Kurt Kreith

Initial value parameters corresponding to positive Green's functions

Archivum Mathematicum, Vol. 27 (1991), No. 3-4, 161--166

Persistent URL: http://dml.cz/dmlcz/107417

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 27 (1991), 161 – 166

INITIAL VALUE PARAMETERS CORRESPONDING TO POSITIVE GREEN'S FUNCTIONS

KURT KREITH

(Received May 15, 1989)

ABSTRACT. Green's functions corresponding to certain fourth order boundary value problems are represented as trajectories in E^2. By adapting previously established conjugacy criteria to such trajectories, the positivity of the corresponding Green's functions is established

1. INTRODUCTION

The use of second order systems to study selfadjoint fourth order differential equations goes back to Whyburn [5] and is implicit in much of the classic paper of Leighton and Nehari [4]. The fact that nonselfadjoint fourth order equations also allow such a systems representation [2] enables one to generalize a variety results regarding conjugate points to more general situations.

In this paper we use systems techniques to establish criteria for the positivity of the Green's function associated with the general real fourth order linear equation

$$(P_2(t)v'' - Q_2(t)v')'' - (P_1(t)v' - Q_1(t)v)' + P_0(t)v = 0 \tag{1.1}$$

and the conjugate point boundary condition

$$v(0) = v'(0) = 0 = v(\eta) = v'(\eta). \tag{1.2}$$

As in [2], it is assumed that the transformation $u(t) = v(t)\exp[-\frac{1}{2}\int_0^t q_2/p_2]$ has been used to reduce (1.1) to the form

$$(p_2(t)u'')'' - (p_1(t)u' - q_1(t)u)' + p_0(t)u = 0. \tag{1.1$'$}$$

Given (1.1)$'$, the transformations

$$y = u;\ x = p_2u'' - \frac{1}{2}[p_1 - \int_0^t q_1]u \tag{1.3)(i}$$

1991 *Mathematics Subject Classification*: 34C10.
Key words and phrases: Green's function, conjugate point, impulse, trajectory.

and

$$(1.3)(ii)\qquad \begin{aligned} A &= \frac{1}{2p_2}[p_1 - \int_o^t q_1]; \quad B = 1/p_2 \\ C &= \frac{1}{4p_2}\left[p_1^2 - \left(\int_0^t q_1\right)^2\right] - \frac{p_1'' - q_1'}{2} - p_0; \; D = \frac{1}{2p_2}[p_1 + \int_0^t q_1] \end{aligned}$$

lead to the second order system

$$(1.4)\qquad \begin{aligned} y'' &= A(t)y + B(t)x \\ x'' &= C(t)y + D(t)x \end{aligned}$$

in which the non-selfadjointness of (1.1) is reflected by the fact that $A \neq D$.

For the purpose of this paper it is assumed that the coefficients of (1.4) are real, continuous, and that $B(t) > 0$ on an interval $[0, \eta_1]$, where η_1 denotes the first conjugate point of 0 relative to (1.1)′ - i.e. η_1 is the smallest $t > 0$ for which (1.2) is realized by a nontrivial solution of (1.1)′. Specific conditions for the existence of such η_1 have been given by numerous authors (see for example [2]).

2. A CLOSED FORM EXAMPLE

Before turning to the general case of (1.1) it is of interest to illustrate the techniques to be applied in a simpler context. To construct the Green's function $G(t, \tau)$ for

$$(2.1)\qquad \begin{aligned} y^{(iv)} &= f(t) \\ y(0) = y'(0) &= 0 = y(\eta) = y'(\eta) \end{aligned}$$

for some fixed η, the fact that $G_{tttt} = 0$ for $t \neq \tau$ can be used to write

$$(2.2)\qquad G(t, \tau) = \frac{1}{2!}at^2 + \frac{1}{3!}bt^3; \quad 0 \leq t \leq \tau.$$

