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TRANSFORMATIONS OF LINEAR HAMILTONIAN SYSTEMS PRESERVING OSCILLATORY BEHAVIOUR

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ABSTRACT. The transformations of linear Hamiltonian systems which preserve oscillatory properties of these systems are investigated. The main result of the paper (Theorem 1) generalizes the well-known duality in oscillatory behaviour of mutually reciprocal Hamiltonian systems.

1. Introduction

The principal concern of this paper is to study relations between oscillatory behaviour of the linear Hamiltonian system

$$Jx' = \mathcal{A}(t)x, \tag{1}$$

where $\mathcal{A} : I = [a, \infty) \to R^{2n\times 2n}$ is a symmetric matrix, $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, I_n is the identity $n \times n$ matrix, and the linear Hamiltonian system

$$Jw' = \mathcal{B}(t)w, \tag{2}$$

which is related to (1) via the transformation

$$x = \mathcal{R}(t)w, \tag{3}$$

where $R(t) \in C^1(I)$ is a $2n \times 2n$ J-unitary matrix, i.e.,

$$\mathcal{R}^T(t)J\mathcal{R}(t) = J \quad \text{or, equivalently} \quad \mathcal{R}(t)J\mathcal{R}^T(t) = J. \tag{4}$$

It is known, see e.g. [2], that if $\mathcal{R}(t)$ is J-unitary then (2) is also a linear Hamiltonian system, i.e., $\mathcal{B}^T(t) = \mathcal{B}(t)$ ("^{T}" denotes the transpose of the matrix indicated). In more details, let

$$x = \begin{pmatrix} u \\ v \end{pmatrix}, w = \begin{pmatrix} y \\ z \end{pmatrix}, \mathcal{A} = \begin{pmatrix} -C & -A^T \\ -A & -B \end{pmatrix}, \mathcal{R} = \begin{pmatrix} H & M \\ K & N \end{pmatrix}, \mathcal{B} = \begin{pmatrix} -\tilde{C} & -\tilde{A}^T \\ -\tilde{A} & -\tilde{B} \end{pmatrix},$$

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then (4) implies

$$H^TK = K^TH,\ M^TN = N^TM,\ H^TN - K^TM = I_n, \tag{5}$$
$$HM^T = MH^T,\ KN^T = NK^T,\ HN^T - MK^T = I_n,$$

systems (1), (2) can be written in the form

$$\begin{aligned} u' &= A(t)u + B(t)v, & y' &= \bar{A}(t)y + \bar{B}(t)z, \\ v' &= -C(t)u - A^T(t)v, & z' &= -\bar{C}(t)y - \bar{A}^T(t)z, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \bar{A} &= N^T(-H' + AH + BK) + M^T(K' + CH + A^TK), \\ \bar{B} &= N^T(-M' + AM + BN) + M^T(N' + CM + A^TN), \\ \bar{C} &= H^T(K' + CH + A^TK) + K^T(-H' + AH + BK). \end{aligned} \tag{7}$$

Recall that two points $t_1, t_2 \in I$ are said to be conjugate relative to (1) if there exists a solution (u, v) of (1) such that $u(t_1) = 0 = u(t_2)$ and $u(t)$ is not identically zero between t_1 and t_2. Equation (1) is said to be oscillatory if for every $b \in [a, \infty)$ the interval $[b, \infty)$ contains at least one pair of distinct points which are conjugate relative to (1). In the opposite case (1) is said to be nonoscillatory.

The simpliest case of the transformation which preserves oscillation behaviour of the transformed linear Hamiltonian systems is the case when $M(t) \equiv 0$ in $\mathcal{R}(t)$, this trivial case will not be taken into consideration here. Another known result is the fact that if the matrices $B(t)$, $C(t)$ are nonnegative definite on I, then the "reciprocal" system

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} -A^T(t) & C(t) \\ -B(t) & A(t) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \tag{8}$$

which is the result of transformation (3) with $\mathcal{R}(t) = J$ applied to (1), is nonoscillatory if and only if (1) is nonoscillatory, see [1, 3, 6, 9, 10].

