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Archivum Mathematicum, Vol. 28 (1992), No. 1-2, 85--94

Persistent URL: http://dml.cz/dmlcz/107440

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SOME FACTORIZATIONS OF MATRIX FUNCTIONS IN SEVERAL VARIABLES

JAROMÍR ŠIMŠA

Dedicated to Professor M. Novotný on the occasion of his seventieth birthday

Abstract. We establish some criteria for a nonsingular square matrix depending on several parameters to be represented in the form of a matrix product of factors which depend on the single parameters.

The purpose of the present work is to find functional and differential equations for matrix-valued functions \( H \) that admit factorization

\[
H(x, y) = F(x) \cdot G(y)
\]

or, more generally,

\[
H(x_1, x_2, \ldots, x_k) = F_1(x_1) \cdot F_2(x_2) \cdot \ldots \cdot F_k(x_k),
\]

where \( \cdot \) stands for the usual matrix multiplication. The history of the scalar version of this problem goes back to the year 1747, when J. d’Alembert [d’Al] recognized that each (smooth) scalar function \( h(x, y) = f(x)g(y) \) has to satisfy the following partial differential equation

\[
h_{xy}h - h_xh_y = 0.
\]

In 1904, C. Stéphanos [St] announced a significant generalization of d’Alembert’s result: scalar functions of the type

\[
h(x, y) = \sum_{k=1}^{n} f_k(x)g_k(y)
\]

1991 Mathematics Subject Classification: 39B42, 26B40.
Key words and phrases: separation of variables, matrix-valued functions, functional and differential equations for factorizable functions.
Received February 14, 1992.
form the space of all solutions of the partial differential equation with the "Wronskian" of order \( n + 1 \)

\[
\begin{vmatrix}
  h & h_y & \cdots & h_{y^n} \\
  h_x & h_{xy} & \cdots & h_{xy^n} \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{x^n} & h_{x^n y} & \cdots & h_{x^n y^n}
\end{vmatrix} = 0
\]

(5)

(for the more precise statement, applications, further extensions and related results see [Neu 1], [Ra], [GR], [Neu 2], [CS 1], [NR] and [CS 2]).

Suppose that matrices \( H, F \) and \( G \) in (1) are of type \( n \times n \) and denote their entries by \( h_{ij}, f_{ij} \) and \( g_{ij} \), respectively, where \( i, j \in \{1, 2, \ldots, n\} \). Then (1) represents a system of \( n^2 \) scalar equalities

\[
h_{ij}(x, y) = \sum_{k=1}^{n} f_{ik}(x)g_{kj}(y),
\]

each of them is of type (4). Consequently, the above mentioned result of Stéphanos yields a necessary (but not sufficient) condition for a (smooth) matrix function \( H \) to have factorization (1): \( \text{each entry } h_{ij} \text{ is a solution of the Wronski equation (5).} \)

We will show here that criteria for factorization (1) can be stated in terms of matrix operations, without taking single entries of the matrix \( H \) and without using equations like (5). However, our procedure is not applicable unless the values of \( H \) are nonsingular, i.e. \( \det(H(x, y)) \neq 0 \) for all \( x \) and \( y \). Let us finish this introductory part by remarking that a smooth matrix \( H \) of type (1) need not satisfy the equation

\[
H_{xy} \cdot H - H_x \cdot H_y = 0,
\]

(6)

a formal matrix analogy of (3). (Equation (6) holds if the matrices \( F \) and \( G \) in (1) commute, which is rather an exceptional case.) The correct version of (6) is given in Theorem 3 below.

Throughout the paper, \( GL_n(\mathbb{K}) \) denotes the group of all \( n \times n \) nonsingular matrices with elements from the field \( \mathbb{K} \), where \( \mathbb{K} \) stands for \( \mathbb{R} \) (reals) or \( \mathbb{C} \) (complex numbers). First we derive a functional equation that characterizes functions (1) without any smoothness condition.

