turns out, that in this case

$$\lim_{\substack{\tau_i \to \tau_0 \\ i = 1, \ldots, k+1}} D^{k+1}(\tau_1, \ldots, \tau_{k+1}; \beta_1, \ldots, \beta_{k+1}) = \tag{3}$$
$$= \lim_{\xi_k \to s_0} \frac{f^{(k)}(\xi_k)}{k!} = \frac{f^{(k)} \tau_0)}{k!}, \qquad k = 0, 1, \ldots, n-1.$$

Now we shall formulate a relation between initial and boundary value problem:

Theorem 3. *Let*

$$\chi(b - a) < 1, \tag{4}$$

$\tau_0 \in \langle a, b)$ *and let an admissible function* $\Phi(t)$ *defined on* E_{τ_0} *be given.*

Let h fulfils the inequalities

$$0 < h < \min\left(b - \tau_0, 1, \frac{1}{1 + L}\right), \tag{5}$$

where

$$L = \max(A_1, \ldots, A_n, B_1, \ldots, B_n).$$

Let the boundary conditions

$$(\tau_v, \bar{\beta}_v) \qquad \tau_{v1}, \tau_{v2}, \ldots, \tau_{vn}; \beta_{v1}, \beta_{v2}, \ldots, \beta_{vn}, \qquad v = 1, 2, \ldots$$

be such that

$$\begin{gathered} \tau_{v1} < \tau_{v1} < \ldots < \tau_{vn}, \qquad v = 1, 2, \ldots, \\ 0 < \tau_{vi} - \tau_0 < h, \qquad i = 1, \ldots, n; v = 1, 2, \ldots, \\ \lim_{v \to \infty} \tau_{vi} = \tau_0, \qquad i = 1, \ldots, n \end{gathered} \tag{6}$$

and

$$\lim_{v \to \infty} D^{k+1}(\tau_{v1}, \ldots, \tau_{vk+1}; \beta_{v1}, \ldots, \beta_{vk+1}) = \frac{\Phi_k(\tau_0)}{k!}, \qquad k = 0, 1, \ldots, n - 1. \tag{7}$$

Then the sequence $\varphi(t; \tau_v, \bar{\beta}_v)$, $v = 1, 2, \ldots$ of solutions of the boundary value problem (E_n), (BV) and the sequences $\varphi^{(k)}(t; \tau_v, \bar{\beta}_v)$, $(k = 1, \ldots, n - 1)$ $v = 1, 2, \ldots$ of theirs derivatives converge uniformly to the solution $\varphi(t; \tau_0, \Phi)$ of the initial value problem (E_n), (IV) resp. to its derivatives $\varphi^{(k)}(t; \tau_0, \Phi)$, $k = 1, \ldots, n - 1$ on $\langle \tau_0, b \rangle$ as $v \to \infty$.

Proof. From (7) we conclude that

$$D^{k+1}(\tau_{v1}, \ldots, \tau_{vk+1}; \beta_{v1}, \ldots, \beta_{vk+1}), \qquad k = 0, \ldots, n - 1; v = 1, 2, \ldots$$

are bounded i.e. there is a positive number M such that

$$\begin{gathered} | n!\, D^{k+1}(\tau_{v1}, \ldots; \tau_{vk+1}; \beta_{v1}, \ldots, \beta_{vk+1}) | \leqq M, \\ k = 0, \ldots, n - 1; v = 1, 2, \ldots \end{gathered} \tag{8}$$

By (2) there are the numbers

$$\xi_{vk}(\tau_{v1}, \ldots, \tau_{vk+1}; \beta_{v1}, \ldots, \beta_{vk+1}) \in (\tau_{v1}, \tau_{vn}), \qquad v = 1, 2, \ldots, \tag{9}$$

such that

$$\begin{gathered} D^{k+1}(\tau_{v1}, \ldots, \tau_{vk+1}; \beta_{v1}, \ldots, \beta_{vk+1}) = \frac{\varphi^{(k)}(\xi_{vk}; \tau_v, \beta_v)}{k!}, \\ k = 0, \ldots, n - 1; v = 1, 2, \ldots \end{gathered} \tag{10}$$