In this paper we shall show that under an additional assumption (which corresponds to the assumption of nonnegativity of B, C) systems (1) and (2) have the same oscilatory behaviour whenever the matrix $M(t)$ in $\mathcal{R}(t)$ is nonsingular on I. In the last section this results is applied to the self-adjoint, second order, differential system

$$(R(t)U')' + P(t)U = 0,$$

where P, $R : I \to R^{n\times n}$ are of the class C^1 and symmetric, $R(t)$ is positive definite. Particularly, we shall find the explicit form of the second order system whose solution is the linear combination $G_1(t)R(t)U' + G_2(t)U$, G_1, $G_2 : I \to R^{n\times n}$.

2. Preliminaries

Consider the matrix linear Hamiltonian system

$$X' = \mathcal{A}(t)X,$$

where the $2n \times 2n$ matrix $X(t)$ consists of $n \times n$ matrices U_1, V_1, U_2, V_2,

$$X(t) = \begin{pmatrix} U_1(t) & U_2(t) \\ V_1(t) & V_2(t) \end{pmatrix}.$$

Since $(X^T(t)JX(t))' = 0$, the matrix $X(t)$ is J-unitary on I whenever it is J-unitary at some $c \in I$. If this is the case, then

$$U_i^T(t)V_i(t) - V_i^T(t)U_i(t) = 0, \quad i = 1,2, \tag{9}$$

$$U_1^T(t)V_2(t) - V_1^T(t)U_2(t) = I_n \tag{10}$$

and also

$$\begin{aligned} U_1(t)U_2^T(t) - U_2(t)U_1^T(t) &= 0, \\ V_1(t)V_2^T(t) - V_2(t)V_1^T(t) &= 0, \\ U_1(t)V_2^T(t) - U_2(t)V_1^T(t) &= I_n. \end{aligned} \tag{11}$$

Any $2n \times n$ solution (U, V) of (1) satisfying (9) is said to be self-conjugate (another terminology is isotropic [4], self-conjoined [11] and prepared [8]).

Recall some known results concerning oscillatory properties and transformations of linear Hamiltonian systems which we shall use later.

Theorem A [6]. *There exists J-unitary matrix $\mathcal{R}(t) \in C^1(I)$ with $M(t) \equiv 0$ which transforms (1) into the so-called trigonometric system, i.e., the system (2), where $\bar{A} = 0, \bar{B}(t) = -\bar{C}(t) =: Q(t)$. Particularly, any $2n \times 2n$ J-unitary solution $X(t)$ of (1) can be expressed in the form*

$$\begin{pmatrix} U_1(t) & U_2(t) \\ V_1(t) & V_2(t) \end{pmatrix} = \mathcal{R}(t) \begin{pmatrix} C(t) & S(t) \\ -S(t) & C(t) \end{pmatrix},$$

where $(S(t), C(t))$ is the $2n \times 2n$ solution of the system

$$S' = Q(t)C, \quad C' = -Q(t)S. \tag{12}$$

Moreover, if the matrix $B(t)$ is nonnegative definite then $Q(t)$ is also nonnegative definite.

Theorem B [10]. *Let the symmetric matrix $Q(t) \in C(I)$ be nonnegative definite on I. Then the trigonometric system (12) is nonoscillatory if and only if $\int^{\infty} Tr\, Q(t)\, dt < \infty$, where $Tr(\ )$ denotes the trace of the matrix indicated.*

Theorem C [7]. *Consider two trigonometric systems*

$$(13)_i \qquad S_i' = Q_i(t)C_i, \ C_i' = -Q_i(t)S_i, \quad i = 1, 2,$$

where $Q_i(t) \in C(I)$ are nonnegative definite $n \times n$ matrices. Let the solutions (S_i, C_i) satisfy

$$(14)_i \qquad S_i^T(t)S_i(t) + C_i^T(t)C_i(t) = I_n, \quad S_i^T(t)C_i(t) = C_i^T(t)S_i(t).$$

If the matrix $C_1(t)S_2^T(t) - C_2(t)S_1^T(t)$ is nonsingular on I then $\int^{\infty} Tr\ Q_1(t)\, dt < \infty$ if and only if $\int^{\infty} Tr\ Q_2(t)\, dt < \infty$, i.e., $(14)_1$ and $(14)_2$ have the same oscillatory behaviour.