**Theorem 1.** Let \( H : X \times Y \rightarrow GL_n(\mathbb{K}) \), where \( X \) and \( Y \) are arbitrary nonempty sets. Choose elements \( x_1 \in X \) and \( y_1 \in Y \). Then the mapping \( H \) has a factorization (1) if and only if it satisfies the functional equation

\[
H(x, y) = H(x, y_1) \cdot H^{-1}(x_1, y_1) \cdot H(x_1, y) \quad \text{for each } x \in X \text{ and } y \in Y.
\]

(7)

Moreover, the factors \( F : X \rightarrow GL_n(\mathbb{K}) \) and \( G : Y \rightarrow GL_n(\mathbb{K}) \) from any representation (1) are exactly pairs of the form

\[
F(x) = H(x, y_1) \cdot C \quad \text{and} \quad G(y) = D \cdot H(x_1, y),
\]

(8)
where $C, D \in GL_n(\mathbb{K})$ are arbitrary constant matrices satisfying $C \cdot D = H^{-1}(x_1, y_1)$.

**Proof.** Let $H$ be as in (1). Setting first $y = y_1$ and then $x = x_1$ in (1), we find that

$$F(x) = H(x, y) \cdot G^{-1}(y_1) \text{ and } G(y) = F^{-1}(x_1) \cdot H(x_1, y)$$

for each $x \in X$ and $y \in Y$. Multiplying these equalities and taking in account that

$$G^{-1}(y_1) \cdot F^{-1}(x_1) = (F(x_1) \cdot G(y_1))^{-1} = H^{-1}(x_1, y_1),$$

we conclude that $H$ satisfies (7) and (8) holds. Conversely, if $H$ satisfies (7) and if $C, D \in GL_n(\mathbb{K})$ are arbitrary matrices satisfying $C \cdot D = H^{-1}(x_1, y_1)$, then

$$(H(x, y_1)C) \cdot (D H(x_1, y)) = H(x, y_1) \cdot H^{-1}(x_1, y_1) \cdot H(x_1, y) = H(x, y)$$

and the proof is complete.

Let us add to Theorem 1 a simple but important rule

$$(9) \quad H \text{ is of type (1) } \Rightarrow \quad H(x_1, y) \cdot H^{-1}(x_2, y) \text{ does not depend on } y,$$

which will be used in next proofs.

Now we turn our attention to the matrix functions $H$ of type (1) which are differentiable in one of both variables, say $x$ in Theorem 2 (for the case of the variable $y$ see Remark 1). We show that such functions are characterized by a mixed functional differential equation.

**Theorem 2.** Let $H : X \times Y \to GL_n(\mathbb{K})$, where $X$ is an interval in $\mathbb{R}$ and $Y$ is a nonempty set. Suppose that the partial derivative $H_x$ exists at each point of $X \times Y$. Then the mapping $H$ has a factorization (1) if and only if it satisfies

$$(10) \quad H_x(x, y) \cdot H^{-1}(x, y) = H_x(x, y_1) \cdot H^{-1}(x, y_1) \text{ for each } x \in X \text{ and } y, y_1 \in Y.$$

**Proof.** (i) If $H$ is as in (1), then

$$H_x(x, y) \cdot H^{-1}(x, y) = (F'(x)G(y)) \cdot (G^{-1}(y)F^{-1}(x)) = F'(x) \cdot F^{-1}(x)$$

for each $y \in Y$, hence the both sides of (10) are equal to $F'(x) \cdot F^{-1}(x)$.

(ii) If $H$ satisfies (10), then

$$\frac{\partial}{\partial x} (H^{-1}(x, y_1)H(x, y)) =$$

$$- H^{-1}(x, y_1)H_x(x, y_1)H^{-1}(x, y_1)H(x, y) + H^{-1}(x, y_1)H_x(x, y) =$$

$$H^{-1}(x, y_1)\left[-H_x(x, y_1)H^{-1}(x, y_1) + H_x(x, y)H^{-1}(x, y)\right]H(x, y) = 0.$$
Thus $H^{-1}(x, y_1) \cdot H(x, y)$ does not depend on $x \in X$, i.e.

$$H^{-1}(x, y_1) \cdot H(x, y) = H^{-1}(x_1, y_1) \cdot H(x_1, y)$$

for each $x \in X$,

where $x_1 \in X$ is a chosen point. Multiplying the last equality by $H(x, y_1)$ from the left, we obtain factorization (7).