Here a and b are "initial value parameters" corresponding to initial data

$$y''(0) = a; \quad y'''(0) = b.$$

The usual continuity and "jump" conditions at $t = \tau$ now imply that

$$(2.3)\qquad G(t, \tau) = \frac{1}{2!}at^2 + \frac{1}{3!}bt^3 + \frac{1}{3!}(t - \tau)^3; \; \tau < t \leq \eta.$$

so that the Green's function can be determined from the terminal value conditions $y(\eta) = y'(\eta) = 0$ - i.e. by solving

$$(2.4)\qquad \begin{aligned} G(\eta, \tau) &= \frac{1}{2!}a\eta^2 + \frac{1}{3!}b\eta^3 + \frac{1}{3!}(\eta - \tau)^3 = 0 \\ G_t(\eta, \tau) &= a\eta + \frac{1}{2!}b\eta^2 + \frac{1}{2!}(\eta - \tau)^2 = 0. \end{aligned}$$

While it is tempting to solve (2.4) for a and b in terms of η and τ it will be instructive, for what follows, to proceed differently. Regarding a and τ as given, we shall instead solve for b and η in terms of a and τ.

When viewed in terms of the associated second order system

$$y'' = x; \quad x'' = 0; \quad (t \neq \tau), \tag{2.5}$$

this latter problem has the following dynamical interpretation in the (x, y)-plane: Given $y(0) = y'(0) = 0$, $x(0) = a > 0$ and a fixed positive value for τ, we seek a "critical initial velocity" $b = x'(0) < 0$ such that the trajectory $(x(t), y(t))$, determined by (2.5) and $x'(\tau^+) - x'(\tau^-) = 1$, will be tangent to the x-axis at some future time $\tilde{t}(a, \tau)$. In order to assure that this tangency occurs at $\tilde{t} = \eta$ for all > 0, one seeks to choose $a(\tau) > 0$ so that $\tilde{t}(a(\tau), \tau) = \eta$.

In the simple case at hand one can readily solve (2.4) to obtain

$$\tilde{t}(a, \tau) = \frac{\tau^{3/2}}{\sqrt{\tau} - \sqrt{a}}. \tag{2.6}$$

Here

$$\lim_{a \to 0} \tilde{t}(a, \tau) = \tau; \quad \lim_{a \to \tau} \tilde{t}(a, \tau) = \infty,$$

so that for $\tau < \eta < \infty$, the existence of such an $a(\tau)$ readily follows from the continuity of $\tilde{t}$ as a function of a. Since one can also verify that the corresponding trajectory $(x(t), y(t))$ satisfies $y(t) > 0$ for $0 < t < \eta$, this construction establishes the positivity of the associated Green's function $G(t, \tau)$ (which can, in this simple case, also be computed explicitly).

The example just considered differs from Theorem 3.4 below in that $\eta_1 = \infty$. In what follows we shall assume $\eta_1 < \infty$ and use similar techniques to conclude that for $0 < \eta < \eta_1$, the corresponding Green's function is positive in $(0, \eta) \times (0, \eta)$.

3. Positivity Criteria

In order to extend the technique of §2 to the general fourth order equation, we shall assume that (1.1) has been transformed into the system form (1.4) and that the four quadrants of the (x, y) plane are labelled I, II, III and IV, respectively. Identifying solutions of (1.3) with their trajectories $\Gamma = (x(t), y(t))$ in the (x, y)-plane, we also assume that the following hypotheses of [2] are satisfied:

(A) If for some $t_0 \geq 0$ the quantities $y(t_0)$, $y'(t_0)$, $x(t_0)$ and $x'(t_0)$ are all nonnegative (but not all zero), then $\Gamma(t)$ remains in I for all $t > t_0$.

(B) No trajectory for (1.3) can remain in II for arbitrary large values of t.

(C) No trajectory in I is asymptotic to the x-axis or y-axis, nor can it have a limit point in the closure of I.

(D) A trajectory for (1.3) cannot go directly from II to I to II.

As shown in [2], these hypotheses lead to the existence of a finite conjugate point relative to (1.1)′ which is realized by a positive solution - i.e. there exists a trajectory $\Gamma(a, b)$ satisfying

$$y(0) = y'(0) = 0; \quad x(0) = a; \quad x'(0) = b \tag{3.1}$$

such that Γ is tangent to the x-axis at $t = \eta_1 < \infty$ and satisfies $y(t) > 0$ for $0 < t < \eta_1$.