Observe that solutions (S_i, C_i) of $(13)_i$ satisfying $(14)_i$ always exist. Indeed, the $2n \times 2n$ matrix $W_i = \begin{pmatrix} C_i & S_i \\ -S_i & C_i \end{pmatrix}$ is a solution of $JW_i' = \begin{pmatrix} -Q_i & 0 \\ 0 & -Q_i \end{pmatrix} W_i$, hence W_i is J-unitary on I whenever it has this property at least at one point and it follows that (S_i, C_i) also satisfy identities

$$(15)_i \qquad S_i(t)S_i^T(t) + C_i(t)C_i^T(t) = I_n, \quad S_i(t)C_i^T(t) = C_i(t)S_i^T(t)\ .$$

3. Transformations preserving oscillatory behaviour

In this section we extend the result concerning duality between oscillatory behaviour of (1) and its reciprocal system (8), mentioned in Section 1.

Theorem 1. *Let the $2n \times 2n$ J-unitary matrix $\mathcal{R}(t) = \begin{pmatrix} H(t) & M(t) \\ K(t) & N(t) \end{pmatrix}$, with $M(t)$ nonsingular on I, transform (1) into (2). If the matrices $B(t)$ and $\bar{B}(t)$ are nonnegative definite on I then (1) is nonoscillatory if and only if (2) is nonoscillatory.*

Proof. Let $X(t)$, $W(t)$ be $2n \times 2n$ J-unitary solutions of (1) and (2), respectively. By Theorem A there exist nonnegative definite $n \times n$ matrices $Q_i(t)$ and $2n \times 2n$ J-unitary matrices $\mathcal{R}_i(t) \in C^1(t)$ of the form $\mathcal{R}_i = \begin{pmatrix} H_i & 0 \\ K_i & H_i^{T-1} \end{pmatrix}$ such that the transformations $X = \mathcal{R}_1(t)W_1$, $W = \mathcal{R}_2(t)W_2$ transform (1) and (2) into the trigonometric systems with the matrices $Q_1(t)$, $Q_2(t)$, respectively. Particularly, if $X = \begin{pmatrix} U_1 & U_2 \\ V_1 & V_2 \end{pmatrix}$, $W = \begin{pmatrix} Y_1 & Y_2 \\ Z_1 & Z_2 \end{pmatrix}$, then

$$(16) \qquad U_1 = H_1C_1, \ U_2 = H_1S_1,$$

$$Y_1 = H_2 C_2,\ Y_2 = H_2 S_2, \tag{17}$$

where (S_i, C_i) are $2n \times n$ matrix solutions of $(13)_i$, satisfying $(14)_i$ and $(15)_i$. Since $\mathcal{R}(t)$ transforms (1) into (2) we have $U_1 = HY_1 + MZ_1$, $U_2 = HY_2 + MZ_2$. Substituting these equalities into (16) we get $HY_1 + MZ_1 = H_1C_1$, $HY_2 + MZ_2 = H_1S_1$. Multiplication of these equalities from the right by $-Y_2^T$ and Y_1^T, respectively, their addition and substitution of (17) into right hand-sides gives

$$H(Y_1Y_2^T - Y_2Y_1^T) + M(Z_2Y_1^T - Z_1Y_2^T) = H_1(C_2S_1^T - C_1S_2^T)H_2^T.$$

As the solution $W(t)$ is J-unitary, using (11) we have $Y_1Y_2^T = Y_2Y_1^T$, $Z_2Y_1^T - Z_1Y_2^T = I_n$. Hence, the matrix $(C_2S_1^T - C_1S_2^T) = H_1^{-1}MH_2^{T-1}$ is nonsingular on I and by Theorem C $(13)_1$, $(13)_2$ are simultaneously oscillatory or nonoscillatory. As the transformations converting (1) into $(13)_1$ and (2) into $(13)_2$ preserve oscillatory behaviour ($M_i(t) \equiv 0$, see the note in Sec. 1), the same statement holds for systems (1) and (2). □

Using the duality in oscillation behaviour between (1) and its reciprocal system, one can prove the following modification of Theorem 1.