**Remark 1.** The reader can easily verify that

$$(11) \quad H^{-1}(x, y) \cdot H_y(x, y) = H^{-1}(x_1, y) \cdot H_y(x_1, y) \quad (x, x_1 \in X, y \in Y)$$

is the analogy of (10) for functions $H$ differentiable in the variable $y$.

Now we state a differential criterion of (1) for mappings $H$ which are smooth in both variables $x$ and $y$.

**Theorem 3.** Let $H : X \times Y \to GL_n(\mathbb{R})$, where $X$ and $Y$ are two intervals in $\mathbb{R}$. Suppose that the partial derivatives $H_x$, $H_y$ and $H_{xy} = (H_x)_y$ exist at each point of $X \times Y$. Then the mapping $H$ has a factorization (1) if and only if it solves the differential equation

$$(12) \quad H_{xy} = H_x \cdot H^{-1} \cdot H_y \quad \text{on the rectangle } X \times Y.$$

**Proof.** If $H$ is as in (1) and the derivatives $H_x$ and $H_y$ exist, then (8) implies that the derivatives $F' = \frac{dF}{dx}$ and $G' = \frac{dG}{dy}$ exist too. So we can write

$$H_x \cdot H^{-1} \cdot H_y = (F'G) \cdot (FG)^{-1} \cdot (FG') = F'GG^{-1}F^{-1}FG' =$$

$$= F'G' = H_{xy},$$

which means that $H$ satisfies (12). Conversely, let $H$ be such that the derivatives $H_x$, $H_y$, $H_{xy} = (H_x)_y$ exist and satisfy (12). Then the product $H_x \cdot H^{-1}$ is differentiable in $y$ and

$$\frac{\partial}{\partial y}(H_x \cdot H^{-1}) =$$

$$= H_{xy}H^{-1} - H_xH^{-1}H_yH^{-1} = (H_{xy} - H_xH^{-1}H_y)H^{-1} = 0$$

on the set $X \times Y$. Hence $H_x \cdot H^{-1}$ does not depend on $y \in Y$, i.e. the mapping $H$ satisfies (10). In view of Theorem 2, $H$ has a factorization (1).

**Remark 2.** In the statement of Theorem 3, the mixed derivative $(H_x)_y$ can be replaced by $(H_y)_x$, because any solution of $(H_y)_x = H_x \cdot H^{-1} \cdot H_y$ satisfies (11).
Now we will solve the problem when a smooth nonsingular matrix function $H$ in $p+q$ variables is factorizable into the form

\[ H(x_1, \ldots, x_p; y_1, \ldots, y_q) = F(x_1, \ldots, x_p) \cdot G(y_1, \ldots, y_q). \]

Let us emphasize that if $H : (X_1 \times \ldots \times X_p) \times (Y_1 \times \ldots \times Y_q) \to GL_n(\mathbb{K})$, then Theorem 1 with vector variables $\mathbf{x} = (x_1, \ldots, x_p)$ and $\mathbf{y} = (y_1, \ldots, y_q)$ yields the following conclusion: the factors $F$ and $G$ from any factorization (13) of the function $H$ are given by

\[ F(x_1, \ldots, x_p) = H(x_1, \ldots, x_p; v_1, \ldots, v_q) \cdot C \]
\[ G(y_1, \ldots, y_q) = D \cdot H(u_1, \ldots, u_p; y_1, \ldots, y_q) \]

where the elements $u_i \in X_i$ and $v_j \in Y_j$ are chosen arbitrarily and the matrices $C, D \in GL_n(\mathbb{K})$ satisfy $C \cdot D = H^{-1}(u_1, \ldots, u_p; v_1, \ldots, v_q)$.