In Theorem 3.4 below it will be shown that for $\eta < \eta_1$, the Green's function corresponding to (1.1)′ and (1.2) satifies $G(t,\tau) > 0$ in $(0,\eta)\times(0,\eta)$. In this regard it will be convenient to denote by $\tilde{\Gamma}(a.b)$ the trajectory determined by (1.4), (3.1) and the "jump condition"

$$x'(\tau^+) - x'(\tau^-) = 1 \ (\text{or } x''(\tau) = \delta(t-\tau)) \tag{3.2}$$

which, in a dynamical context, corresponds to "a unit impulse in the positive x-direction at $t = \tau$." The desired positivity properties will follow from the fact that $\tilde{\Gamma}(a,b)$ has qualitative properties analogous to those of $\Gamma(a,b)$ [3].

3.1 Lemma. *Given $\tau \in (0,\eta_1)$ and $a > 0$, there exists $\tilde{b}(a,\tau)$ such that $\tilde{\Gamma}(a,\tilde{b})$ is tangent to the x-axis at some $\tilde{\eta} > 0$ and $y(t) > 0$ in $(0,\tilde{\eta})$.*

Proof. Since $y(0) = y'(0) = 0$ and $x(0) = a > 0$ is fixed, we can choose $x'(0) = b$ sufficiently negative so that $\tilde{\Gamma}(a,b)$ exits the half-plane $y > 0$ at an orbitrarily small value of t. On the other hand, for $b > 0$ it follows from (A) that $\tilde{\Gamma}(a,b)$ will remain in I for all t. Therefore, as in [3], there exists

$$\tilde{b} = \inf\{b \cdot \Gamma(a,b) \text{ remains in I } \cup \text{ II for all } t > 0\}$$

such that $\tilde{\Gamma}(a,\tilde{b})$ is tangent to the x-axis some $\tilde{\eta} > 0$ and satisfies $y(t) > 0$ for $0 < t < \tilde{\eta}$.

It remains to show that, given $\eta < \eta_1$, one can choose $a(\tau,\eta)$ so that $\tilde{\eta}(a,\tau) = \eta$. This will be done by a continuity argument analogous to that of §2. □

3.2 Lemma. *With $\tilde{\eta}(a,\tau)$ defined as in Lemma 3.1,*

$$\lim_{a\to 0} \tilde{\eta}(a,\tau) = \tau. \tag{3.3}$$

Proof. By (A), the trajectory Γ_0 of (1.3) satisfying $y(\tau) = y'(\tau) = x(\tau) = 0$; $x'(t) = 1$ will stay in I for all $t > \tau$. Writing $\tilde{\Gamma}(a,\tilde{b}(a,\tau)) = (\tilde{x}_a(t), \tilde{y}_a(t))$ and noting that $\lim_{a\to 0}\tilde{b}(a,\tau) = 0$, it follows that $\tilde{y}_a(\tau)$, $\tilde{y}'_a(\tau)$, $\tilde{x}_a(\tau)$ and $\tilde{x}'_a(\tau^-)$ all tend to zero as $a \to 0$, while $\lim_{a\to 0}\tilde{x}'_a(\tau^+) = 1$. Therefore, as $a \to 0$, $\tilde{\Gamma}(a,\tilde{b}(a,\tau))$ approaches the trajecory Γ_0 which lies in I, and if $\tilde{t}_a$ is a zero of $\tilde{y}_a(t)$, then $\lim_{a\to 0}\tilde{t}_a = 0$. Since $\tilde{\eta}(a,\tau)$ <u>is</u> such a zero (indeed a double zero) of $\tilde{y}_a(t)$, (3.3) follows. □

3.3 Lemma. *With $\tilde{\eta}(a,\tau)$ defined as in Lemma 3.1, let η_1 be the first conjugate point of 0 with respect to (1.1)′. Then*

$$\lim_{a\to\infty} \tilde{\eta}(a,\tau) = \eta_1.$$

Proof. If $\Gamma(a,b)$ is a trajectory which realizes the conjugate point η_1 then, for any $c>0$, so does $\Gamma(ca,cb)$. Regarding a small impulse at $t=\tau$ as a perturbation of $\Gamma(a,b)$, we use the construction of Lemma 3.1 to choose ε sufficiently small so that

$$\begin{aligned} y'' &= A(t)y + B(t)x \\ x'' &= C(t)y + D(t)x + \varepsilon\delta(t-\tau) \end{aligned}$$

has a trajectory $\tilde{\Gamma}_\varepsilon(a,\tilde{b})$ with double zeros at $t=0$ and $t=\eta_\varepsilon$ and for which $\lim_{\varepsilon\to 0}\eta_\varepsilon = \eta_1$. It then follows that $(\frac{1}{\varepsilon}x(t), \frac{1}{\varepsilon}y(t))$ corresponds to a Green's function trajectory $\Gamma(\frac{1}{\varepsilon}a, \frac{1}{\varepsilon}b)$ with the same double zeros - i.e. at $t=0$ and the other arbitrarily close to $t=\eta_1$. □

As in §2, a continuity argument now enables one to combine Lemma 3.2 and 3.3 to obtain the following.