Theorem 2. *Let the $2n \times 2n$ J-unitary matrix $\mathcal{R}(t) \in C^1(I)$ of the form $\mathcal{R} = \begin{pmatrix} H & M \\ K & L \end{pmatrix}$ transform (1) into (2). Suppose that the matrices B, C, $\bar{B}$, $\bar{C}$ in (1) and (2) are nonnegative definite on I and the matrix $K(t)$ in $\mathcal{R}(t)$ is nonsingular. Then (1) is nonoscillatory if and only if (2) is nonoscillatory.*

Proof. The idea of the proof is the same as in Theorem 1. Suppose that (1) is nonoscillatory. Since B, C are nonnegative definite, system (8) — the reciprocal system to (1) — is also nonoscillatory. Now, apply to (8) and (2) the transformation which converts these system into trigonometric system, but in the form which preserves zeros of the second component z of a solution (y, z), i.e., $\mathcal{R} = \begin{pmatrix} H_i & M_i \\ 0 & H_i^{T-1} \end{pmatrix}$. Then by the same argument as in the above proof, nonsingularity of the matrix $K(t)$ (which now plays the same role as $M(t)$ above) and the statement of Theorem C imply that the system reciprocal to (1) has the same oscillation behaviour as (8) (and hence as (1)). Since the matrices B, C are nonnegative definite, the same statement holds for the system reciprocal to the system reciprocal to (2), i.e., for system (2). □

4. Second order systems

Consider the second order system

$$(R(t)U')' + P(t)U = 0, \tag{18}$$

where R, $P : I \to R^{n \times n}$ are symmetric, of the class $C^1(I)$ and $R(t)$ is positive definite. We shall look for a second order system, whose solution is the "linear combination" $G_1(t)R(t)U' + G_2(t)U$, where $G_1,\ G_2 : I \to R^{n\times n}$ are of the class $C^2(I)$. The results of this section generalize the results of [12], where a similar problem was studied in the scalar case, i. e., $n = 1$.

Since (18) can be written in the form of the linear Hamiltonian system with $U = U,\ V = RU'$, we can use the results of the preceding section with $G_1(t) = M(t),\ G_2(t) = H(t)$. Hence , in order to obtain a second order system which is also self-adjoint and has the same oscillatory behaviour, the following assumptions are needed

$$G_1(t)G_2^T(t) = G_2(t)G_1^T(t), \tag{19}$$

$$G_1(t) \text{ is nonsingular on } I. \tag{20}$$

The main difficulty is to find the matrices $K,\ N$, which, together with $H = G_2,\ M = G_1$ form the J-unitary matrix, in such a way that the resulting linear Hamiltonian system will be equivalent to a second order system. To overcome this difficulty, we proceed as follows. Let $Y_1,\ Y_2 : I \to R^{n\times n}$ be the solutions of the second order system (which may be generally nonself-adjoint)

$$Y'' + P_1(t)Y' + P_0(t)Y = 0, \tag{21}$$

where $P_0,\ P_1 : I \to R^{n\times n}$. It follows that

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix}' = \begin{pmatrix} 0 & I_n \\ -P_0 & -P_1 \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix}$$

and if Y_1, Y_2 form the basis of the solution space of (21) then

$$\begin{pmatrix} 0 & I_n \\ -P_0 & -P_1 \end{pmatrix} = \begin{pmatrix} Y_1' & Y_2' \\ Y_1'' & Y_2'' \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix}^{-1}$$

Hence, to compute the matrices P_0, P_1 if the solutions Y_1, Y_2 are known, we need to compute the inverse matrix to the Wronski matrix of the solutions Y_1, Y_2.