**Theorem 4.** Let $X = X_1 \times \ldots \times X_p$ and $Y = Y_1 \times \ldots \times Y_q$ be the Cartesian products of real intervals $X_1, \ldots, X_p$ and $Y_1, \ldots, Y_q$, respectively. Suppose that a mapping $H : X \times Y \to GL_n(\mathbb{K})$ has the partial derivatives $H_{x_i}, H_{y_j}$, and $H_{x_iy_j}$ (in some order of differentiation) on the set $X \times Y$, $1 \leq i \leq p$ and $1 \leq j \leq q$. Then the mapping $H$ has a factorization (13) if and only if it satisfies the system of $pq$ differential equations

\[ H_{x_iy_j} = H_{x_i} \cdot H^{-1} \cdot H_{y_j} (1 \leq i \leq p, 1 \leq j \leq q) \text{ on the set } X \times Y. \]

**Proof.** Consider the partial functions $H_{ij} : X_i \times Y_j \to GL_n(\mathbb{K})$ defined by

\[ H_{ij}(x_i, y_j) = H(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p; y_1, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_q) \]

on condition that the other variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p$ and $y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_q$ are assumed to be fixed.

(i) If $H$ is as in (13), then $H_{ij}(x_i, y_j) = F_i(x_i) \cdot G_j(y_j)$, where

\[ F_i(x_i) = F(x_1, \ldots, x_p) \] \[ G_j(y_j) = G(y_1, \ldots, y_q) \]

Applying Theorem 3 (or Remark 2) to each function $H_{ij}$, we conclude that $H$ satisfies (14).

(ii) Suppose that $H$ solves (14). Then Theorem 3 (or Remark 2) implies that each partial function $H_{ij}$ is of type (1) on the set $X_i \times Y_j$. Choose $u_1 \in X_1$ and $v \in Y$ and define a mapping $\Phi_1 : X \to GL_n(\mathbb{K})$ by

\[ \Phi_1(\mathbf{x}) = H(x_1, \ldots, x_p; v) \cdot H^{-1}(u_1, x_2, \ldots, x_p; v) , \]

for each $\mathbf{x} = (x_1, \ldots, x_p) \in X$. According to the rule (9) applied to $H_{ij}$, where $1 \leq j \leq q$, the matrix product

\[ H(x_1, \ldots, x_p; y_1, \ldots, y_q) \cdot H^{-1}(u_1, x_2, \ldots, x_p; y_1, \ldots, y_q) \]
does not depend on any of the variables \( y_1, \ldots, y_q \), i.e. it equals to \( \Phi(x_1, \ldots, x_p) \).

This leads to the factorization

\[
H(x; y) = \Phi_1(x) \cdot H(u_1, x_2, \ldots, x_p; y),
\]

for each \( x = (x_1, \ldots, x_p) \in X \) and \( y \in Y \). In the case when \( p > 1 \), we repeat the previous procedure to the function \( H(x_2, \ldots, x_p; y) = H(u_1, x_2, \ldots, x_p; y) \) to obtain the factorization

\[
H(u_1, x_2, \ldots, x_p; y) = \Phi_2(x_2, \ldots, x_p) \cdot H(u_1, u_2, x_3, \ldots, x_p; y)
\]

(with a chosen \( u_2 \in X_2 \)), etc. After \( p \) repetitions we conclude that \( H \) is of the form (13) in which

\[
F(x_1, \ldots, x_p) = \Phi_1(x_1, \ldots, x_p) \cdot \Phi_2(x_2, \ldots, x_p) \cdot \ldots \cdot \Phi_p(x_p)
\]

and \( G(y_1, \ldots, y_q) = H(u_1, \ldots, u_p; y_1, \ldots, y_q) \). This completes the proof.

Now we start to deal with the factorization problem (2). To state an extension of Theorem 1 as Theorem 5, we introduce the following notation. Given a function

\[
H : X_1 \times X_2 \times \ldots \times X_k \rightarrow GL_n(\mathbb{K})
\]

and chosen \( k \) elements \( u_i \in X_i \), \( 1 \leq i \leq k \), we define the \( k \)-tuple of partial functions \( H_i : X_i \rightarrow GL_n(\mathbb{K}) \) by

\[
H_i(x) = H(u_1, \ldots, u_{i-1}, x, u_{i+1}, \ldots, u_k) \quad (x \in X_i, \ 1 \leq i \leq k).
\]