Thus we have

$$| \varphi^{(k)}(\tau_0; \bar{\tau}_v, \bar{\beta}_v) - \Phi_k(\tau_0) | \leqq | \varphi^{(k)}(\tau_0; \bar{\tau}_v, \bar{\beta}_v) - \varphi^{(k)}(\xi_{vk}; \bar{\tau}_v, \bar{\beta}_v) | +$$
$$+ | k!\, D^{k+1}(\tau_{v1}, \dots, \tau_{vk+1}; \beta_{v1}, \dots, \beta_{vk+1}) - \Phi_k(\tau) |,$$
$$k = 0, \dots, n-1; v = 1, 2, \dots$$

from where by Mean Value Theorem we get

$$(11) \qquad | \varphi^{(k)}(\tau_0; \bar{\tau}_v, \bar{\beta}_v) - \Phi_k(\tau_0) | \leqq (\tau_{vn} - \tau_0) \max_{t \in \langle \tau_0, \tau_0 + h \rangle} | \varphi^{(k+1)}(t; \bar{\tau}_v, \bar{\beta}_v) | +$$
$$+ | k!\, D^{k+1}(\tau_{v1}, \dots, \tau_{vk+1}; \beta_{v1}, \dots, \beta_{vk+1}) - \Phi_k(\tau_0) |,$$
$$k = 0, \dots, n-1; v = 1, 2, \dots$$

Further, by Theorem 2 for each $(\bar{\tau}_v, \bar{\beta}_v)$, $v = 1, 2, \dots$ there is unique function $\psi_v \in H_\Phi^{\tau_0}$, $\psi_v = (\psi_{v0}, \dots, \psi_{vn-1})$ such that $\varphi(t; \tau_0, \psi_v) = \varphi(t; \bar{\tau}_v, \bar{\beta}_v)$, $t \in \langle \tau_0, b \rangle$, $v = 1, 2, \dots$ i.e. there are constants c_{vk}, $k = 0, 1, \dots, n-1$; $v = 1, 2, \dots$ such that

$$(12) \qquad \psi_{vk}(t) = \Phi_k(t) + c_{vk}, \; t \in E_{\tau_0}, \qquad k = 0, 1, \dots, n-1; v = 1, 2, \dots$$

Thus the equality

$$(13) \qquad \psi_{vk}(\tau_0) = \Phi_k(\tau_0) + c_{vk}, \qquad k = 0, 1, \dots, n-1; v = 1, 2, \dots$$

holds. By (12) and (13) we have

$$\psi_{vk}(t) = \Phi_k(t) + (\varphi^{(k)}(\tau_0; \bar{\tau}_v, \bar{\beta}_v) - \Phi_k(\tau_0)), \qquad t \in E_{\tau_0},$$
$$k = 0, 1, \dots, n-1; v = 1, 2, \dots$$

from where by (11) we get

$$(14) \qquad | \psi_{vk}(t) - \Phi(t) | \leqq (\tau_{vn} - \tau_0) \max_{t \in \langle \tau_0, \tau_0 + h \rangle} | \varphi^{(k+1)}(t; \bar{\tau}_v, \bar{\beta}_v) | +$$
$$+ | k!\, D^{k+1}(\tau_{v1}, \dots, \tau_{vk+1}; \beta_{v1}, \dots, \beta_{vk+1}) - \Phi_k(\tau_0) |,$$
$$t \in E_{\tau_0}, \qquad k = 0, 1, \dots, n-1.$$

To show that $\psi_{vk}(t)$, $k = 0, 1, \dots, n-1$; $v = 1, 2, \dots$ uniformly converge to $\Phi_k(t)$ on E_{τ_0} as $v \to \infty$ it suffices to show (because of (7), (8) and (14)) that there is a constant C which is not dependent on $\bar{\tau}_v, \bar{\beta}_v$ such that

$$(15) \qquad p_i(\bar{\tau}_v, \bar{\beta}_v) = \max_{t \in \langle \tau_0, \tau_0 + h \rangle} | \varphi^{(i)}(t; \bar{\tau}_v, \bar{\beta}_v) | \leqq C, \qquad i = 1, \dots, n, v = 1, 2, \dots$$

We have

$$| \varphi^{(k)}(t; \bar{\tau}_v, \bar{\beta}_v) | \leqq | \varphi^{(k)}(\xi_{vk}; \bar{\tau}_v, \bar{\beta}_v) | + | \varphi^{(k)}(t; \bar{\tau}_v, \bar{\beta}_v) - \varphi^{(k)}(\xi_{vk}; \bar{\tau}_v, \bar{\beta}_v) |,$$
$$k = 0, 1, \dots, n-1, v = 1, 2, \dots$$