Let U, V, be two self-conjugate solutions of (18) (i. e., $U^{T'}RU = U^TRU'$, $V^{T'}RV = V^TRV'$) for which $U^T(t)R(t)V'(t) - U^{T'}(t)R(t)V(t) = I_n$. This identity, together with self-conjugacy of U and V imply

$$\begin{gathered} UV^T = VU^T,\ U'V^{T'} = V'U^{T'}, \\ UV^{T'} - VU^{T'} = R^{-1}. \end{gathered} \tag{22}$$

Denote

$$Y_1 = G_1RU' + G_2U, \quad Y_2 = G_1RV' + G_2V.$$

We have $Y_1Y_2^T - Y_2Y_1^T = (G_1RU' + G_2U)(V^{T'}RG_1^T + V^TG_2^T) - (G_1RV' + G_2V)$ $(U^{T'}RG_1^T + U^TG_2^T) = G_1R(U'V^{T'} - V'U^{T'})RG_1^T + G_1R(U'V^T - V'U^T)G_2^T +$

$G_2(UV^{T'} - VU^{T'})RG_1^T + G_2(UV^T - VU^T)G_2^T = -G_1G_2^T + G_2G_1^T = 0$. Similarly $Y_1Y_2^{T'} - Y_2Y_1^{T'} = G_1PG_1^T - G_1G_2^{T'} + G_2G_1^{T'} + G_2R^{-1}G_2^T$. If Y_1, Y_2 form the basis of the solution space of (21), the last matrix is nonsingular, see [11], denote it R_1^{-1}. Observe that in view of (19) this matrix is symmetric. Further denote $Q = Y_2'Y_1^{T'}$, $L = Y_1'Y_2^{T''} - Y_2'Y_1^{T''}$. Then $Y_1Y_2^{T''} - Y_2Y_1^{T''} = (R_1^{-1})' + Q - Q^T$ and $(Q^T - Q)' = Y_1''Y_2^{T'} - Y_2''Y_1^{T'} + Y_1'Y_2^{T''} - Y_1'Y_2^{T''} = -L^T + L$, hence the matrix $Q' - L^T$ is symmetric. Now,

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix} \begin{pmatrix} Y_2^{T'} & -Y_2^T \\ -Y_1^{T'} & Y_1^T \end{pmatrix} = \begin{pmatrix} R_1^{-1} & 0 \\ Q^T - Q & R_1^{-1} \end{pmatrix},$$

hence

$$\begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix}^{-1} = \begin{pmatrix} Y_2^{T'} & -Y_2^T \\ -Y_1^{T'} & Y_1^T \end{pmatrix} \begin{pmatrix} R_1 & 0 \\ R_1(Q - Q^T)R_1 & R_1 \end{pmatrix}.$$

It follows

$$\begin{pmatrix} 0 & I_n \\ -P_0 & -P_1 \end{pmatrix} = \begin{pmatrix} Y_1' & Y_2' \\ Y_1'' & Y_2'' \end{pmatrix} \begin{pmatrix} Y_2^{T'} & -Y_2^T \\ -Y_1^{T'} & Y_1^T \end{pmatrix} \begin{pmatrix} R_1 & 0 \\ R_1(Q - Q^T)R_1 & R_1 \end{pmatrix} =$$
$$= \begin{pmatrix} Q^T - Q & R_1^{-1} \\ -L^T & (R_1^{-1})' + Q^T - Q \end{pmatrix} \begin{pmatrix} R_1 & 0 \\ R_1(Q - Q^T)R_1 & R_1 \end{pmatrix}$$

and thus the equation we look for is of the form $Y'' - [(R_1^{-1})'R_1 + (Q^T - Q)R_1]Y' - [-L^T + (R_1^{-1})'R_1(Q^T - Q) + (Q^T - Q)R_1(Q - Q^T)]R_1Y = 0$. Multiplying this equation from the left by R_1, after some computation we get

(24)
$$[R_1Y' + R_1QR_1Y]' - R_1Q^TR_1Y' + R_1[L^T - Q' - QR_1'R_1^{-1} - R_1^{-1}R_1'Q^T - $$
$$- (Q^T - Q)R_1(Q - Q^T)]R_1Y = 0.$$