**Theorem 5.** Let \( H : X_1 \times \ldots \times X_k \rightarrow GL_n(\mathbb{K}) \), where \( X_1, \ldots, X_k \) are \( k \geq 2 \) non-empty sets. Consider partial functions (15) for a fixed \( k \)-tuple \( u_1, \ldots, u_k \). Then the mapping \( H \) has a factorization (2) if and only if it satisfies the equation

\[
H(x) = H_1(x_1) \cdot H_0^{-1} \cdot H_2(x_2) \cdot H_0^{-1} \cdot \ldots \cdot H_0^{-1} \cdot H_k(x_k)
\]

for any \( x = (x_1, \ldots, x_k) \in X_1 \times \ldots \times X_k \),

where \( H_0 = H(u_1, \ldots, u_k) \). Moreover, the factors \( F_i : X_i \rightarrow GL_n(\mathbb{K}) \) from any factorization (2) are given by

\[
F_i(x) = H_1(x) \cdot C_i, \quad F_i(x) = D_i^{-1} \cdot H_i(x) \cdot C_i \quad (1 < i < k)
\]

and \( F_k(x) = D_{k-1} \cdot H_k(x) \),

where \( C_i, D_i \in GL_n(\mathbb{K}) \) are arbitrary constant matrix satisfying

\[
C_1 \cdot D_1 = C_2 \cdot D_2 = \ldots = C_{k-1} \cdot D_{k-1} = H_0^{-1}.
\]

**Proof.** If \( H \) is as in (2), then one can check inductively that

\[
H_1(x_1) H_0^{-1} \ldots H_0^{-1} H_i(x_i) = F_1(x_1) \ldots F_i(x_i) F_{i+1}(u_{i+1}) \ldots F_k(u_k),
\]
for each \(i = 2, 3, \ldots, k\). This equality with \(i = k\) proves (16). Conversely, if the
mapping \(H\) satisfies (16), then it is clearly of type (2), with factors \(F_1 = H_1\) and \(F_i = H_{i-1}^{-1}H_i, i = 2, 3, \ldots, k\) (as well as with factors (17) under condition (18)). So
it remains to show that the factors \(F_i\) from any factorization must be of type (17).
Indeed, it follows from (2) that \(F_1(x) = H_1(x)H_0^{-1}F_1(u_1),\)
\[
F_i(x) = F_{i-1}^{-1}(u_{i-1})F_{i-2}^{-1}(u_{i-2}) \ldots F_1^{-1}(u_1)H_i(x)H_0^{-1}F_1(u_1)F_2(u_2) \ldots F_i(u_i)
\]
for \(i = 2, 3, \ldots, k - 1\) and \(F_k(x) = F_{k-1}^{-1}(u_{k-1}) \ldots F_1^{-1}(u_1)H_k(x).\) So \(F_i\) are of type
(17), with matrices
\[
H = H_0^{-1}F_1(u_1)F_2(u_2) \ldots F_i(u_i) \quad \text{and} \quad D_i = F_i^{-1}(u_i)F_{i-1}^{-1}(u_{i-1}) \ldots F_1^{-1}(u_1)
\]
that satisfy (18). The proof is complete.

Comparing (16) with (7), the reader may analogously presume that
(19)
\[
H_{x_1, x_2 \ldots, x_k} = H_{x_1} \cdot H^{-1}_{x_2} \cdot H^{-1}_{x_3} \cdot \ldots \cdot H^{-1}_{x_k}
\]
is a good generalization of (12) for the factorization problem (2). We disprove this
conjecture in the following

**Theorem 6.** Let \(X_1, \ldots, X_k\) be \(k \geq 2\) intervals in \(R\) and let the mapping \(F_i : \)
\(X_i \to GL_n(\mathbb{R})\) be differentiable at each point of \(X_i, 1 \leq i \leq k.\) Then the mapping \(H\) defined by (2) is a solution of (19) on the set \(X_1 \times \ldots \times X_k.\) However, in the
case when \(k \geq 3,\) equation (19) has such solutions which are not of the form (2).