3.4 Theorem. *If the coefficients of* (1.4) *satisfy hypothesis* (A) - (D) *above, then for any* $\eta<\eta_1$ *the Green's function corresponding to* (1.1)$'$ *and* (1.2) *is positive in* $(0,\eta)X(0,\eta)$.

Proof. For any $\tau\in(0,\eta_1)$ we have $\lim_{a\to 0}\tilde{\eta}(a,\tau)=0$ and $\lim_{a\to\infty}\tilde{\eta}(a,\tau)=\eta_1$. Thus given $\eta\in(0,\eta_1)$, we can choose $\tilde{a}(\tau,\eta)>0$ so that $\tilde{\eta}(\tilde{a}(\tau,\eta),\tau)=\eta$, where this double zero is realized by a solution $y_\tau(t)$ which is positive in $(0,\eta)$ for all $\tau\in(0,\eta)$. Writing $y_\tau(t)=G(t,\tau)$, it follows that $G(t,\tau)>0$ in $(0,\eta)X(0,\eta)$. □

Examples and Generalizations. Specific criteria on the coefficients of (1.4) which assure that (A) - (D) are satisfied and given in [2; Theorem 3.4]. These are, in effect, "positivity conditions" on

$$B=\frac{1}{p_2} \text{ and } C=\frac{1}{4p_2}\left[p_1^2-\left(\int_0^t q_1\right)^2\right]-\frac{p_1''-q_1'}{2}-p_0$$

and "smallness and nonnegativeness conditions" on

$$A=\frac{1}{2p_2}\left[p_1-\int_0^t q_1\right] \text{ and } D=\frac{1}{2p_2}\left[p_1+\int_0^t q_1\right].$$

4.1 Theorem [2]. *If the coefficients of* (1.4) *satisfy*
(i) $C(t)\geq A(t)\geq 0$ *in* $[0,\infty)$
(ii) $B(t)\geq D(t)\geq 0$ *in* $[0,\infty)$
(iii) $v''+\min\{B(t)-D(t),C(t)-A(t)\}v$ *is oscillatory at* $t=\infty$,
(iv) $\int^\infty t\,B(t)\,dt=\int^\infty t\,C(t)\,dt=\infty$
Then conditions (A) - (D) *are satisfied.*

Using (1.3) one can readily verify, for example, that

$$u^{(iv)}-[u'-(\sin t)u]'+p_0(t)u=0$$

satisfies these hypotheses whenever $-p_0(t) \geq \frac{1}{2}$ on $[0, \infty)$. Other more complex examples are readily constructed using (1.3) (ii).

It should also be noted that boundary conditions other than (1.2) can be accommodated by these techniques. In particular, one can establish criteria for the positivity of Green's functions associated with focal and "hinged beam" problems, and these tend to require weaker hypotheses than Theorem 3.4 above.

References

[1] M.A. Krasnoselski, *Positive Solutions of Operator Equations*, Groningen, Nordhoff, 1984.

[2] K. Kreith, *A dynamical criterion for conjugate points*, Pacific. J. Math. 58 (1975), 123-132.

[3] K. Kreith, *Conjugate points of fourth order differential equations with impulses*, Differ. and Integ. Eq. 1 (1988), 391-398.

[4] W. Leighton and Z. Nehari, *On the oscillation of solutions of selfadjoint linear differential equations of fourth order*, Trans. Amer. Math. Soc. 89 (1958), 325-377.

[5] W.M. Whyburn, *On selfadjoint ordinary differential equations of the fourth order*, Amer. J. Math. 52 (1930), 171-196.

Kurt Kreith
Department of Mathematics
University of California
Davis, CA 95616, USA