This equation is self-adjoint since both R_1 and $(L^T - Q' - QR_1'R_1^{-1} - R_1^{-1}R_1'Q^T - (Q^T - Q)R_1(Q - Q^T))$ are symmetric. If Y_1, Y_2 are given by (23), a routine computation gives

(25)
$$Q^T - Q = G_1'PG_1^T - G_1'G_2{}^{T'} - G_1PR^{-1}G_2^T + G_2'R^{-1}G_2^T,$$
$$L = G_1'P'G_1^T - G_1'R^{-1}G_2^{T''} + G_1'PR^{-1}G_2^T - G_1PG_1^{T''} + G_1PR^{-1}PG_1^T -$$
$$-2G_1PR^{-1}G_2^{T'} + G_1PR^{-1}R'R^{-1}G_2^T + G_1'G_2^{T''} - G_2R^{-1}PG_1^T + 2G_2'R^{-1}G_2^T -$$
$$-G_2'R^{-1}R'R^{-1}G_2^T + G_2R^{-1}P'G_1^T - G_2R^{-1}G_2^{T''} +$$
$$+2G_1'PG_1^{T'} + 2G_2R^{-1}PG_1^{T'}.$$

The following theorem summarizes the preceding computations.

Theorem 3. *Let G_1, $G_2 : I \to R^{n\times n}$ satisfy (19), (20) and suppose that the (symmetric) matrix $R_1^{-1} = (G_1PG_1^T - G_1G_2^{T'} + G_2G_1^{T'} + G_2R^{-1}G_2T)$ is positive definite on I. If U is a solution of (18) then the matrix $Y = G_1RU' + G_2U$ is a solution of the self-adjoint equation (24), where the matrices Q, L are given by (25). Moreover, this equation is nonoscillatory if and only if (18) is nonoscillatory.*

Remark 1. If we set $Z = R_1Y' + R_1QR_1Y$, equation (24) can be written in the form of a linear Hamiltonian system. Substituting for Y' and Y in the last equation, we get $Z = (R_1G_1' + R_1GR_1^{-1} + R_1QR_1G_1)RU' + (R_1G_2' - R_1G_1P)U$, hence the transformation

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} G_2 & G_1 \\ R_1G_2' - R_1G_1P & R_1G_1' + R_1G_2R_1^{-1} + R_1QR_1G_1 \end{pmatrix} \begin{pmatrix} U \\ RU' \end{pmatrix}$$

transforms the linear Hamiltonian system corresponding to (18) into the linear Hamiltonian system corresponding to (24). By a direct computation one can verify that the transformation matrix is J-unitary.

Remark 2. In the special case $G_1(t) = I_n$, $G_2(t) = G$, G being a constant symmetric $n \times n$ matrix, the results of this section has applications in the theory of singular quadratic functionals [5]. In this simple case the matrices Q and L are of the form $Q = PR^{-1}G$, $L = PR^{-1}P + PR^{-1}R'R^{-1}G + GR^{-1}P' + GGR^{-1}PR^{-1}G$ and equation (24) takes the form

$$\begin{gathered}
\left[(P+GR^{-1}G)^{-1}Y' + (P+GR^{-1}G)^{-1}PR^{-1}G(P+GR^{-1}G)^{-1}Y\right]' - \\
-(P+GR^{-1}G)^{-1}GR^{-1}P(P+GR^{-1}G)^{-1}Y' + (P+GR^{-1}G)\left[PR^{-1}G- \right. \\
-G(R^{-1})'P + P(R^{-1})'G + GR^{-1}PR^{-1}G + PR^{-1}G(P'+ \\
+G(R^{-1})'G)(P+GR^{-1}G)^{-1} + (P+GR^{-1}G)^{-1}(P'+ \\
+G(R^{-1})'G)GR^{-1}P - (GR^{-1}P - PR^{-1}G)(P+ \\
\left. +GR^{-1}G)^{-1}(PR^{-1}G - GR^{-1}P)\right](P+GR^{-1}G)^{-1}Y = 0.
\end{gathered}$$

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