**Proof.** If \(H\) is as in (2), with differential factors \(F_i,\) then one can check inductively that
\[
H_{x_1}H^{-1}_{x_2}H^{-1}_{x_3} \ldots H^{-1}_{x_k} = F'_1F'_2 \ldots F'_iF'_{i-1}F'_{i-2} \ldots F'_1
\]
for \(i = 1, 2, \ldots, k - 1\) and, in the last step,
\[
H_{x_1}H^{-1}_{x_2}H^{-1}_{x_3} \ldots H^{-1}_{x_k} = F'_1F'_2 \ldots F'_k = H_{x_1, x_2 \ldots, x_k}.
\]
Hence \(H\) solves (19). On the other side, a smooth mapping \(H = H(x_2, \ldots, x_k)\)
(which does not depend on \(x_1\)) is an example of a solution of (19), which is not of
type (2) in general (provided that \(k \geq 3\)). The proof is complete.

We finish our paper by showing that the factorization problem (2) can be reduced to a family of problems (1). This reduction (described in Theorem 7) enables to formulate differential criteria of factorizations (2), based on the preceding results
on factorizations (1) - see Remark 3. Given a mapping \(H : X_1 \times \ldots \times X_k \to GL_n(\mathbb{R}),\)
let us introduce the families of \(\binom{k}{2}\) partial functions \(H_{\alpha \beta} : X_\alpha \times X_\beta \to GL_n(\mathbb{R}),\)
where \(1 \leq \alpha < \beta \leq k,\) defined by
(20) \[H_{\alpha \beta}(x_\alpha, x_\beta) = H(x_1, \ldots, x_k)\] for any \(x_\alpha \in X_\alpha\) and \(x_\beta \in X_\beta\)
on condition that the other variables \(x_i \in X_i (1 \leq i \leq k, i \neq \alpha\) and \(i \neq \beta)\) are assumed
to be fixed.
Theorem 7. Let $H : X_1 \times \ldots \times X_k \to GL_n(\mathbb{K})$, where $X_1, \ldots, X_k$ are $k \geq 3$ nonempty sets. The mapping $H$ has a factorization (2) if and only if each partial function $H_{\alpha\beta}$ ($1 \leq \alpha < \beta \leq k$, see (20)) is of the form

$$H_{\alpha\beta}(x_{\alpha}, x_{\beta}) = \Phi(x_{\alpha}) \cdot \Psi(x_{\beta}) \quad (x_{\alpha} \in X_{\alpha} \text{ and } x_{\beta} \in X_{\beta}),$$

(21)

for each $(k-2)$-tuple of the other variables $x_i \in X_i \ (1 \leq i \leq k, i \neq \alpha \text{ and } i \neq \beta)$.

Proof. We will proceed in a similar way as in the proof of Theorem 4. If $H$ is as in (2) and $1 \leq \alpha < \beta \leq k$, then (21) holds with

$$\Phi(x_{\alpha}) = F_1(x_1) \cdot \ldots \cdot F_\alpha(x_\alpha) \text{ and } \Psi(x_{\beta}) = F_{\alpha+1}(x_{\alpha+1}) \cdot \ldots \cdot F_k(x_k).$$

Conversely, suppose that each $\binom{k}{2}$-tuple of partial functions $H_{\alpha\beta}$ of a given function $H$ satisfies (21). Choose fixed elements $u_i \in X_i, \ 1 \leq i \leq k-1$. In view of the rule (9) applied to each $H_{\alpha\beta}$, the relations

$$F_i(x_i) = H(u_1, \ldots, u_{i-1}, x_i, \ldots, x_k) \cdot H^{-1}(u_1, \ldots, u_i, x_{i+1}, \ldots, x_k) \ (1 \leq i \leq k-1)$$

determine $k-1$ mappings $F_i : X_i \to GL_n(\mathbb{K})$ (in a correct way). Moreover, the identity $F_1(x_1) \cdot \ldots \cdot F_i(x_i) = H(x_1, \ldots, x_k) \cdot H^{-1}(u_1, \ldots, u_i, x_{i+1}, \ldots, x_k)$ holds for $i = 1, 2, \ldots, k-1$. So putting $F_k(x_k) = H(u_1, \ldots, u_{k-1}, x_k)$, we get factorization (2). This completes the proof.