From this by (8), (10) and (15) we get

$$(16) \qquad p_k(\bar{\tau}_v, \bar{\beta}_v) \leqq M + hp_{k+1}(\bar{\tau}_v, \bar{\beta}_v), \qquad k = 0, 1, \ldots, n-1, v = 1, 2, \ldots$$

From (16) and (5) we get

$$(17) \qquad \sum_{k=0}^{n-1} p_k(\bar{\tau}_v, \beta_v) \leqq nM + h \sum_{i=1}^{n} p_i(\bar{\tau}_v, \beta_v)$$

and

$$(18) \qquad p_k(\bar{\tau}_v, \bar{\beta}_v) \leqq M(h_0 + \cdots + h^{n-k-1}) + h^{n-k} p_n(\bar{\tau}_v, \bar{\beta}_v) \leqq nM + hp_n(\bar{\tau}_v, \bar{\beta}_v),$$
$$k = 0, 1, \ldots, n-1, v = 1, 2, \ldots$$

On the other hand $\varphi(t; \bar{\tau}_v, \bar{\beta}_v)$ is a solution of (E_n). Thus (by (1) and (5)) the inequalities

$$(19) \qquad |\varphi^{(n)}(t; \bar{\tau}_v, \beta_v)| \leqq L \sum_{k=0}^{n-1} |\varphi^{(k)}(t; \bar{\tau}_v, \beta_v)|, \qquad t \in \langle \tau_0, b \rangle, v = 1, 2, \ldots,$$

$$p_n(\tau_v, \beta_v) \leqq L \sum_{k=0}^{n-1} p_k(\tau_v, \beta_v), \qquad v = 1, 2, \ldots$$

hold.

Now, from (17) and (19) we get

$$\sum_{k=0}^{n-1} p_k(\bar{\tau}_v, \beta_v) \leqq nM + h \sum_{k=0}^{n-1} p_k(\bar{\tau}_v, \beta_v) + hp_n(\bar{\tau}_v, \beta_v) \leqq$$
$$\leqq nM + h(1 + L) \sum_{k=0}^{n-1} p_k(\bar{\tau}_v, \beta_v),$$

from where

$$(1 - h(1 + L)) \sum_{k=0}^{n-1} p_k(\bar{\tau}_v, \beta_v) \leqq nM.$$

Since (5) holds, we have

$$(20) \qquad \sum_{k=0}^{n-1} p_k(\bar{\tau}_v, \beta_v) \leqq \frac{nM}{1 - h(1 + L)}.$$

At last, from (18), (20) and (19) we conclude

$$p_j(\bar{\tau}_v, \beta_v) \leqq \frac{nM}{1 - h(1 + L)}, \qquad j = 0, 1, \ldots, n, v = 1, 2, \ldots$$

Thus (15) is valid with the constant $C = \dfrac{nM}{1 - h(1 + L)}$ and thus $\psi_{vk}(t)$, $k = 0, 1, \ldots, n-1$, $v = 1, 2, \ldots$ uniformly converge to $\Phi_k(t)$ on E_{τ_0} as $v \to \infty$. From this fact by theorem on continuous dependence of solutions on initial conditions we have that $\varphi^{(k)}(t; \bar{\tau}_v, \bar{\beta}_v)$, $k = 0, 1, \ldots, n-1$ uniformly converge to

$$\varphi^{(k)}(t; \tau_0, \Phi) \qquad \text{on} \qquad \langle \tau_0, b \rangle.$$

The proof of theorem is complete.

REFERENCES

[1] A. Haščák, *Disconjugacy and Multipoint Boundary Value Problems for Linear Differential Equations with Delay*. Czech. Math. J., 39 (14) 1989, 70–77.

[2] A. Haščák, *Tests for Disconjugacy and Strict Disconjugacy of Linear Differential Equations with Delays*. Czech. Math. J., 39 (114) 1989, 225–231.

[3] А. О. Гельфонд, *Исчисление конечных разностей*. Гос. Изд. техническо-теоретической литературы. Москва 1952 Ленинград.

[4] С. Б. Норкин, *Дифференциальные уравнения второго порядка с запаздывающим аргументом*. Наука, Москва 1965.

Alexander Haščák
Katedra matematickej analýzy MFF UK
Mlynská dolina
842 15 Bratislava