Remark 3. A criterion for each factorization (21) can be stated by applying one of Theorems 1 – 3 (or even Theorem 4 if some of the variables $x_1, \ldots, x_k$ are multidimensional). For example, if $X_1, \ldots, X_k$ are real intervals, then (smooth) mappings $H : X_1 \times \ldots \times X_k \to GL_n(\mathbb{K})$ of type (2) form the set of all solutions of the differential system

$$H_{x_\alpha x_\beta} = H_{x_\alpha} \cdot H^{-1} \cdot H_{x_\beta} \quad (1 \leq \alpha < \beta \leq k) \text{ on } X_1 \times \ldots \times X_k.$$

Remark 4. The reader may ask whether the system of $\binom{k}{2}$ conditions (21) can be reduced to a subsystem, say the subsystem of $(k-1)$ conditions (21) with

$$(\alpha, \beta) \in \{(1, 2), (2, 3), \ldots, (k-1, k)\}.$$ 

The negative answer follows from the following example. Given indices $p$ and $q$ ($1 \leq p < q \leq k$), define a mapping $H = H(x_1, \ldots, x_k) = \Phi(x_p, x_q)$, where $\Phi$ is a matrix-valued function in two variables which does not permit factorization (1). It is obvious that such a mapping $H$ has $\binom{k}{2}$ factorizations (21) excepting the only one, that with $\alpha = p$ and $\beta = q$.

Remark 5. Let us mention an open problem which generalizes the subject of the present paper: Given $k$ surjective mappings $\varphi_i : X \to Y_i, \ 1 \leq i \leq k$, find...
some necessary and sufficient conditions for a mapping $H : X \to GL_n(\mathbb{K})$ to be factorizable into

\begin{equation}
H(x) = F_1(\varphi_1(x)) \cdot F_2(\varphi_2(x)) \cdot \ldots \cdot F_k(\varphi_k(x)),
\end{equation}

with some factors $F_i : Y_i \to GL_n(\mathbb{K})$. We are able to solve it only in the case when the mapping $\varphi : X \to Y_1 \times Y_2 \times \ldots \times Y_k$ defined by

$$
\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_k(x)) \quad (x \in X)
$$

is a bijection. Then (22) can be solved by transforming $\tilde{H}(y) = H(\varphi^{-1}(y))$ to a problem treated here: $\tilde{H}(y_1, y_2, \ldots, y_k) = F_1(y_1) \cdot F_2(y_2) \cdot \ldots \cdot F_k(y_k)$. As an example of this procedure, we derive functional and differential equations for matrix functions of the form

\begin{equation}
H(x, y) = F(x + y) \cdot G(x - y).
\end{equation}

**Corollary 1.** (i) Let $S$ be an abelian group divisible by 2. A given mapping $H : S \times S \to GL_n(\mathbb{K})$ has a factorization (23) if and only if it satisfies the equation

$$
H(x, y) = H \left( \frac{x + y}{2}, \frac{x + y}{2} \right) \cdot H^{-1}(0, 0) \cdot H \left( \frac{x - y}{2}, \frac{y - x}{2} \right)
$$

for any $x, y \in S$.

(ii) Suppose that a mapping $H : \mathbb{R} \times \mathbb{R} \to GL_n(\mathbb{K})$ has the second order differential $d^2H$ at each point of the plane $\mathbb{R} \times \mathbb{R}$. Then $H$ has a factorization (23) if and only if it satisfies the differential equation

$$
H_{xx} - H_{yy} = (H_x + H_y) \cdot H^{-1} \cdot (H_x - H_y) \quad \text{on} \ \mathbb{R} \times \mathbb{R}.
$$

**Proof.** Corollary 1 is an immediate consequence of Theorems 1 and 3 applied to the mapping

$$
\tilde{H}(u, v) = H \left( \frac{u + v}{2}, \frac{u - v}{2} \right).
$$

**Acknowledgement.** I wish to thank to Prof. Jiří Vanžura who motivated me to study the problem of matrix factorization (1).
References


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